

# THOM PROSPECTRA FOR LOOPGROUP REPRESENTATIONS

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ABSTRACT. This is very much an account of work in progress. We sketch the construction of an Atiyah dual (in the category of  $\mathbb{T}$ -spaces) for the free loop space of a manifold; the main technical tool is a kind of Tits building for loop groups, discussed in detail in an appendix. Together with a new localization theorem for  $\mathbb{T}$ -equivariant  $K$ -theory, this yields a construction of the elliptic genus in the string topology framework of Chas-Sullivan, Cohen-Jones, Dwyer, Klein, and others. We also show how the Tits building can be used to construct the dualizing spectrum of the loop group. Using a tentative notion of equivariant  $K$ -theory for loop groups, we relate the equivariant  $K$ -theory of the dualizing spectrum to recent work of Freed, Hopkins and Teleman.

## Introduction

If  $P \rightarrow M$  is a principal bundle with structure group  $G$  then  $LP \rightarrow LM$  is a principal bundle with structure group

$$LG = \text{Maps}(S^1, G),$$

and if the tangent bundle of  $M$  is defined by a representation  $V$  of  $G$  then the tangent bundle of  $LM$  is defined by the representation  $LV$  of  $LG$ . The circle group  $\mathbb{T}$  acts on all these spaces.

This is a report on the beginnings of a theory of differential topology for such objects. Note that if we want the structure group  $LG$  to be connected, we need  $G$  to be 1-connected; thus  $SU(n)$  is preferable to  $U(n)$ . This helps explain why Calabi-Yau manifolds are so central in string theory, and this note is written assuming this simplifying hypothesis.

Alternately, we could work over the universal cover of  $LM$ ; then  $\pi_2(M)$  would act on everything by deck translations, and our topological invariants become modules over the Novikov ring  $\mathbb{Z}[H_2(M)]$ . From the point of view we're developing, these translations seem to be what really underlies modularity, but this issue, like several others, will be backgrounded here.

The circle action on the free loop space defines a structure much closer to classical differential geometry than one finds on more general (eg) Hilbert manifolds; this action defines something like a Fourier filtration on the tangent space of this infinite-dimensional manifold, which is in some sense locally finite. This leads to a host of new kinds of geometric invariants, such as the Witten genus; but this filtration is unfamiliar, and has been difficult to work with [17]. The main conceptual result of this note [which was motivated by ideas of Cohen, Godin, and Segal] is the definition of a canonical equivariant ‘thickening’ of a free loop space, where

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the pulled-back tangent bundle admits a canonical filtration by finite-dimensional equivariant bundles. This thickening involves a contractible  $LG$ -space called the affine Tits building  $\mathbf{A}(LG)$ . This space occurs under various guises in nature: it is a homotopy colimit of homogeneous spaces with respect to a finite collection of compact Lie subgroups of  $LG$ . It is also the affine space of principal  $G$ -connections on the trivial bundle over  $S^1$ . We explore its structure in the appendix.

In the first section below, we recall why the Spanier-Whitehead dual of a finite CW-space is a ring-spectrum, and sketch the construction (due to Milnor and Spanier, and Atiyah) of a model for that dual, when the space is a smooth compact manifold. Our goal is to produce an analog of this construction for a free loop space, which captures as much as possible of its string-topological algebraic structure. In the second section, we introduce the technology used in our construction: pro-spectra associated to filtered infinite-dimensional vector bundles, and the topological Tits building which leads to the construction of such a filtration for the tangent bundle.

In §3 we observe that recent work of Freed, Hopkins, and Teleman on the Verlinde algebra can be reformulated as a conjectural duality between  $LG$ -equivariant  $K$ -theory of a certain dualizing spectrum for  $LG$  constructed from its Tits building, and positive-energy representations of  $LG$ . In §4 we use a new strong localization theorem to study the equivariant  $K$ -theory of our construction, and we show how this recovers the Witten genus from a string-topological point of view.

We plan to discuss actions of various string-topological operads [15] on our construction in a later paper; that work is in progress.

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## 1. THE ATIYAH DUAL OF A MANIFOLD

If  $X$  is a finite complex, then the function spectrum  $F(X, S^0)$  is a **ring-spectrum** (because  $S^0$  is). If  $X$  is a manifold  $M$ , Spanier-Whitehead duality says that

$$F(M_+, S^0) \sim M^{-TM} .$$

If  $E \rightarrow X$  is a vector bundle over a compact space, we can define its Thom space to be the one-point compactification

$$X^E := E_+ .$$

There is always a vector bundle  $E_\perp$  over  $X$  such that

$$E \oplus E_\perp \cong \mathbf{1}_N$$

is trivial, and following Atiyah, we write

$$X^{-E} := S^{-N} X^{E_\perp} .$$

With this notation, the Thom collapse map for an embedding  $M \subset \mathbb{R}^N$  is a map

$$S^N = \mathbb{R}_+^N \rightarrow M^\nu = S^N M^{-TM} ,$$

and the midpoint construction

$$M_+ \wedge M^\nu \rightarrow \mathbb{R}_+^N = S^N$$

defines the equivalence with the functional dual. More generally, a smooth map  $f : M \rightarrow N$  of compact closed orientable manifolds has a Pontrjagin-Thom dual map

$$f_{PT} : N^{-TN} \rightarrow M^{-TM}$$

of spectra; in particular, the map  $S^0 \rightarrow M^{-TM}$  dual to the projection to a point defines a kind of fundamental class, and the dual to the diagonal of  $M$  makes  $M^{-TM}$  into a ring-spectrum [13].

**Prospectus:** Chas and Sullivan [10] have constructed a very interesting product on the homology of a free loop space, suitably desuspended, motivated by string theory. Cohen and Jones [15] saw that this product comes from a ring-spectrum structure on

$$LM^{-TM} := LM^{-e^*TM}$$

where

$$e : LM \rightarrow M$$

is the evaluation map at  $1 \in S^1$ . Unfortunately this evaluation map is not  $\mathbb{T}$ -equivariant, so the Chas-Sullivan Cohen-Jones spectrum is not in general a  $\mathbb{T}$ -spectrum. The full Atiyah dual constructed below promises to capture some of this equivariant structure. The Chas-Sullivan Cohen-Jones spectrum and the full Atiyah dual live in rather different worlds: our prospectus is an equivariant object, whose multiplicative properties are not yet clear, while the CSCJ spectrum has good multiplicative properties, but it is not a  $\mathbb{T}$ -spectrum. In some vague sense our object resembles a kind of center for the Chas-Sullivan-Cohen-Jones spectrum, and we hope that a better understanding of the relation between open and closed strings will make it possible to say something more explicit about this.

## 2. PROBLEMS & SOLUTIONS

For our constructions, we need two pieces of technology:

Cohen, Jones, and Segal [16](appendix) associate to a filtration

$$\mathbf{E} : \cdots \subset E_i \subset E_{i+1} \subset \cdots$$

of an infinite-dimensional vector bundle over  $X$ , a pro-object

$$X^{-\mathbf{E}} : \cdots \rightarrow X^{-E_{i+1}} \rightarrow X^{-E_i} \rightarrow \cdots$$

in the category of spectra. [A rigid model for such an object can be constructed by taking  $\mathbf{E}$  to be a bundle of Hilbert spaces, which are trivializable by Kuiper's theorem. Choose a trivialization  $\mathbf{E} \cong H \times X$  and an exhaustive filtration  $\{H_k\}$  of  $H$  by finite-dimensional vector spaces; then we can define

$$X^{-E_i} = \lim S^{-H_k} X^{H_k \cap E_i^\perp},$$

with  $E_i^\perp$  the orthogonal complement of  $E_i$  in the trivialized bundle  $\mathbf{E}$ .] This pro-object will, in general, depend on the choice of filtration. We will be interested in the **direct** systems associated to such a pro-object by a cohomology theory; of course in general the colimit of this system can be very different from the cohomology of the limit of the system of pro-objects.

**Example 2.1.** If  $X = \mathbb{C}P_\infty$ ,  $\eta$  is the Hopf bundle, and  $\mathbf{E}$  is

$$\infty\eta : \cdots \subset (k-1)\eta \subset k\eta \subset (k+1)\eta \subset \cdots$$

then the induced maps of cohomology groups are multiplication by the Euler classes of the bundles  $E_{i+1}/E_i$ , so

$$H^*(\mathbb{C}P_\infty^{-\infty\eta}, \mathbb{Z}) := \operatorname{colim}\{Z[t], t - \text{mult}\} = \mathbb{Z}[t, t^{-1}].$$

We would like to apply such a construction to the tangent bundle of a free loop space. Unfortunately, these tangent bundles do **not**, in general, possess any such nice filtration by finite-dimensional ( $\mathbb{T}$ -equivariant) subbundles [17]! However, such a splitting **does** exist in a neighborhood of the **constant** loops:

$$M = LM^{\mathbb{T}} \subset LM$$

has normal bundle

$$\nu(M \subset LM) = TM \otimes_{\mathbb{C}} (\mathbb{C}[q, q^{-1}]/\mathbb{C})$$

(at least, up to completions; and assuming things complex for convenience). Here small perturbations of a constant loop are identified with their Fourier expansions

$$\sum_{n \in \mathbb{Z}} a_n q^n,$$

with  $q = e^{i\theta}$ . The related fact, that  $TLM$  is defined by the representation  $LV$  of  $LG$  looped up from the finite-dimensional representation  $V$  of  $G$ , will be important below: for  $LV$  is **not** a positive-energy representation of  $LG$ .

The main step toward our resolution of this problem depends on the following result, proved in §7 below. Such constructions were first studied by Quillen, and were explored further by S. Mitchell [26]. The first author has studied these buildings for a general Kac-Moody group [23]; most of the properties of the affine building used below hold for this larger class.

**Theorem 2.2.** *The topological affine Tits building*

$$\mathbf{A}(LG) := \operatorname{hocolim}_I LG/H_I$$

of  $LG$  is  $\mathbb{T}\tilde{\times}LG$ -equivariantly contractible. In other words, given any compact subgroup  $K \subset \mathbb{T}\tilde{\times}LG$ , the fixed point space  $\mathbf{A}(LG)^K$  is contractible.

[Here  $I$  runs over certain proper subsets of roots of  $G$ , and the  $H_I$  are certain **compact** ‘parabolic’ subgroups of  $LG$  (see §7.2).]

**Remark 2.3.** The group  $LG$  admits a universal central extension  $\mathbb{L}G$ . The natural action of the rotation group  $\mathbb{T}$  on  $LG$  lifts to  $\mathbb{L}G$ , and the  $\mathbb{T}$ -action preserves the subgroups  $H_I$ . Hence  $\mathbf{A}(LG)$  admits an action of  $\mathbb{T}\tilde{\times}\mathbb{L}G$ , with the center acting trivially. We can therefore express  $\mathbf{A}(LG)$  as

$$\mathbf{A}(LG) = \operatorname{hocolim}_I \mathbb{L}G/\mathbb{H}_I$$

where  $\mathbb{H}_I$  is the induced central extension of  $H_I$ .

OTHER DESCRIPTIONS OF  $\mathbf{A}(LG)$ 

This Tits building has other descriptions as well. For example:

1.  $\mathbf{A}(LG)$  can be seen as the classifying space for proper actions with respect to the class of compact Lie subgroups of  $\mathbb{T} \tilde{\times} LG$ .
2. It also admits a more differential-geometric description as the smooth infinite dimensional manifold of holonomies on  $S^1 \times G$  (see Appendix): Let  $\mathcal{S}$  denote the subset of the space of smooth maps from  $\mathbb{R}$  to  $G$  given by

$$\mathcal{S} = \{g(t) : \mathbb{R} \rightarrow G, g(0) = 1, g(t+1) = g(t) \cdot g(1)\};$$

then  $\mathcal{S}$  is homeomorphic to  $\mathbf{A}(LG)$ . The action of  $h(t) \in LG$  on  $g(t)$  is given by  $hg(t) = h(t) \cdot g(t) \cdot h(0)^{-1}$ , where we identify the circle with  $\mathbb{R}/\mathbb{Z}$ . The action of  $x \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$  is given by  $xg(t) = g(t+x) \cdot g(x)^{-1}$ .

3. The description given above shows that  $\mathbf{A}(LG)$  is equivalent to the affine space  $\mathcal{A}(S^1 \times G)$  of connections on the trivial  $G$ -bundle  $S^1 \times G$ . This identification associates to the function  $f(t) \in \mathcal{S}$ , the connection  $f'(t)f(t)^{-1}$ . Conversely, the connection  $\nabla_t$  on  $S^1 \times G$  defines the function  $f(t)$  given by transporting the element  $(0, 1) \in \mathbb{R} \times G$  to the point  $(t, f(t)) \in \mathbb{R} \times G$  using the connection  $\nabla_t$  pulled back to the trivial bundle  $\mathbb{R} \times G$ .

**Remark 2.4.** These equivalent descriptions have various useful consequences. For example, the model given by the space  $\mathcal{S}$  of holonomies says that given a finite cyclic group  $H \subset \mathbb{T}$ , the fixed point space  $\mathcal{S}^H$  is homeomorphic to  $\mathcal{S}$ . Moreover, this is a homeomorphism of  $LG$ -spaces, where we consider  $\mathcal{S}^H$  as an  $LG$ -space and identify  $LG$  with  $LG^H$  in the obvious way. Notice also that  $\mathcal{S}^{\mathbb{T}}$  is  $G$ -homeomorphic to the model of the adjoint representation of  $G$  defined by  $\text{Hom}(\mathbb{R}, G)$ .

Similarly, the map  $\mathcal{S} \rightarrow G$  given by evaluation at  $t = 1$  is a principal  $\Omega G$  bundle, and the action of  $G = LG/\Omega G$  on the base  $G$  is given by conjugation. This allows us to relate our work to that of Freed, Hopkins and Teleman in the following section.

Finally, the description of  $\mathbf{A}(LG)$  as the affine space  $\mathcal{A}(S^1 \times G)$  implies that the fixed point space  $\mathbf{A}(LG)^K$  is contractible for any compact subgroup  $K \subseteq \mathbb{T} \tilde{\times} LG$ .

If  $E \rightarrow B$  is a principal bundle with structure group  $LG$ , then (motivated by ideas of [14]) we construct a ‘thickening’ of  $B$ :

**Definition 2.5.** *The thickening of  $B$  associated to the bundle  $E$  is the  $LG$ -space*

$$B_{\dagger}(E) = E \times_{LG} \mathbf{A}(LG) = \text{hocolim}_I E/H_I.$$

We will omit  $E$  from the notation, when the defining bundle is clear from context.

**Remark 2.6.** If  $P \rightarrow M$  is a principal  $G$  bundle, then  $LP \rightarrow LM$  is a principal  $LG$  bundle. In this case, the description above gives  $L_{\dagger}M := LM_{\dagger}(LP)$  a smooth structure:

$$L_{\dagger}M = \{(\gamma, \omega) \mid \gamma \in LM, \omega \in \mathcal{A}(\gamma^*(P))\}$$

where  $\mathcal{A}(\gamma^*(P))$  is the space of connections on the pullback bundle  $\gamma^*(P)$ .

Let  $\mathbb{T} \tilde{\times} LG$  be the extension of the central extension of  $LG$  by  $\mathbb{T}$  acting as rotations. On restriction to the subgroup  $\mathbb{T} \tilde{\times} \mathbb{H}_I$ , a unitary representation  $U$  of  $\mathbb{T} \tilde{\times} LG$

decomposes into a sum of finite dimensional representations. We want to construct a Thom  $\mathbb{T}\tilde{\times}\mathbb{L}G$ -prospectrum  $\mathbf{A}(LG)^{-U}$ .

We consider the decomposition of the restriction of  $U$  to  $\mathbb{T}\tilde{\times}\mathbb{H}_I$  as a sum of irreducibles:

$$U|_{\mathbb{T}\tilde{\times}\mathbb{H}_I} \cong \oplus U_I(\alpha)$$

and let

$$U_I(k) = \oplus \{U_I(\alpha) \mid \dim U_I(\alpha) \leq k\}.$$

Then we can define  $S_I^{-U}$  to be the Thom  $\mathbb{T}\tilde{\times}\mathbb{H}_I$ -prospectrum associated to the filtered (equivariant) vector bundle

$$\mathbf{U}_I : \cdots \subset U_I(k) \subset U_I(k+1) \subset \cdots$$

over a point. If  $I \subset J$  then  $\mathbb{H}_I$  maps naturally to  $\mathbb{H}_J$ , and there is a corresponding morphism

$$\mathbf{U}_J \rightarrow \mathbf{U}_I$$

of filtered vector bundles, given by inclusions  $U_J(k) \rightarrow U_I(k)$ .

**Definition 2.7.** We define  $\mathbf{A}(LG)^{-U}$  to be the  $\mathbb{T}\tilde{\times}\mathbb{L}G$ -prospectrum

$$\mathbf{A}(LG)^{-U} = \text{hocolim}_I \mathbb{L}G_+ \wedge_{\mathbb{H}_I} S_I^{-U},$$

where  $\mathbb{L}G_+$  denotes  $\mathbb{L}G$ , with a disjoint basepoint.

Homotopy colimits in the category of prospectra can be defined in general, using the model category structure of [11].

**Remark 2.8.** Given any principal  $LG$ -bundle  $E \rightarrow B$ , and a representation  $U$  of  $LG$ , we define the Thom prospectrum of the virtual bundle associated to the representation  $-U$  to be

$$B_!^{-U} = E_+ \wedge_{LG} \mathbf{A}(LG)^{-U} = \text{hocolim}_I E_+ \wedge_{H_I} S_I^{-U}.$$

In particular, if  $P$  is the refinement of the frame bundle of  $M$  via a representation  $V$  of  $G$ , then the tangent bundle of  $LM$  is defined by the representation  $LV$  of  $LG$ .

**Definition 2.9.** The Atiyah dual  $LM^{-\mathbf{T}LM}$  of  $LM$  is the pro-spectrum  $L_!M^{-LV}$ .

We will explore this object further in §6.

### 3. THE DUALIZING SPECTRUM OF $\mathbb{L}G$

The dualizing spectrum of a topological group  $K$  is defined [24] as the  $K$ -homotopy fixed point spectrum:

$$D_K = K_+^{hK} = F(EK_+, K_+)^K$$

where  $K_+$  is the suspension spectrum of the space  $K_+$ , endowed with a right  $K$ -action. The dualizing spectrum  $D_K$  admits a  $K$ -action given by the residual left  $K$ -action on  $K_+$ . If  $K$  is a compact Lie group, then it is known that  $D_K$  is the one point compactification of the adjoint representation  $S^{Ad(K)}$ . It is also known that there is a  $K \times K$ -equivariant homotopy equivalence

$$K_+ \cong F(K_+, D_K).$$

It follows from the compactness of  $K_+$  that for any free  $K_+$ -spectrum  $E$ , we have the  $K$ -equivariant homotopy equivalence

$$E \cong F(K_+, E \wedge_{K_+} D_K) .$$

It is our plan to understand the dualizing spectrum for the (central extension of the) loop group.

**Theorem 3.1.** *There is an equivalence*

$$D_{LG} \cong \operatorname{holim}_I LG_+ \wedge_{H_I} S^{Ad(H_I)}$$

of left  $LG$ -spectra.

*Proof.* We have the sequence of equivalences:

$$D_{LG} = F(ELG_+, LG_+)^{LG} \cong F(ELG_+ \wedge \mathbf{A}(LG)_+, LG_+)^{LG} .$$

The final space may be written as

$$\operatorname{holim}_I F(ELG_+ \wedge_{H_I} LG_+, LG_+)^{LG} = \operatorname{holim}_I F(ELG_+, LG_+)^{H_I} .$$

Now recall the equivalence of  $H_I \times H_I$ -spectra:

$$(1) \quad LG_+ \cong F(H_{I+}, LG_+ \wedge_{H_I} D_{H_I}) .$$

Taking  $H_I$ -homotopy fixed points implies a left  $H_I$ -equivalence

$$F(ELG_+, LG_+)^{H_I} = (LG_+)^{hH_I} \cong LG_+ \wedge_{H_I} S^{Ad(H_I)} ;$$

where we have used equation (1) at the end. Replacing this term into the homotopy limit completes the proof.  $\square$

Similarly, we have:

**Theorem 3.2.** *There is an equivalence*

$$D_{\mathbb{L}G} \cong \operatorname{holim}_I \mathbb{L}G_+ \wedge_{\mathbb{H}_I} S^{Ad(\mathbb{H}_I)}$$

of left  $\mathbb{L}G$ -spectra.

**Remark 3.3.** The diagram underlying  $D_{LG}$  or  $D_{\mathbb{L}G}$  can be constructed in the category of spaces. Given an inclusion  $I \subseteq J$ , the orbit of a suitable element in  $Ad(\mathbb{H}_J)$  gives an embedding  $\mathbb{H}_J/\mathbb{H}_I \subset Ad(\mathbb{H}_J)$ , and the Pontrjagin-Thom construction for this embedding defines an  $\mathbb{H}_J$ -equivariant map

$$S^{Ad(\mathbb{H}_J)} \longrightarrow \mathbb{H}_{J+} \wedge_{\mathbb{H}_I} S^{Ad(\mathbb{H}_I)}$$

which extends to the map

$$\mathbb{L}G_+ \wedge_{\mathbb{H}_J} S^{Ad(\mathbb{H}_J)} \longrightarrow \mathbb{L}G_+ \wedge_{\mathbb{H}_I} S^{Ad(\mathbb{H}_I)}$$

required for the diagram. Moreover, composites of these maps can be made compatible up to homotopy.

A CONJECTURAL RELATIONSHIP WITH THE WORK OF FREED, HOPKINS AND  
TELEMAN

The discussion below assumes the existence of a hypothetical  $\mathbb{L}G$ -equivariant  $K$ -theory, whose value on a point in degree zero is the Grothendieck group of positive energy representations (or equivalently, the group of characters of integrable representations). The symmetric monoidal category whose objects are finite direct sums of irreducible positive energy representations, and whose morphism spaces consist of the (nonequivariant) isomorphisms of the vector spaces underlying the representation (given the compactly generated topology) defines a candidate for a spectrum representing such a functor: this is a topological category with an  $\mathbb{L}G$ -action which respects the symmetric monoidal structure.

This hypothesis provides us with a convenient language. We expect to return to the underlying technical issues in a later paper.

The center of  $\mathbb{L}G$  acts trivially on  $D_{\mathbb{L}G}$ , defining a second grading on  $K_{\mathbb{L}G}^*(D_{\mathbb{L}G})$ ; we will use a formal variable  $z$  to keep track of the grading, so

$$K_{\mathbb{L}G}^*(D_{\mathbb{L}G}) = \bigoplus_n K_{\mathbb{L}G}^{*,n}(D_{\mathbb{L}G})z^n.$$

The spectral sequence for the cohomology of a cosimplicial space, in the case of  $K_{\mathbb{L}G}^*(D_{\mathbb{L}G})$ , has

$$E_2^{i,j} = \operatorname{colim}_I^i K_{\mathbb{H}_I}^j(S^{Ad(\mathbb{H}_I)}).$$

This spectral sequence respects the second grading given by powers of  $z$ . In a sequel to this paper, we will show that this spectral sequence collapses to give

$$K_{\mathbb{L}G}^*(D_{\mathbb{L}G}) = \operatorname{colim}_I K_{\mathbb{H}_I}^*(S^{Ad(\mathbb{H}_I)}) \cong \operatorname{colim}_I K_{\mathbb{H}_I}^{*-r-1}(pt),$$

where  $r$  is the rank of  $G$ . Therefore, this group admits a natural Thom class given the system  $\{S(Ad(\mathbb{H}_I))\}$  of spinor bundles for the adjoint representations of the parabolics  $\mathbb{H}_I$ . In section 11 of [19], the authors construct an explicit map between the Verlinde algebra and this colimit, as follows:

To a positive energy representation corresponding to a dominant character  $\lambda$ , we associate the  $\mathbb{L}G$ -equivariant bundle given by  $\mathcal{L}_{-\lambda-\rho} \otimes S(N)$ , where  $\mathcal{L}_{-\lambda-\rho}$  is the canonical line bundle over the coadjoint orbit of the regular element  $\lambda + \rho$ , and  $S(N)$  is the spinor bundle of the normal bundle to the coadjoint orbit. Such an orbit is of the form  $\mathbb{L}G/\mathbb{H}$  for some parabolic subgroup  $\mathbb{H}$ , and its normal bundle is  $Ad(\mathbb{H})$ , so this element defines a class in  $K_{\mathbb{L}G}(D_{\mathbb{L}G})$ . The same can be done for antidominant weights. This suggests the following:

**Conjecture 3.4.** For following map is an isomorphism in homogeneous degree  $z^n$ , for  $n \neq 0$ :

$$\bigoplus_{k \geq 0} V_k z^{k+h} \bigoplus_{k \geq 0} V_k z^{-(k+h)} \cong \operatorname{colim}_I K_{\mathbb{H}_I}(pt) \rightarrow K_{\mathbb{L}G}^{r+1}(D_{\mathbb{L}G}) = \bigoplus_n K_{\mathbb{L}G}^{*,n}(D_{\mathbb{L}G})z^n$$

where  $V_k$  is the Verlinde algebra of level  $k$ ,  $h$  is the dual Coxeter number of  $G$ , and  $r$  is its rank.

**Example 3.5.** To illustrate this in an example, consider the case  $G = SU(2)$ . In this case  $r = 1$ ,  $h(G) = 2$ . Here the groups  $\mathbb{H}_I$  are given by

$$\mathbb{H}_0 = SU(2) \times S^1, \quad \mathbb{H}_1 = S^1 \times SU(2), \quad \mathbb{H}_0 \cap \mathbb{H}_1 = T = S^1 \times S^1.$$

The respective representation rings may be identified by restriction with subalgebras of  $K_T(pt) = \mathbb{Z}[u^{\pm 1}, z^{\pm 1}]$  :

$$K_{\mathbb{H}_0}(pt) = \mathbb{Z}[u + u^{-1}, (z/u)^{\pm 1}], \quad K_{\mathbb{H}_1}(pt) = \mathbb{Z}[z^{\pm 1}, u + u^{-1}] .$$

Now consider the two pushforward maps involved in the colimit:

$$\varphi_0 : K_T(pt) \rightarrow K_{\mathbb{H}_0}(pt), \quad \varphi_1 : K_T(pt) \rightarrow K_{\mathbb{H}_1}(pt)$$

A quick calculation shows that for  $k > 0$ , we have

$$\varphi_j(z^k) = \begin{cases} (z/u)^k \text{Sym}^k(u + u^{-1}), & j = 0 \\ z^k, & j = 1 \end{cases}$$

$$\varphi_j(z^k u^{-1}) = \begin{cases} (z/u)^k \text{Sym}^{k-1}(u + u^{-1}), & j = 0 \\ 0, & j = 1, \end{cases}$$

where  $\text{Sym}^k(V)$  denotes the  $k$ -th symmetric power of the representation  $V$ , e.g.  $\text{Sym}^k(u + u^{-1}) = u^k + \dots + u^{-k}$ .

We also have a similar formula for negative exponents:

$$\varphi_j(z^{-k}) = \begin{cases} -(u/z)^k \text{Sym}^{k-2}(u + u^{-1}), & j = 0 \\ z^{-k}, & j = 1 \end{cases}$$

$$\varphi_j(z^{-k} u^{-1}) = \begin{cases} -(u/z)^k \text{Sym}^{k-1}(u + u^{-1}), & j = 0 \\ 0, & j = 1. \end{cases}$$

The colimit is the cokernel of

$$\varphi_1 \oplus \varphi_0 : K_T(pt) \longrightarrow K_{\mathbb{H}_1}(pt) \oplus K_{\mathbb{H}_0}(pt)$$

Now consider the decomposition

$$\mathbb{Z}[u^{\pm 1}, z^{\pm 1}] = \mathbb{Z}[u + u^{-1}, z^{\pm 1}] \oplus u^{-1}\mathbb{Z}[u + u^{-1}, z^{\pm 1}] .$$

It is easy to check from this that the cokernel for nontrivial powers of  $z$  is isomorphic to the cokernel of  $\varphi_0$  restricted to  $u^{-1}\mathbb{Z}[u + u^{-1}, z^{\pm 1}]$  and hence is

$$\bigoplus_{k \geq 0} \frac{\mathbb{Z}[u + u^{-1}]}{\langle \text{Sym}^{k+1}(u + u^{-1}) \rangle} (z/u)^{k+2} \bigoplus_{k \geq 0} \frac{\mathbb{Z}[u + u^{-1}]}{\langle \text{Sym}^{k+1}(u + u^{-1}) \rangle} (u/z)^{k+2}$$

which agrees with the classical result [18].

**Remark 3.6.** We can calculate the equivariant  $K$ -homology  $K_{\mathbb{L}G_*}(\mathbf{A}(LG))$  using the same spectral sequence. This establishes an isomorphism between  $K_{\mathbb{L}G}^*(D_{\mathbb{L}G})$  and  $K_{\mathbb{L}G_*}(\mathbf{A}(LG))$ . Results of [19] suggest that the latter group calculates the Verlinde algebra, which is yet another motivation for the conjecture. Recall also that  $\mathbf{A}(LG)$  is the classifying space for **proper** actions (i.e. with compact isotropy) so our conjecture is a topological analog of the Baum-Connes conjecture for finite groups [7]

**Question.** Given a manifold  $LM$ , with frame bundle  $LP$ , we can construct a spectrum

$$D_{LM} := \text{holim}_I LP_+ \wedge_{H_I} S^{Ad(H_I)}$$

It would be very interesting to understand something about  $K_T(D_{LM})$ .

## 4. LOCALIZATION THEOREMS

If  $E$  is a  $\mathbb{T}$ -equivariant complex-oriented multiplicative cohomology theory, and  $X$  is a  $\mathbb{T}$ -space, we have contravariant ( $j^*$ ) and covariant ( $j^!$ ) homomorphisms associated to the fixedpoint inclusion

$$j : X^{\mathbb{T}} \subset X ,$$

satisfying

$$j^* j^!(x) = x \cdot e_{\mathbb{T}}(\nu) ;$$

if the Euler class of the normal bundle  $\nu$  is invertible, this leads to a close relation between the cohomology of  $X$  and  $X^{\mathbb{T}}$ .

More generally, if  $f : M \rightarrow N$  is an equivariant map, then its Pontrjagin-Thom transfer is related to the analogous transfer defined by its restriction

$$f^{\mathbb{T}} : M^{\mathbb{T}} \rightarrow N^{\mathbb{T}}$$

to the fixedpoint spaces, by a ‘clean intersection’ formula:

$$j_N^* \circ f^!(-) = f^{\mathbb{T}!}(j_M^*(-) \cdot e_{\mathbb{T}}(\nu(f)|_{M^{\mathbb{T}}})) .$$

**Definition 4.1.** *The fixed-point orientation defined by the Thom class*

$$\mathrm{Th}^{\dagger}(\nu(f^{\mathbb{T}})) = \mathrm{Th}(\nu(f^{\mathbb{T}})) \cdot e_{\mathbb{T}}(\nu(f)|_{M^{\mathbb{T}}})$$

for the normal bundle of the inclusion of fixed-point spaces is the product of the usual Thom class with the equivariant Euler class of the full normal bundle restricted to the fixed-point space.

Since  $f^{\mathbb{T}!}(-) = f_{PT}^{\mathbb{T}*}(- \cdot \mathrm{Th}(\nu(f^{\mathbb{T}})))$ , in this new notation the clean intersection formula becomes

$$j_N^* \circ f^! = f^{\mathbb{T}\dagger} \circ j_M^*$$

with a new Pontrjagin-Thom transfer

$$f^{\mathbb{T}\dagger}(-) = f_{PT}^{\mathbb{T}*}(- \cdot \mathrm{Th}^{\dagger}(\nu(f^{\mathbb{T}}))) .$$

In the case of most interest to us (free loopspaces), we identified the normal bundle above, in §2; using that description, we have

$$e_{\mathbb{T}}(\nu(M \subset LM)) = \prod_{0 \neq k \in \mathbb{Z}; i} (e(L_i) +_E [k](q)),$$

where the  $L_i$  are the line bundles in a formal decomposition of  $TM$ ,  $q$  is the Euler class of the standard one-dimensional complex representation of  $\mathbb{T}$ , and  $+_E$  is the sum with respect to the formal group law defined by the orientation of  $E$ . It may not be immediately obvious, but it turns out that such a formula implies that the fixed-point orientation defined above will have good multiplicative properties.

Such Weierstrass products sometimes behave better when ‘renormalized’, by dividing by their values on constant bundles [2]. If  $E$  is  $K_{\mathbb{T}}$  with the usual complex (Todd) orientation, we have

$$e(L) +_K [k](q) = 1 - q^k L ;$$

but for our purposes things turn out better with the Atiyah-Bott-Shapiro **spin** orientation; in that case the corresponding Euler class is

$$(q^k L)^{1/2} - (q^k L)^{-1/2} .$$

The square roots make sense under the simple-connectivity assumptions on  $G$  mentioned in the introduction:

To be precise, let  $V$  be a representation of  $LG$  with an intertwining action of  $\mathbb{T}$ . We restrict ourselves to representations  $V$  which (for lack of a better name [?]) we call **symmetric**, i.e. such that  $V$  is equivalent to the representation of  $LG$  obtained by composing  $V$  with the involution of  $LG$  which reverses the orientation of the loops. The restriction of the representation  $V$  to the constant loops  $\mathbb{T} \times G \subset \mathbb{T} \tilde{\times} LG$  has a decomposition

$$V = V^{\mathbb{T}} \oplus \sum_{k \neq 0} V_k q^k$$

where  $V_k$  are representations of  $G$ , and  $q$  denotes the fundamental representation of  $\mathbb{T}$ . Let  $V(m)$  be the finite dimensional subrepresentation

$$V(m) = V^{\mathbb{T}} \oplus \sum_{0 < |k| \leq m} V_k q^k ;$$

the symmetry assumption implies that  $V_k = V_{-k}$  as representations of  $G$ , so this can be rewritten

$$V(m) = V^{\mathbb{T}} \oplus \sum_{0 < k \leq m} V_k (q^k \oplus q^{-k}) .$$

At this point we need the following

**Proposition 4.2.** *If  $G$  is a compact Lie group, and  $W$  is an  $m$ -dimensional complex spin representation of  $G$ , then the representation  $\tilde{W} = W \otimes (q^k \oplus q^{-k})$  of  $\mathbb{T} \times G$  admits a canonical spin structure.*

*Proof.* The representation  $q^k \oplus q^{-k}$  of  $\mathbb{T}$  admits a unique spin structure. Since  $W$  is also endowed with a spin structure, the representation  $W \otimes (q^k \oplus q^{-k})$  admits a canonical spin structure defined by their tensor product.  $\square$

**Remark 4.3.** *If  $G$  is simply connected, then any representation  $W$  of  $G$  admits a unique spin structure.*

This justifies the square roots of the formal line bundles appearing in the restriction of  $TLM$  to  $M$ . The resulting renormalized Euler class

$$\prod_{k \neq 0} \frac{(q^k L)^{1/2} - (q^k L)^{-1/2}}{q^{k/2} - q^{-k/2}}$$

is a product of terms of the form

$$(1 - q^k L)(-q^k L)^{-1/2}(q^{-k} L)^{1/2}(1 - q^k L^{-1})$$

divided by terms of the form

$$(1 - q^k)(-q^k)^{-1/2}(q^{-k})^{1/2}(1 - q^k) ,$$

(where now all  $k$ 's are positive) yielding a unit

$$\epsilon_{\mathbb{T}}(L) = \prod_{k \geq 1} \frac{(1 - q^k L)(1 - q^k L^{-1})}{(1 - q^k)^2}$$

in the ring  $\mathbb{Z}[L^{\pm}][[q]]$ .

Following 4.1, we can reformulate the localization theorem in terms of a new orientation, obtained by multiplying the ABS Thom class by this unit, to get **precisely** the Mazur-Tate normalization

$$\sigma(L, q) = (L^{1/2} - L^{-1/2}) \prod_{k \geq 1} \frac{(1 - q^k L)(1 - q^k L^{-1})}{(1 - q^k)^2}$$

for the Weierstrass sigma-function as Thom class for a line bundle  $L$ . This extends by the splitting principle to define the orientation giving the Witten genus [32].

## 5. ONE MORAL OF THE STORY

Since the early 80's physicists have been trying to interpret

$$M \mapsto K_{\mathbb{T}}(LM)$$

as a kind of elliptic cohomology theory; but of course we know better, because we know that mapping-space constructions (such as free loop spaces) don't preserve cofibrations.

Now it is an easy exercise in commutative algebra to prove that

$$\mathbb{Z}((q)) := \mathbb{Z}[[q]][q^{-1}]$$

is flat over

$$K_{\mathbb{T}} = \mathbb{Z}[q^{\pm}],$$

for the completion of a Noetherian ring, eg  $\mathbb{Z}[q]$ , at an ideal, eg  $(q)$ , defines a flat [6](§10.14)  $\mathbb{Z}[q]$ -algebra  $\mathbb{Z}[[q]]$ . Flat modules pull back to flat modules [8] (Ch I §2.7), so it follows that  $\mathbb{Z}((q))$  is flat over the localization

$$\mathbb{Z}[q^{\pm}] := \mathbb{Z}[q][q^{-1}] = \mathbb{Z}[q, q^{-1}].$$

The Weierstrass product above is a genuine formal power series in  $q$ , so for questions involving the Witten genus it is formally easier to work with the functor defined on finite  $\mathbb{T}$ -CW spaces by

$$X \mapsto K_{\mathbb{T}}^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) := K_{\mathbb{T}}^*(X).$$

This takes cofibrations to long exact sequence of  $\mathbb{Z}((q))$ -modules. Its real virtue, however, is that it satisfies a strong localization theorem:

**Theorem 5.1.** *If  $X$  is a finite  $\mathbb{T}$ -CW space, then restriction to the fixedpoints defines an isomorphism*

$$j^* : K_{\mathbb{T}}^*(X) \cong K_{\mathbb{T}}^*(X^{\mathbb{T}}).$$

Proof, by skeletal induction; based on the

**Lemma 5.2.** *If  $C \subset \mathbb{T}$  is a proper closed subgroup, then*

$$K_{\mathbb{T}}^*(\mathbb{T}/C) = 0.$$

*Proof.* If  $C$  is cyclic of order  $n \neq 1$ , then

$$K_{\mathbb{T}}(\mathbb{T}/C) = K_{\mathbb{T}}(\mathbb{T}/C) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = \mathbb{Z}[q]/(q^n - 1) \otimes_{\mathbb{Z}[q]} \mathbb{Z}((q))$$

is zero, since

$$-1 = (q^n - 1) \cdot \sum_{k \geq 0} q^{nk} = 0 .$$

On the other hand, if  $C = \{0\}$ , then

$$K_{\mathbb{T}}(T) = \mathbb{Z} \otimes_{\mathbb{Z}[q]} \mathbb{Z}((q)) ,$$

with  $\mathbb{Z}$  a  $\mathbb{Z}[q]$ -module via the specialization  $q \rightarrow 1$ ; but by a similar argument, the resulting tensor product again vanishes.  $\square$

The functor  $K_{\mathbb{T}}^*$  extends to an equivariant cohomology theory on the category of  $\mathbb{T}$ -CW spaces, which sends a general (large) object  $X$  to the pro- $\mathbb{Z}((q))$ -module

$$\{K_{\mathbb{T}}^*(X_i) \mid X_i \in \text{finite } \mathbb{T}\text{-CW} \subset X\} ,$$

[5] (appendix). We can thus extend the claim above:

**Corollary 5.3.** *For a general  $\mathbb{T}$ -CW-space  $X$ , the restriction-to-fixedpoints map*

$$K_{\mathbb{T}}^*(X) \rightarrow K_{\mathbb{T}}^*(X^{\mathbb{T}})$$

*is an isomorphism of pro-objects. Moreover, if the fixedpoint space  $X^{\mathbb{T}}$  is a finite CW-space, then the pro-object on the left is isomorphic to the **constant** pro-object on the right.*

The free loop space  $LX$  of a CW-space  $X$  is weakly  $\mathbb{T}$ -homotopy equivalent to a  $\mathbb{T}$ -CW-space, by a map which preserves the fixed-point structure [25] (§1.1).

**Theorem 5.4.**

$$M \mapsto K_{\mathbb{T}}(LM) := K_{\text{Tate}}(M)$$

*is a cohomology theory, after all!*

**Remark 5.5.** This seems to be what the physicists have been trying to tell us all along: they probably thought (as the senior author did [27]) that the formal completion was a minor technical matter, not worth making any particular fuss about. Of course our construction is a completion of a much smaller (elliptic) cohomology theory, whose coefficients are modular forms, with the completion map corresponding to the  $q$ -expansion. The geometry underlying modularity is still [9] quite mysterious.

**Remark 5.6.** A cohomology theory defined on finite spectra extends to a cohomology theory on all spectra [1]; moreover, any two extensions are equivalent, and the equivalence is unique up to phantom maps. For the case at hand, it is clear that this cohomology theory is equivalent to the formal extension  $K((q))$ , where  $K$  is complex  $K$ -theory, with  $q$  a parameter in degree zero. Hovey and Strickland [20] have shown that an evenly graded spectrum does not support phantom maps, so our cohomology theory is uniquely equivalent, as a homotopy functor, to  $K((q))$ .

However, there is more to our construction than a simple homotopy functor: it comes with a natural (fixed-point) orientation, which defines a systematic theory of Thom isomorphisms. In the terminology of [4], it is represented by an **elliptic**

spectrum, associated to the Tate curve over  $\mathbb{Z}((q))$ ; its natural orientation is defined by the  $\sigma$ -function of §4.

**Remark 5.7.** Let  $H_{\mathbb{T}}$  denote  $\mathbb{T}$ -equivariant singular Borel cohomology with rational coefficients; then  $H_{\mathbb{T}}^*(pt) = \mathbb{Q}[t]$ , where  $t$  has degree two. We have a localization theorem

$$H_{\mathbb{T}}^*(X)[t^{-1}] = H_{\mathbb{T}}^*(X^{\mathbb{T}})[t^{-1}] = H^*(X^{\mathbb{T}}) \otimes \mathbb{Q}[t^{\pm}]$$

for finite complexes, and can therefore play the same game as before, and observe that there is a unique extension  $\hat{H}_{\mathbb{T}}(-)$  of  $H_{\mathbb{T}}(-)[t^{-1}]$  to all  $\mathbb{T}$ -spectra, with a localization theorem valid for an arbitrary  $\mathbb{T}$ -spectrum  $X$ :

$$\hat{H}_{\mathbb{T}}^*(X) = \hat{H}_{\mathbb{T}}^*(X^{\mathbb{T}}).$$

Jones and Petrack [21] have constructed such a theory over the real numbers, together with the analogous fixedpoint orientation - which in their case is (a rational version of) the  $\hat{A}$ -genus.

Results of this sort (which relate oriented equivariant cohomology theories on free loopspaces to cohomology theories on the fixedpoints, with related (but distinct) formal groups, is part of an emerging understanding of what homotopy theorists call ‘chromatic redshift’ phenomena, cf. [2, 3, 31].

**Remark 5.8.** Our completion of equivariant  $K$ -theory is the natural repository for characters of **positive-energy** representations of loop groups; it is **not** preserved by the orientation-reversing involution  $\lambda \mapsto \lambda^{-1}$  of  $\mathbb{T}$ . It is in some sense a **chiral** completion.

**Remark 5.9.** The completion theorem above is a specialization of Segal’s original localization theorem [29], which says that  $K_{\mathbb{T}}(X)$ , considered as a sheaf over the multiplicative groupscheme  $\text{Spec } K_{\mathbb{T}} = \mathbb{G}_m$  (cf. [28]), has for its stalks over generic (ie nontorsion) points, the  $K$ -theory of the fixed point space. The Tate point

$$\text{Spec } \mathbb{Z}((q)) \rightarrow \text{Spec } \mathbb{Z}[q^{\pm}]$$

is an example of such a generic point, perhaps too close to zero (or infinity) to have received the attention it seems to deserve.

In fact, we can play a similar game for any oriented theory  $E_{\mathbb{T}}$ . Let  $\mathcal{E}$  denote the union of the one-point compactifications of all representations of  $\mathbb{T}$  which do not contain the trivial representation (cf. [25]). Then the theory  $E_{\mathbb{T}} \wedge \mathcal{E}$  satisfies a strong localization theorem.

## 6. TOWARD PONTRJAGIN-THOM DUALITY

One might hope for a construction which associates to a map  $f : M \rightarrow N$  of manifolds (with suitable properties), a morphism

$$Lf^{PT} : LN^{-\mathbf{T}LN} \rightarrow LM^{-\mathbf{T}LM}$$

of prospectra. This seems out of reach at the moment, but some of the constructions sketched above can be interpreted as partial results in this direction.

In particular, there is at least a **cohomological** candidate for a PT dual

$$j^{PT} : LM^{-\mathbf{T}LM} \rightarrow M^{-TM}$$

to the fixedpoint inclusion  $j : M \subset LM$ . To describe it, we should first observe that there is a morphism

$$M^{-TLM} \rightarrow L_!M^{-TLM}$$

of prospectra, constructed by pulling back the tangent bundle of  $LM$  along the fixedpoint inclusion. To be more precise we need to note that the  $\mathbb{T}$ -fixedpoints of the thickened loop space is a bundle  $L_!M^{\mathbb{T}} \rightarrow M$  with contractible fiber; the choice of a section defines a composition

$$\tilde{j} : M^{-TLM} \sim ((L_!M)^{\mathbb{T}})^{-TLM} \rightarrow L_!M^{-TLM} ;$$

of course  $M^{-TLM} = (M^{-TM})^{-\nu}$ , where  $\nu = T_M \otimes (\mathbb{C}[q^{\pm}]/\mathbb{C})$  is the normal bundle described in §2.

Now according to the localization theorem above, the map induced on  $K_{\hat{\mathbb{T}}}$  by  $\tilde{j}$  is an isomorphism, so it makes sense to define

$$j^! := (\tilde{j}^*)^{-1} \circ \phi_{\nu}^{-1} : K_{\hat{\mathbb{T}}}(M^{-TM}) \rightarrow K_{\hat{\mathbb{T}}}(M^{-TM-\nu}) \rightarrow K_{\hat{\mathbb{T}}}(LM^{-\mathbf{T}LM}) ,$$

where  $\phi_{\nu}$  is the Thom pro-isomorphism associated to the filtered vector bundle  $\nu$ .

A good general theory of PT duals would provide us with a commutative diagram

$$\begin{array}{ccc} LN^{-\mathbf{T}LN} & \xrightarrow{Lf^{PT}} & LM^{-\mathbf{T}LM} \\ \downarrow j_N^{PT} & & \downarrow j_M^{PT} \\ N^{-TN} & \xrightarrow{f^{PT}} & M^{-TM} , \end{array}$$

so it follows from the constructions above that

$$Lf^! := j_N^! \circ f^! \circ (j_M^!)^{-1}$$

defines a formally consistent theory of PT duals for  $K_{\hat{\mathbb{T}}}$ .

For example, (the evaluation at 1 of) the composition

$$K_{\text{Tate}}(M) = K_{\hat{\mathbb{T}}}(LM) \cong K_{\hat{\mathbb{T}}}(LM^{-\mathbf{T}LM}) \rightarrow K_{\hat{\mathbb{T}}}(M^{-TM}) \rightarrow K_{\mathbb{T}}(S^0) \cong K_{\text{Tate}}(pt)$$

is (the  $q$ -expansion of) the Witten genus; more generally, our *ad hoc* construction  $Lf^!$  for  $K_{\hat{\mathbb{T}}}$  agrees with the covariant construction  $f^{\dagger}$  defined for the underlying manifold by the fixedpoint (or  $\sigma$ ) orientation.

The **un**completed  $K$ -theory  $K_{\mathbb{T}}(LM^{-\mathbf{T}LM})$  is also accessible, through the spectral sequence of a colimit, but our understanding of it is at an early stage. It is of course not a cohomological functor of  $M$ , but it does not seem unreasonable to hope that some of its aspects may be within reach through similar PT-like constructions.

These fragmentary constructions suffice to show that  $K_{\hat{\mathbb{T}}}(LM^{-\mathbf{T}LM})$  has enough of a Frobenius (or ambialgebra) structure to define a two-dimensional topological field theory, which assigns to a closed surface of genus  $g$ , the class

$$\pi^{\dagger} \epsilon_{\mathbb{T}}(TM)^g \in \mathbb{Z}((q)) ,$$

with  $\epsilon_{\mathbb{T}}$  the characteristic class defined in §4, and  $\pi^{\dagger}$  the pushforward of  $M$  to a point defined by the fixed-point ( $\sigma$ ) orientation. When  $g = 0$  this is the Witten genus of  $M$ , and when  $g = 1$  it is the Euler characteristic.

Finally: our construction is, from its beginnings onward, formulated in terms of **closed** strings. Stolz and Teichner [30] have produced a deeper approach to a theory of elliptic objects, which promises to incorporate interesting aspects of **open** strings as well. However, their theory is in some ways quite complicated; and our hope is that their global theory, combined with the quite striking computational simplicity of the very local theory sketched here, will lead to something **really** interesting.

## 7. APPENDIX: THE TITS BUILDING OF A LOOP GROUP

Let  $G$  be a simply-connected compact Lie group of rank  $n$ . Let  $LG$  denote the loop group of  $G$ . For the sake of convenience, we will work with a smaller (but equivalent) model for  $LG$ , which we now describe:

Let  $G_{\mathbb{C}}$  be the complexification of the group  $G$ . Since  $G_{\mathbb{C}}$  has the structure of a complex affine variety, we may define the group  $L_{alg}G_{\mathbb{C}}$  to be the group of polynomial maps from  $\mathbb{C}^*$  to  $G_{\mathbb{C}}$ . Let  $L_{alg}G$  be the subgroup of  $L_{alg}G_{\mathbb{C}}$  consisting of maps taking the unit circle into  $G \subset G_{\mathbb{C}}$ . The inclusion  $L_{alg}G \subset LG$  is a homotopy equivalence in the category of  $\mathbb{T}$ -spaces. We begin by making our constructions with  $L_{alg}G$ . We then use these to draw conclusions about the (smooth) Tits building  $\mathbf{A}(LG)$

Fix a maximal torus  $T$  of  $G$ , and let  $\alpha_i$ ,  $1 \leq i \leq n$  be a set of simple roots. We let  $\alpha_0$  denote the highest root. Each root  $\alpha_i$ ,  $0 \leq i \leq n$  determines a compact subgroup  $G_i$  of  $G$ . More explicitly,  $G_i$  is the semisimple factor in the centralizer of the codimension one subtorus given by the kernel of  $\alpha_i$ . Each  $G_i$  may be canonically identified with  $SU(2)$  via an injective map

$$\varphi_i : SU(2) \longrightarrow G.$$

We use these groups  $G_i$  to define corresponding compact subgroups  $G_i$  of  $L_{alg}G$  as follows:

$$G_i = \left\{ z \mapsto \varphi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \quad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2), \quad i > 0$$

$$G_0 = \left\{ z \mapsto \varphi_0 \begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix} \right\} \quad \text{if} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2),$$

**Remark 7.1.** Note that each  $G_i$  is a compact subgroup of  $L_{alg}G$  isomorphic to  $SU(2)$ . Moreover,  $G_i$  belongs to the subgroup  $G$  of constant loops if  $i \geq 1$ . The circle group  $\mathbb{T}$  preserves each  $G_i$ , acting trivially on  $G_i$  for  $i \geq 1$ , and nontrivially on  $G_0$ .

**Definition 7.2.** For any proper subset  $I \subset \{0, 1, \dots, n\}$ , define the parabolic subgroup  $H_I \subset L_{alg}G$  to be the group generated by  $T$  and the groups  $G_i$ ,  $i \in I$ . For the empty set, we define  $H_I$  to be  $T$ . It follows from 7.1 that each  $H_I$  is preserved under the action of  $\mathbb{T}$ .

**Remark 7.3.** The groups  $H_I$  are compact Lie [26]. Moreover,  $H_I$  is isomorphic to its image in  $G$ , under the evaluation map  $ev(1) : L_{alg}G \rightarrow G$ . Notice that for  $I = \{1, \dots, n\}$ ,  $H_I = G$ . Notice also that  $\mathbb{T}$  acts nontrivially on  $H_I$  if and only if  $0 \in I$ .

We are now ready to define the Tits building  $\mathbf{A}(L_{alg}G)$ .

**Definition 7.4.** Let  $\mathbf{A}(L_{alg}G)$  be the homotopy colimit:

$$\mathbf{A}(L_{alg}G) = \text{hocolim}_{I \in \mathcal{C}} L_{alg}G/H_I$$

where  $\mathcal{C}$  denotes the poset category of proper subsets of  $\{0, 1, \dots, n\}$ .

We now come to the main theorem:

**Theorem 7.5.** The space  $\mathbf{A}(L_{alg}G)$  is  $\mathbb{T}\tilde{\times}L_{alg}G$ -equivariantly contractible. In other words, given a compact subgroup  $K \subset \mathbb{T}\tilde{\times}L_{alg}G$ , then the fixed point space  $\mathbf{A}(L_{alg}G)^K$  is contractible.

*Proof.* A proof of the contractibility of  $\mathbf{A}(L_{alg}G)$  was given in [26]. We use some of the ideas from that paper, but our proof is different in flavour.

Mitchell expresses the space  $\mathbf{A}(L_{alg}G)$  as the following:

$$\mathbf{A}(L_{alg}G) = (L_{alg}G/T \times \Delta)/\sim$$

where  $\Delta$  is the  $n$ -simplex, and  $(aT, x) \sim (bT, y)$  if and only if  $x = y \in \overset{\circ}{\Delta}_I$  and  $aH_I = bH_I$ . Here we have indexed the walls of  $\Delta$  by the category  $\mathcal{C}$ , and denoted the interior of  $\Delta_I$  by  $\overset{\circ}{\Delta}_I$ .

Let  $L_{alg}\mathcal{G} \oplus \mathbb{R}d$  be the Lie algebra of the extended loop group  $\mathbb{T}\tilde{\times}L_{alg}G$ . Consider the affine subspace

$$\mathbf{A} = L_{alg}\mathcal{G} + d \subset L_{alg}\mathcal{G} \oplus \mathbb{R}d.$$

The adjoint action of  $L_{alg}G$  on  $\mathbf{A}$  is given by

$$Ad_{f(z)}(\lambda(z) + d) = Ad_{f(z)}(\lambda(z)) + zf'(z)f(z)^{-1} + d$$

This action extends to an affine action of  $\mathbb{T}\tilde{\times}L_{alg}G$ . The identification of  $\mathbf{A}$  with  $\mathbf{A}(L_{alg}G)$  is given as follows. Let  $\Delta$  be identified with the affine alcove:

$$\Delta = \{(h + d) \in Lie(T) + d \mid \alpha_i(h) \geq 0, i > 0, \alpha_0(h) \leq 1\}.$$

General facts about Loop groups [22, 26] show that the surjective map

$$L_{alg}G \times \Delta \longrightarrow \mathbf{A}, \quad (f(z), y) \mapsto Ad_{f(z)}(y)$$

has isotropy  $H_I$  on the subspace  $\Delta_I$ . Hence it factors through a  $\mathbb{T}\tilde{\times}L_{alg}G$ -equivariant homeomorphism between  $\mathbf{A}(L_{alg}G)$  and the affine space  $\mathbf{A}$ . Notice that any compact subgroup  $K \subset \mathbb{T}\tilde{\times}L_{alg}G$  admits a fixed point on  $\mathbf{A}(L_{alg}G)$ . Hence, the space  $\mathbf{A}(L_{alg}G)^K$  is also affine. This completes the proof.  $\square$

We now define the smooth Tits building

**Definition 7.6.** Let  $\mathbf{A}(LG)$  be the homotopy colimit:

$$\mathbf{A}(LG) = \text{hocolim}_{I \in \mathcal{C}} LG/H_I = LG \times_{L_{alg}G} \mathbf{A}(L_{alg}G).$$

It is clear from the proof of the above theorem, that  $\mathbf{A}(LG)$  is  $\mathbb{T}\tilde{\times}LG$ -equivariantly homeomorphic to the affine space  $LG \times_{L_{alg}G} \mathbf{A}$  which is homeomorphic to the affine space  $\mathcal{A}(S^1 \times G)$  of connections on the trivial bundle  $S^1 \times G$ . This shows that:

**Theorem 7.7.** The smooth Tits building  $\mathbf{A}(LG)$  is  $\mathbb{T}\tilde{\times}LG$ -equivariantly contractible.

Recall [26] that  $\mathbf{A}(L_{alg}G)$  is homeomorphic to the space

$$\mathcal{S}_{alg} = \{g(t) : [0, 1] \rightarrow G \mid g(t) = f(e^{2\pi it}) \cdot \exp(tX); f(z) \in \Omega_{alg}G, X \in Lie(G)\}$$

We have a corresponding smooth version

$$\mathcal{S} = \{g(t) : [0, 1] \rightarrow G \mid g(t) = f(e^{2\pi it}) \cdot \exp(tX); f(z) \in \Omega G\} = LG \times_{L_{alg}G} \mathcal{S}_{alg}$$

which is clearly homeomorphic to  $\mathbf{A}(LG)$ . It remains to identify  $\mathcal{S}$  with the space

$$\mathcal{S} = \{g(t) : \mathbb{R} \rightarrow G, g(0) = 1, g(t+1) = g(t) \cdot g(1)\};$$

This is straightforward, and is left to the reader.

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