$(-1,-1)$-BALANCED FREUDENTHAL KANTOR TRIPLE SYSTEMS AND NONCOMMUTATIVE JORDAN ALGEBRAS

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Abstract. A noncommutative Jordan algebra of a specific type is attached to any $(-1,-1)$-balanced Freudenthal Kantor triple system, in such a way that the triple product in this system is determined by the binary product in the algebra. Over fields of characteristic zero, the simple noncommutative Jordan algebras of this type are classified.

1. Introduction

The well-known Tits-Kantor-Koecher construction [Tit62, Kan64, Koe67] relates Jordan systems to 3-graded Lie algebras. In [Kan73], several models of exceptional Lie algebras with a 5-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

are given, based on *generalized Jordan triple systems*:

Definition 1.1. A vector space J over a field F, endowed with a trilinear operation $J \times J \times J \to J$, $(x, y, z) \mapsto xyz$, is said to be a *generalized Jordan triple system* (GJTS for short) if it satisfies the identity:

$$uv(xyz) = (uvx)yz - x(vuy)z + xy(uvz) \tag{1.2}$$

for any $u, v, x, y, z \in J$.

A (linear) *Jordan triple system* is then a generalized Jordan triple system with the added constraint:

$$xyz = zyx \tag{1.3}$$

for any x, y, z.

Unless otherwise stated, all the algebras and algebraic systems considered will be assumed to be defined over a ground field F of characteristic not 2.

Given two elements a, b in a GJTS J, consider the linear maps $l_{a,b}, k_{a,b}$: $J \to J$ given by $l_{a,b}c = abc$, $k_{a,b}c = acb + bca$. Thus, equation (1.2) is equivalent to

$$[l_{u,v}, l_{x,y}] = l_{l_{u,v}x,y} - l_{x,l_{v,u}y} \tag{1.4}$$

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for any u, v, x, y.

Definition 1.5. A generalized Jordan triple system J is said to be a $(-1,-1)$ *Freudenthal Kantor triple system* ($(-1,-1)$-FKTS for short) if it satisfies

$$l_{d,c}k_{a,b} + k_{a,b}l_{c,d} = k_{k_{a,b}c,d} \tag{1.6}$$

for any $a, b, c, d \in J$.

The more general concept of (ε, δ) Freudenthal Kantor triple system was introduced in [YO84] for $\varepsilon, \delta = \pm 1$.

In case there is a nonzero symmetric bilinear form $\langle .|. \rangle$ defined on a $(-1,-1)$-FKTS such that

$$k_{a,b}c = \frac{1}{2}\langle a|b \rangle c \tag{1.7}$$

for any a, b, c, the $(-1,-1)$-FKTS is said to be *balanced*.

In this case, equations (1.6) and (1.7) simplfy to

$$xxy = xyx = \langle x|x \rangle y \tag{1.8}$$

for any x, y.

Therefore, a $(-1,-1)$-*balanced Freudenthal Kantor triple system* (or $(-1,-1)$-BFKTS for short) is a vector spaced over a field F endowed with a trilinear operation xyz and a nonzero symmetric bilinear form $\langle .|. \rangle$ such that both (1.2) and (1.8) hold.

Some simple $(-1,-1)$-BFKTS's were used in [KO03] to give models of the simple complex exceptional Lie superalgebras of classical type: $D(2,1;\alpha)$, $G(3)$ and $F(4)$. Furthermore, this close relationship between $(-1,-1)$-BFKTS's and some Lie superalgebras was used in [EKO03] to provide the classification of the simple finite dimensional $(-1,-1)$-BFKTS's over fields of characteristic zero.

The aim of this paper is to show that a quadratic noncommutative Jordan algebra in a specific variety can be attached to any $(-1,-1)$-BFKTS in such a way that the triple product is determined by the (binary) multiplication of the algebra. The classification of the simple finite dimensional quadratic noncommutative Jordan algebras over fields of characteristic 0 in this variety will be deduced too from the known classification of the simple $(-1,-1)$-BFKTS's.

The next section will be devoted to introduce the variety $\mathcal{V}$ of noncommutative Jordan algebras that will be relevant for our purposes. Then in Section 3, the relationship between some GJTS's and algebras in $\mathcal{V}$ will be studied, while in Section 4 the attention will be restricted to $(-1,-1)$-BFKTS's. The last Section will deal with the classification of the simple finite dimensional quadratic noncommutative Jordan algebras in the variety $\mathcal{V}$ over fields of characteristic 0.

2. A variety of noncommutative Jordan algebras

Given any algebra A over a field F (always of characteristic $\neq 2$), let L_x and R_x denote the left and right multiplications by x: $L_x(y) = xy$, $R_x(y) = yx$; and let (x, y, z) denote the *associator* of the elements x, y, z: $(x, y, z) = (xy)z - x(yz)$.

Recall that the algebra A is a *noncommutative Jordan algebra* if it is flexible:

$$(x, y, x) = 0 \tag{2.1}$$

for any $x, y \in A$, and satisfies the Jordan identity:

$$(x, y, x^2) = 0 \tag{2.2}$$

for any $x, y \in A$.

The Jordan identity is equivalent to the condition $[L_x, R_{x^2}] = 0$ for any $x \in A$, while the flexibility amounts to $[L_x, R_x] = 0$ for any $x \in A$. Also, by flexibility

$$(x, x, y) + (y, x, x) = 0$$

for any $x, y \in A$, or $L_{x^2} - L_x^2 = R_{x^2} - R_x^2$. Thus, in the presence of flexibility, $[L_x, R_{x^2}] = 0$ if and only if

$$[L_x, L_{x^2}] = 0 \tag{2.3}$$

for any $x \in A$.

Given two elements x, y of a flexible algebra A, consider the linear map $A \to A$ given by:

$$\begin{aligned} D_{x,y} &= L_{[x,y]} - [L_x, L_y] \\ &= (x, y, \,.\,) - (y, x, \,.\,) \\ &= (\,.\,, x, y) - (\,.\,, y, x) \\ &= -R_{[x,y]} - [R_x, R_y] \end{aligned} \tag{2.4}$$

Notice that (2.3) is equivalent to the condition $D_{x^2,x} = 0$.

Let $\mathcal{V}$ be the variety of those noncommutative Jordan algebras A over a field F satisfying that

$$D_{x,y} \text{ is a derivation of } A \text{ for any } x, y \in A. \tag{2.5}$$

This is the variety that will be relevant in what follows.

Theorem 2.6. *Let A be an algebra in $\mathcal{V}$, then for any $x, y, z \in A$,*

$$D_{xy,z} + D_{yz,x} + D_{zx,y} = 0. \tag{2.7}$$

Proof. The fact that $D_{x,y} = L_{[x,y]} - [L_x, L_y] \in \operatorname{Der} A$ (the Lie algebra of derivations of A) for any $x, y \in A$ is equivalent to the validity of

$$\big[L_{[x,y]}, L_z\big] - [[L_x, L_y], L_z] = L_{(x,y,z)-(y,x,z)}$$

for any $x, y, z \in A$. Permute cyclically x, y, z and add the resulting equations to get

$$\begin{aligned} &\big[L_{[x,y]}, L_z\big] + \big[L_{[y,z]}, L_x\big] + \big[L_{[z,x]}, L_y\big] \\ &\qquad = L_{(x,y,z)+(y,z,x)+(z,x,y)-(y,x,z)-(z,y,x)-(x,z,y)}. \end{aligned} \tag{2.8}$$

But in any algebra,

$$\begin{aligned} &(x, y, z) + (y, z, x) + (z, x, y) - (y, x, z) - (z, y, x) - (x, z, y) \\ &\qquad = [[x, y], z] + [[y, z], x] + [[z, x], y], \end{aligned}$$

so (2.8) is equivalent to

$$\big[L_{[x,y]}, L_z\big] + \big[L_{[y,z]}, L_x\big] + \big[L_{[z,x]}, L_y\big] = L_{[[x,y],z]+[[y,z],x]+[[z,x],y]},$$

or to

$$D_{[x,y],z} + D_{[y,z],x} + D_{[z,x],y} = 0. \tag{2.9}$$

But $D_{x^2,x} = 0$ for any $x \in A$, since A is a noncommutative Jordan algebra, so that, by linearization, $D_{x\circ y}, x + D_{x^2,y} = 0$ for any $x, y \in A$, where $x \circ y = xy + yx$, and

$$D_{x\circ y,z} + D_{y\circ z,x} + D_{z\circ x,y} = 0 \tag{2.10}$$

for any $x, y, z \in A$.

The result now follows by adding up (2.9) and (2.10). □

The algebras A endowed with a skew symmetric bilinear map $D : A \times A \to \operatorname{Der} A$, $(x, y) \mapsto D_{x,y}$ satisfying (2.7) have been named *generalized structurable algebras* in [Kam92]. The variety of generalized structurable algebras includes the most usual varieties of nonassociative algebras.

Corollary 2.11. *Any algebra in $\mathcal{V}$ is a generalized structurable algebra.*

3. Generalized Jordan triple systems

It will be shown in this section the close connection of the algebras in the variety $\mathcal{V}$ with some generalized Jordan triple systems:

Theorem 3.1. *Let J be a generalized Jordan triple system over a field F of characteristic $\neq 2, 3$ which contains an element $e \in J$ such that:*

(i) *$eee = e$,*
(ii) *$eex = xee$ for any $x \in J$,*
(iii) *the map $U_e : x \mapsto exe$ is onto.*

Then the homotope algebra $J^{(e)}$, defined on the vector space J with multiplication given by $x \cdot y = xey$ for any $x, y \in J$, belongs to the variety $\mathcal{V}$ and is unital with $1 = e$. Moreover, the map $x \mapsto \bar{x} = exe$ is an involution of $J^{(e)}$ and the triple product in J satisfies

$$xyz = x \cdot (\bar{y} \cdot z) - \bar{y} \cdot (x \cdot z) + (\bar{y} \cdot x) \cdot z, \tag{3.2}$$

for any $x, y, z \in J$.

Conversely, let $(A, \cdot)$ be a unital algebra in $\mathcal{V}$ over a field F of characteristic $\neq 2$, with unity element 1_A and endowed with an involution $x \mapsto \bar{x}$, and define a triple product on A by means of (3.2)*. Then A becomes a GJTS and satisfies conditions* (i)–(iii) *above with $e = 1_A$ and $U_e x = \bar{x}$ for any $x \in A$.*

Proof. Let J be a GJTS satisfying the conditions above. Then (1.2), together with (i) and (ii), give:

$$\begin{aligned} ee(exe) &= (eee)xe - e(eex)e + ex(eee) \\ &= 2exe - e(eex)e, \end{aligned} \tag{3.3}$$

$$\begin{aligned} exe = ex(eee) &= (exe)ee - e(xee)e + ee(exe) \\ &= 2ee(exe) - e(eex)e. \end{aligned} \tag{3.4}$$

Since the characteristic of F is $\neq 3$, (3.3) and (3.4) imply

$$exe = ee(exe) = e(eex)e \tag{3.5}$$

for any $x \in J$. Since $U_e : x \mapsto exe$ is onto, we get

$$x = eex\,(= xee), \tag{3.6}$$

so that

$$x = e \cdot x = x \cdot e$$

for any $x \in J$ and e is the unity element of the homotope algebra $J^{(e)}$.

Now, by (3.6) and (1.2),

$$\begin{aligned} x = xe(eee) &= (xee)ee - e(exe)e + ee(xee) \\ &= 2x - e(exe)e \end{aligned}$$

so $\bar{\bar{x}} = x$ for any $x \in J$, where $\bar{x} = U_e x = exe$. Also, for any $x, y \in J$,

$$\begin{cases} x \cdot y = xe(eey) = (xee)ey - e(exe)y + ee(xey) \\ \qquad\qquad\qquad\;\; = x \cdot y - e\bar{x}y + x \cdot y, \\ x \cdot y = xe(yee) = (xey)ee - y(exe)e + ye(xee) \\ \qquad\qquad\qquad\;\; = x \cdot y - y\bar{x}e + y \cdot x. \end{cases}$$

Hence, for any $x, y \in J$,

$$\begin{cases} exy = \bar{x} \cdot y, \\ xye = x \cdot \bar{y}, \end{cases} \tag{3.7}$$

while

$$\begin{aligned} \bar{x} \cdot \bar{y} = ex(eye) &= (exe)ye - e(xey)e + ey(exe) \\ &= \bar{x}ye - \overline{x \cdot y} + ey\bar{x} \\ &= \bar{x} \cdot \bar{y} - \overline{x \cdot y} + \bar{y}\bar{x}, \end{aligned}$$

and this shows that $\overline{x \cdot y} = \bar{y} \cdot \bar{x}$ for any $x, y \in J$. Therefore, the map $x \mapsto \bar{x}$ is an involution of $J^{(e)}$.

Besides, for any $x, y, z \in J$,

$$\begin{aligned} y \cdot (x \cdot z) = ye(xez) &= (yex)ez - x(eye)z + xe(yez) \\ &= (y \cdot x) \cdot z - x\bar{y}z + x \cdot (y \cdot z), \end{aligned}$$

so, substituting y by $\bar{y}$ we obtain (3.2).

But also, because of (3.7),

$$\begin{aligned} z \cdot (x \cdot \bar{y}) = e\bar{z}(xye) &= (e\bar{z}x)ye - x(\bar{z}ey)e + xy(e\bar{z}e) \\ &= (z \cdot x) \cdot \bar{y} - x \cdot \overline{(\bar{z} \cdot y)} + xyz, \end{aligned}$$

so

$$xyz = x \cdot (\bar{y} \cdot z) - (z \cdot x) \cdot \bar{y} + z \cdot (x \cdot \bar{y}) \tag{3.8}$$

for any x, y, z, since $x \mapsto \bar{x}$ is an involution. Equations (3.2) and (3.8) yield, for any $x, y, z \in J$,

$$(\bar{y}, x, z)^{\cdot} = -(z, x, \bar{y})^{\cdot}, \tag{3.9}$$

where $(a, b, c)^{\cdot} = (a \cdot b) \cdot c - a \cdot (b \cdot c)$ is the associator in $J^{(e)}$. Identity (3.9) is equivalent to the flexible law in $J^{(e)}$.

Moreover, (3.2) is equivalent to

$$l_{x,y} = [L_x, L_{\bar{y}}] + L_{\bar{y} \cdot x} = -D_{x,\bar{y}} + L_{x \cdot \bar{y}}, \tag{3.10}$$

where L_a denotes the left multiplication by a in $J^{(e)}$ and $D_{x,y} = L_{[x,y]} - [L_x, L_y]$ as in (2.4). Notice that $L_a = l_{a,e}$ and $L_{\bar{z}} = l_{e,z}$ (because of (3.7)). Thus,

$$\begin{aligned}[D_{x,y}, l_{z,t}] &= -[l_{x,\bar{y}}, l_{z,t}] + [l_{x\cdot y,e}, l_{z,t}] \\ &= -l_{x\bar{y}z,t} + l_{z,\bar{y}xt} + l_{(x\cdot y)ez,t} - l_{z,e(x\cdot y)t} \\ &= l_{-x\bar{y}z+(x\cdot y)\cdot z,t} + l_{z,\bar{y}xt-(\bar{y}\cdot\bar{x})\cdot t} \\ &= l_{D_{x,y}z,t} - l_{z,D_{\bar{y},\bar{x}}t}\end{aligned} \tag{3.11}$$

for any $x, y, z, t \in J$. With $t = e$, this shows that $[D_{x,y}, L_z] = L_{D_{x,y}z}$, since $D_{a,b}e = 0$ for any $a, b \in J$. Hence $D_{x,y} \in \operatorname{Der} J^{(e)}$ for any $x, y \in J$.

Finally, (3.10) shows that $l_{x,\bar{x}} = L_{x\cdot x}$, so

$$\begin{aligned}[L_x, L_{x\cdot x}] &= [l_{x,e}, l_{x,\bar{x}}] \\ &= l_{xex,\bar{x}} - l_{x,ex\bar{x}} \\ &= l_{x\cdot x,\bar{x}} - l_{x,\bar{x}\cdot\bar{x}} \quad \text{by (3.7)}, \\ &= \big([L_{x\cdot x}, L_x] + L_{x\cdot(x\cdot x)}\big) - \big([L_x, L_{x\cdot x}] + L_{(x\cdot x)\cdot x}\big) \quad \text{by (3.10)}, \\ &= -2[L_x, L_{x\cdot x}] \quad \text{by flexibility.}\end{aligned}$$

Since the characteristic is $\neq 3$, this shows that $[L_x, L_{x^{\cdot 2}}] = 0$ for any $x \in J$ which, together with the flexible law, shows that $J^{(e)}$ is a noncommutative Jordan algebra, thus completing the proof of the first part of the Theorem.

Conversely, let $(A, \cdot)$ be a unital algebra in $\mathcal{V}$ endowed with an involution $x \mapsto \bar{x}$. Let $e = 1_A$ be the unity element and use (3.2) to define a triple product on A. Then

$$\begin{aligned}xey &= x \cdot (\bar{e} \cdot y) - \bar{e} \cdot (x \cdot y) + (\bar{e} \cdot x) \cdot y \\ &= x \cdot y - x \cdot y + x \cdot y,\end{aligned}$$

so $x \cdot y = xey$ for any $x, y \in A$, and hence $eee = e$ and $eex = x = xee$ for any $x \in A$. Also,

$$exe = e \cdot (\bar{x} \cdot x) - \bar{x} \cdot (e \cdot e) + (\bar{x} \cdot e) \cdot e = \bar{x}$$

for any $x \in A$. Hence, conditions (i)–(iii) are satisfied.

Finally, with $D_{x,y}$ defined by (2.4), we obtain for any $x, y, z \in A$,

$$D_{x,y}\bar{z} = \overline{D_{\bar{x},\bar{y}}z} = -\overline{D_{\bar{y},\bar{x}}z}, \tag{3.12}$$

because

$$(x, y, \bar{z})^{\cdot} = -\overline{(z, \bar{y}, \bar{x})^{\cdot}} = \overline{(\bar{x}, \bar{y}, z)^{\cdot}}$$

by flexibility. Since $D_{x,y} \in \operatorname{Der} A$, (3.12) and (3.2) show that

$$[D_{x,y}, l_{a,b}] = l_{D_{x,y}a,b} - l_{a,D_{\bar{y},\bar{x}}b} \tag{3.13}$$

for any $x, y, a, b \in A$. Now, (2.7) amounts to

$$\begin{aligned}&[L_{a\cdot b}, L_c] + [L_{b\cdot c}, L_a] + [L_{c\cdot a}, L_b] \\ &\qquad = L_{[a\cdot b,c]+[b\cdot c,a]+[c\cdot a,b]}\end{aligned} \tag{3.14}$$

for any $a, b, c \in A$. But

$$\begin{aligned}
&L_{[a\cdot b,c]+[b\cdot c,a]+[c\cdot a,b]}\\
&\quad = L_{(a,b,c)^\cdot+(b,c,a)^\cdot+(c,a,b)^\cdot}\\
&\quad = L_{(a,b,c)^\cdot-(b,a,c)^\cdot+(b,c,a)^\cdot}\\
&\quad = L_{D_{a,b}c} + L_{(b,c,a)^\cdot}\\
&\quad = [D_{a,b}, L_c] + L_{(b,c,a)^\cdot}\\
&\quad = -[[L_a, L_b], L_c] + [L_{[a,b]}, L_c] + L_{(b,c,a)^\cdot},
\end{aligned}$$

thus (3.14) becomes,

$$[L_c, [L_a, L_b] + L_{b\cdot a}] = \big([L_{c\cdot a}, L_b] + L_{b\cdot(c\cdot a)}\big) - \big([L_a, L_{b\cdot c}] + L_{(b\cdot c)\cdot a}\big).$$

Substituting b by $\bar{b}$ and using (3.2), this last equation is equivalent to

$$[L_c, l_{a,b}] = l_{c\cdot a,b} - l_{a,\bar{c}\cdot b} \tag{3.15}$$

for any $a, b, c \in A$. From (3.10), (3.13) and (3.15) we obtain

$$\begin{aligned}
[l_{x,y}, l_{a,b}] &= [-D_{x,\bar{y}} + L_{x\cdot\bar{y}}, l_{a,b}]\\
&= -l_{D_{x,\bar{y}}a,b} + l_{a,D_{y,\bar{x}}b} + l_{(x\cdot\bar{y})\cdot a,b} - l_{a,(y\cdot\bar{x})\cdot b}\\
&= l_{l_{x,y}a,b} - l_{a,l_{y,x}b}\\
&= l_{xya,b} - l_{a,yxb},
\end{aligned}$$

thus proving that A is a GJTS with the triple product defined by (3.2) □

Remark 3.16. Notice that in the proof of the first part of the Theorem above, the restriction on the characteristic to be $\neq 3$ has only been used to prove (3.6) and $[L_x, L_{x^{\cdot 2}}] = 0$ for any $x \in J$.

As a particular case, for Jordan triple systems, the following known result is recovered [Loo71, 1.4, 3.2]:

Corollary 3.17. *Let J be a Jordan triple system over a field F of characteristic $\neq 2, 3$ which contains an element $e \in J$ such that*

$$exe = x \tag{3.18}$$

for any $x \in J$. Then, the homotope algebra $J^{(e)}$ (with the product $x\cdot y = xey$) is a unital Jordan algebra with $1 = e$. Moreover, for any $x, y, z \in J$:

$$xyz = x \cdot (y \cdot z) - y \cdot (x \cdot z) + (y \cdot x) \cdot z. \tag{3.19}$$

Conversely, if A is a unital Jordan algebra with unity e, then the triple product xyz defined by (3.19) *becomes a Jordan triple system satisfying* (3.18)*.*

Proof. For a Jordan triple system J satisfying (3.18), all the conditions (i)–(iii) in Theorem 3.1 are automatically satisfied. Moreover, for any $x \in J$, $\bar{x} = exe = x$ so that the involution law $\overline{x \cdot y} = \bar{y} \cdot \bar{x}$ is equivalent to the commutative law $x \cdot y = y \cdot x$. □

4. $(-1,-1)$-balanced Freudenthal Kantor triple systems

An algebra Q over the field F is said to be *quadratic* if it is unital and for any $x \in Q$, 1, x and x^2 are linearly dependent. Then the set of *vectors* $V = \{x \in Q \setminus F1 : x^2 \in F1\}$ is a subspace of Q with $Q = F1 \oplus V$ [Osb62]. For any $u, v \in V$,

$$uv = -(u|v)1 + u \times v, \tag{4.1}$$

where $(.|.)$ is a bilinear form and $\times$ is an anticommutative multiplication on V. The triple $\Bigl(V, (.|.), \times\Bigr)$ determines the algebra Q, so we will write $Q = Q\Bigl(V, (.|.), \times\Bigr)$. Moreover, for any $x \in Q$,

$$x^2 - T(x)x + N(x)1 = 0 \tag{4.2}$$

where T is a linear form and N a quadratic form on A, called the *norm*, given for any $\alpha \in F$ and $u \in V$ by

$$\begin{cases} T(\alpha 1 + u) = 2\alpha \\ N(\alpha 1 + u) = \alpha^2 + (u|u) \end{cases} \tag{4.3}$$

In particular, $T(x) = N(x,1)$ for any x, where $N(x,y) = N(x+y) - N(x) - N(y)$ is the associated symmetric bilinear form.

It is well-known [Osb62] that the bilinear form $(.|.)$ is symmetric if and only if the map $x \mapsto \bar{x} = T(x)1 - x$ is an involution (the *standard involution*), and that the quadratic algebra Q is flexible if and only if $(.|.)$ is symmetric and $(u \times v|w) = (u|v \times w)$ for any $u, v, w \in V$. Notice that if $(.|.)$ is symmetric, it is determined by N and that any flexible quadratic algebra is a noncommutative Jordan algebra.

Theorem 4.4. *Let S be a $(-1,-1)$-balanced Freudenthal Kantor triple system over a field F and let $e \in S$ such that $\langle e|e\rangle \neq 0$ ($\langle .|.\rangle$ as in (1.8)). Define a binary product on S by*

$$x \cdot y = \frac{1}{\langle e|e\rangle} exy$$

for any $x, y \in S$. Then $(S, \cdot)$ is a quadratic algebra in the variety $\mathcal{V}$ with norm given by $N(x) = \frac{\langle x|x\rangle}{\langle e|e\rangle}$ for any $x \in S$. Moreover, the original triple product on S is related to the binary product by

$$xyz = \langle e|e\rangle\Bigl((\bar{x} \cdot y) \cdot z - \bar{x} \cdot (y \cdot z) + y \cdot (\bar{x} \cdot z)\Bigr) \tag{4.5}$$

for any $x, y, z \in S$, where $x \mapsto \bar{x}$ denotes the standard involution of the quadratic algebra $(S, \cdot)$.

Conversely, let $(Q, \cdot)$ be a quadratic algebra in $\mathcal{V}$ with norm N and define a triple product on Q by the formula

$$xyz = (\bar{x} \cdot y) \cdot z - \bar{x} \cdot (y \cdot z) + y \cdot (\bar{x} \cdot z), \tag{4.6}$$

where $x \mapsto \bar{x} = N(x,1)1 - x$ is the standard involution on Q. Then Q, with this triple product, is a $(-1,-1)$-BFKTS with associated nonzero symmetric bilinear form given by $\langle x|x\rangle = N(x)$ for any $x \in Q$.

Proof. Let S be a $(-1,-1)$-BFKTS and let $e\in S$ with $\langle e|e\rangle\neq 0$. Define a new triple product on S by the formula

$$\widetilde{xyz}=\frac{1}{\langle e|e\rangle}yxz.$$

Then the map $\tilde{l}_{x,y}: z\mapsto \widetilde{xyz}$ equals

$$\begin{aligned}\tilde{l}_{x,y}&=\frac{1}{\langle e|e\rangle}l_{y,x}\\&=\frac{1}{\langle e|e\rangle}\Big(2\langle x|y\rangle id-l_{x,y}\Big),\end{aligned}$$

where $l_{x,y}$ is the 'left multiplication' on S (see (1.8)). Thus, for any $x,y,z,t\in S$,

$$\begin{aligned}[\tilde{l}_{x,y},\tilde{l}_{z,t}]&=\frac{1}{\langle e|e\rangle^2}[l_{y,x},l_{t,z}]\\&=\frac{1}{\langle e|e\rangle^2}\Big(l_{yxt,z}-l_{t,xyz}\Big)\\&=\frac{1}{\langle e|e\rangle^2}\Big(l_{t,yxz-2\langle x|y\rangle z}-l_{xyt-2\langle x|y\rangle t,z}\Big)\\&=\frac{1}{\langle e|e\rangle^2}\Big(l_{t,yxz}-l_{xyt,z}\Big)\\&=\tilde{l}_{\widetilde{xyz},t}-\tilde{l}_{z,\widetilde{yxt}}.\end{aligned}\tag{4.7}$$

Therefore, $(S,\widetilde{xyz})$ is a GJTS. Moreover,

$$\begin{cases}\widetilde{eex}=\dfrac{1}{\langle e|e\rangle}eex=x,\\[2mm]\widetilde{xee}=\dfrac{1}{\langle e|e\rangle}exe=x,\end{cases}$$

so (3.6) is satisfied, and

$$\begin{aligned}\bar{x}=\widetilde{exe}=\frac{1}{\langle e|e\rangle}xee&=\frac{1}{\langle e|e\rangle}\Big(-eex+2\langle e|x\rangle e\Big)\\&=-x+N(e,x)e\end{aligned}$$

with N the quadratic form given by $N(x)=\frac{\langle x|x\rangle}{\langle e|e\rangle}$ for any $x\in S$. Define the algebra $(S,\cdot)$ by means of

$$x\cdot y=\frac{1}{\langle e|e\rangle}exy=\widetilde{xey}$$

for any $x,y\in S$. This algebra $(S,\cdot)$ is unital with $1_S=e$ and for any $x\in S$:

$$\begin{aligned}x\cdot x=\frac{1}{\langle e|e\rangle}exx&=\frac{1}{\langle e|e\rangle}\Big(2\langle e|x\rangle-xex\Big)\\&=N(e,x)x-N(x)e,\end{aligned}$$

so $(S,\cdot)$ is quadratic. Now by Theorem 3.1 and Remark 3.16, the algebra $(S,\cdot)$ is flexible (and hence noncommutative Jordan) and satisfies that $D_{x,y}\in$

$\mathrm{Der}(S,\cdot)$ for any x,y. Finally, from (3.2),

$$\begin{aligned} xyz &= \langle e|e\rangle \widetilde{yxz} \\ &= \langle e|e\rangle \Big(y\cdot(\bar{x}\cdot z) - \bar{x}\cdot(y\cdot z) + (\bar{x}\cdot y)\cdot z \Big) \end{aligned}$$

for any $x,y,z\in S$, thus completing the proof of the first part.

Conversely, if $(Q,\cdot)$ is a quadratic algebra in $\mathcal{V}$ with norm N and we use (4.6) to define a triple product on Q, then

$$\begin{cases} xxy = (\bar{x}\cdot x)\cdot y - \bar{x}\cdot(x\cdot y) + x\cdot(\bar{x}\cdot y) = N(x)y \\ xyx = (\bar{x}\cdot y)\cdot x - \bar{x}\cdot(y\cdot x) + y\cdot(\bar{x}\cdot x) = N(x)y \end{cases}$$

since $x\cdot\bar{x} = \bar{x}\cdot x = N(x)1$, $\bar{x}\cdot(x\cdot y) = x\cdot(\bar{x}\cdot y)$ and $(\bar{x},y,x)^{\cdot} = 0$ for any x, as $\bar{x}\in F1+Fx$ and $(Q,\cdot)$ is flexible. Also, with $\widetilde{xyz} = yxz = (\bar{y}\cdot x)\cdot z - \bar{y}\cdot(x\cdot z) + x\cdot(\bar{y}\cdot z)$, Theorem 3.1 shows that $(Q,\widetilde{xyx})$ is a GJTS and, as in (4.7), this shows that so is (Q,xyz), as required. □

We close this section with a result relating the simplicity of a quadratic algebra in $\mathcal{V}$ and of the associated $(-1,-1)$-BFKTS, constructed in the previous Theorem.

Theorem 4.8. *Let $(Q,\cdot)$ be a quadratic algebra in $\mathcal{V}$ and let (Q,xyz) be the associated $(-1,-1)$-BFKTS with triple product given by* (4.6)*. Then (Q,xyz) is simple if and only if either $(Q,\cdot)$ is simple or $(Q,\cdot)$ is isomorphic to the direct product of two copies of the ground field F: $(Q,\cdot)\cong F\times F$.*

Proof. Assume first that (Q,xyz) is simple and let $0\neq I$ be an ideal of $(Q,\cdot)$. Let $x\mapsto\bar{x}$ be the standard involution. Then either $I=\bar{I}$ and hence, by (4.6), I is an ideal of (Q,xyz), so $I=Q$ by simplicity, or $I\neq\bar{I}$. In the latter case, $I+\bar{I}$ and $I\cap\bar{I}$ are ideals of $(Q,\cdot)$ closed under the involution so, by the previous argument, $I\cap\bar{I}=0$ and $I+\bar{I}=Q$. Besides, $I\neq\bar{I}$, so there is some element $x\in I$ with $N(x,1)\neq 0$. But for any $y\in Q$, by (4.2),

$$x\cdot y + y\cdot x = N(x,1)y + N(y,1)x - N(x,y)1,$$

so $N(x,1)y - N(x,y)1\in I$ and hence $\{y\in Q : N(x,y)=0\}\subseteq I$ and the codimension of I (which coincides with the codimension of $\bar{I}$) is 1. The only possibility is that $\dim I = \dim\bar{I} = 1$ and $Q = I\oplus\bar{I}$. In particular both I and $\bar{I}$ are one-dimensional quotients of the unital algebra Q, so both I and $\bar{I}$ are isomorphic, as algebras, to the ground field F, and $(Q,\cdot)\cong F\times F$, as required.

Conversely, any ideal I of (Q,xyz) satisfies that for any $x\in I$, $\bar{x} = x11\in I$, so that I is an ideal of $(Q,\cdot)$ closed under the involution, and hence I is trivial in both cases: $(Q,\cdot)$ simple or $(Q,\cdot)\cong F\times F$. □

5. Simple algebras

According to Theorems 4.4 and 4.8, to obtain the simple finite dimensional quadratic algebras in $\mathcal{V}$ it is enough to consider the simple finite dimensional $(-1,-1)$-BFKTS's S with an element $e\in S$ such that $\langle e|e\rangle = 1$ and to define the associated quadratic algebras $(S,\cdot)$ where

$$x\cdot y = exy \tag{5.1}$$

for any $x,y\in S$. The element e becomes the unity element of $(S,\cdot)$.

The classification of the simple finite dimensional $(-1,-1)$-BFKTS's over fields of characteristic zero was obtained in [EKO03]. Here we will review the list of examples that appear in [EKO03, Section 3] and obtain the associated quadratic algebras in $\mathcal{V}$. This will set the stage for the classification in the last section.

5.(i) Orthogonal type:

Let S be a vector space endowed with a symmetric bilinear form $\langle .|. \rangle$ and an element $e \in S$ such that $\langle e|e \rangle = 1$. Then S becomes a $(-1,-1)$-BFKTS with the triple product

$$xyz = \langle z|x \rangle y - \langle z|y \rangle x + \langle x|y \rangle z$$

for any $x, y, z \in S$. Therefore, (5.1) becomes

$$x \cdot y = \langle e|y \rangle x + \langle e|x \rangle y - \langle x|y \rangle e$$

so that $(S, \cdot) = Fe \oplus V$, with $V = (Fe)^{\perp}$ (the orthogonal subspace to Fe relative to $\langle .|. \rangle$), is the Jordan algebra of a quadratic form: for any $u, v \in V$ and $\alpha, \beta \in F$,

$$(\alpha e + u) \cdot (\beta e + v) = (\alpha\beta - \langle u|v \rangle)e + (\alpha v + \beta u). \tag{5.2}$$

5.(ii) Unitarian type:

Let K be a quadratic étale F-algebra; that is, either K is a quadratic field extension of F (recall that the characteristic of F is not 2) or it is isomorphic to $F \times F$; and let S be a left K-module endowed with a hermitian form $h : S \times S \to K$ and an element $e \in S$ with $h(e, e) = 1$. Thus, h is F-bilinear and

$$h(\alpha x, y) = \alpha h(x, y)$$
$$h(x, y) = \overline{h(y, x)}$$

for any $\alpha \in F$ and $x, y \in S$, where $\alpha \mapsto \bar{\alpha}$ is the nontrivial F-automorphism of K.

Then S is a $(-1,-1)$-BFKTS with the triple product

$$xyz = h(z, x)y - h(z, y)x + h(x, y)z$$

for any $x, y, z \in S$. Thus (5.1) becomes here:

$$x \cdot y = h(y, e)x + h(e, x)y - h(y, x)e \tag{5.3}$$

for any $x, y \in S$; so that $(S, \cdot) = Ke \oplus W$, with $W = (Ke)^{\perp} = \{x \in S : h(e, x) = 0\}$ and for any $\alpha, \beta \in K$ and $u, v \in W$:

$$(\alpha e + u) \cdot (\beta e + v) = (\alpha\beta - h(v, u))e + (\bar{\alpha} v + \beta u). \tag{5.4}$$

Therefore, $(S, \cdot)$ is the *structurable algebra* associated to the hermitian form $-h|_W$ (see [All78, § 8, Example (iii)].

5.(iii) Symplectic type:

Change K to H, a quaternion algebra over F, in the unitarian type; so that now S is a left H-module endowed with a hermitian form $h : S \times S \to H$ and an element $e \in S$ with $h(e, e) = 1$. (Here $\alpha \mapsto \bar{\alpha}$ denotes the standard involution in H.) As before, $S = He \oplus W$ with $W = (He)^{\perp}$, but now (5.4) becomes

$$(\alpha e + u) \cdot (\beta e + v) = \big(\bar{\alpha}\beta + \beta(\alpha - \bar{\alpha}) - h(v, u)\big)e + (\bar{\alpha} v + \beta u). \tag{5.5}$$

for any $\alpha, \beta \in H$ and $u, v \in W$.

In order to deal with the remaining types, some preliminaries are needed.

Given a quadratic algebra $Q = Q\Big(V, (.|.), \times\Big)$ and a nonzero scalar $\mu \in F$, we will denote by $Q^{[\mu]}$ the quadratic algebra

$$Q^{[\mu]} = Q\Big(V, \mu(.|.), \times\Big).$$

(Same anticommutative multiplication on V, but bilinear forms scaled by μ.)

There is a related construction in the literature. Given any algebra A and a scalar $\alpha \in F$, the *scalar mutation* $A^{(\alpha)}$ (see [Alb48, McC66]) is the algebra defined on the same vector space but with the new product

$$x \overset{\alpha}{\cdot} y = \alpha xy + (1-\alpha)yx$$

for any $x, y \in A$. For a flexible quadratic algebra $Q = Q\Big(V, (.|.), \times\Big)$, it follows immediately from (4.1) that $Q^{(\alpha)} = Q\Big(V, (.|.), (2\alpha-1)\times\Big)$ (same bilinear form, but anticommutative multiplication scaled by $2\alpha - 1$). Also, for any $0 \neq \nu \in F$, the linear endomorphism of $Q = F1 \oplus V$, given by $\varphi(1) = 1$ and $\varphi(v) = \nu v$ for any $v \in V$, gives an isomorphism $Q^{[\nu^2]} \cong Q\Big(V, (.|.), \nu^{-1}\times\Big)$, so that for $\alpha \neq \frac{1}{2}$, the scalar mutation $Q^{(\alpha)}$ is isomorphic to $Q^{\left[\frac{1}{(2\alpha-1)^2}\right]}$.

5.(iv) $\mathbf{D}_{\boldsymbol{\mu}}$-type:

Let S be a four dimensional vector space endowed with a nondegenerate symmetric bilinear form $\langle .|. \rangle$ and an element e with $\langle e|e \rangle = 1$, and let $\phi : S \times S \times S \times S \to F$ be a nonzero alternating multilinear form (unique up to multiplication by a nonzero scalar). Define the alternating triple product $[xyz]$ on S by means of

$$\phi(x, y, z, t) = \langle [xyz] | t \rangle$$

for any $x, y, z, t \in S$. Then [EKO03, Lemma 3.2] there exists a nonzero scalar $\mu \in F$ such that

$$\langle [a_1a_2a_3] | [b_1b_2b_3] \rangle = \mu \det\Big(\langle a_i | b_j \rangle\Big) \tag{5.6}$$

for any $a_i, b_i \in S$ $(i = 1, 2, 3)$.

In this case, S becomes a $(-1,-1)$-BFKTS with the triple product

$$xyz = [xyz] + \langle z|x \rangle y - \langle z|y \rangle x + \langle x|y \rangle z$$

for any $x, y, z \in S$. Thus, (5.1) becomes

$$x \cdot y = [exy] + \langle e|y \rangle x + \langle e|x \rangle y - \langle x|y \rangle e,$$

so that $(S, \cdot) = Fe \oplus V$ with $V = (Fe)^\perp$ and for any $\alpha, \beta \in F$ and $u, v \in V$,

$$(\alpha e + u) \cdot (\beta e + v) = (\alpha\beta - \langle u|v \rangle)e + (\alpha v + \beta u + u \times v), \tag{5.7}$$

where $u \times v = [euv]$. That is $(S, \cdot) = Q\Big(V, \langle.|.\rangle, \times\Big)$. From (5.6) and since $\langle e|e\rangle = 1$, it follows that for any $u, v \in V$,

$$\langle u \times v | u \times v\rangle = \mu \begin{vmatrix} \langle u|u\rangle & \langle u|v\rangle \\ \langle u|v\rangle & \langle v|v\rangle \end{vmatrix},$$

so that

$$(u \times v | u \times v) = \begin{vmatrix} (u|u) & (u|v) \\ (u|v) & (v|v) \end{vmatrix}, \tag{5.8}$$

where $(u|v) = \mu\langle u|v\rangle$ for any $u, v \in V$. The above equation (5.8) shows that $\times$ is a vector cross product on V relative to the nondegenerate symmetric bilinear form $(.|.)$ (see [BG67]) and, therefore, the quadratic algebra $H = Q\Big(V, (.|.), \times\Big)$ is a quaternion algebra over F. But then we conclude that

$$(S, \cdot) = Q\Big(V, \langle.|.\rangle, \times\Big) = Q\Big(V, \mu^{-1}(.|.), \times\Big) = H^{[\mu^{-1}]}.$$

That is, $(S, \cdot)$ is a quadratic algebra obtained from a quaternion algebra by scaling the bilinear form on the subspace of vectors.

Conversely, it is straightforward to check that for any quaternion algebra H and nonzero scalar $\nu \in F$, the quadratic algebra $H^{[\nu]}$ belongs to $\mathcal{V}$.

5.(v) G-type:

Let C be a Cayley algebra (that is, an eight dimensional unital composition algebra) over F with norm n and trace t and let $S = C_0 = \{x \in C : t(x) = 0\}$. Let $0 \neq \alpha \in F$ and consider the nondegenerate symmetric bilinear form and the triple product on S given by:

$$\begin{cases} \langle x|y\rangle = -2\alpha t(xy) \\ xyz = \alpha\Big(D_{x,y}(z) - 2t(xy)z\Big) \end{cases}$$

for any $x, y, z \in S$, where

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y]$$

(a derivation of C). We refer to [Sch95, Chapter III] for the basic properties of Cayley algebras. Assume that there is an element $e \in S$ with $\langle e|e\rangle = 1$. Then $t(e^2) = -2n(e) = -\frac{1}{2\alpha}$, so

$$\begin{cases} \langle x|y\rangle = \dfrac{t(xy)}{t(e^2)}, \\ xyz = \dfrac{1}{4n(e)}\Big(D_{x,y}(z) - 2t(xy)z\Big). \end{cases}$$

Here $K = F1 + Fe$ is a quadratic étale subalgebra of C and $S = Fe \oplus V$, where $V = \{x \in C_0 : t(ex) = 0\} = \{x \in C : t(Kx) = 0\}$. For any $u, v \in V$, (5.1) becomes

$$u \cdot v = \frac{1}{4n(e)} D_{e,u}(v),$$

but $D_{x,y}(z) = [[x, y], z] + 3(x, z, y)$ ([Sch95, (3.70)]), so

$$\begin{aligned} D_{e,u}(v) &= [[e, u], v] + 3(e, v, u) \\ &= -2[ue, v] + 3(u, e, v) \quad (\text{since } eu + ue = 0 \text{ as } t(eu) = 0) \\ &= (ue)v + 2v(ue) - 3u(ev). \end{aligned}$$

Also, $t(eu) = t(ev) = 0$ so that, by alternativity, for any $u, v \in V$,

$$\begin{aligned} &(ue)v = -(uv)e, \\ &v(ue) = -v(eu) = e(vu) = -n(u,v)e - e(uv), \\ &u(ev) = -e(uv). \end{aligned}$$

Hence $D_{e,u}(v) = -2n(u,v)e + [e, uv]$ and (5.1) becomes

$$(\alpha e + u) \cdot (\beta e + v) = \Bigl(\alpha\beta - \frac{n(u,v)}{2n(e)}\Bigr)e + \Bigl(\alpha v + \beta u + \frac{1}{4n(e)}[e, uv]\Bigr) \tag{5.9}$$

for any $\alpha, \beta \in F$ and $u, v \in V$.

Now ([Jac58, §6], [EM95, §§ 2,3]) for any $u, v \in V$,

$$uv = -\sigma(u,v) + u * v, \tag{5.10}$$

where $\sigma(x,y) = \frac{1}{2}\Bigl(n(x,y) - \frac{1}{n(e)}n(ex,y)e\Bigr)$ for any $x, y \in V$. Then $\sigma : V \times V \to K$ is a hermitian form, $*$ is anticommutative and

$$\begin{cases} \mu(x * y) = (\bar{\mu}x) * y, \\ \sigma(x, y * z) = \overline{\sigma(x * y, z)}, \\ (x * y) * z = \sigma(x,z)y - \sigma(y,z)x, \end{cases} \tag{5.11}$$

for any $\mu \in K$ and $x, y \in V$. The quadratic algebra $B = F1 \oplus V = Q\Bigl(V, -n(.|.), *\Bigr)$ is a *color algebra* (see [EM95]) and any color algebra is obtained in this way. For the origin and basic properties of color algebras one may consult [EM95] and the references therein.

Proposition 5.12. *The linear map $\varphi : B^{[-2]} \to (S, \cdot)$ given by $\varphi(1) = e$ and $\varphi(u) = -2eu$ for any $u \in V$ is an isomorphism of algebras.*

Proof. Since e is the unity element of $(S, \cdot)$ and $B^{[-2]} = Q\Bigl(V, 2n(.|.), *\Bigr)$, it is enough to prove that for any $u, v \in V$,

$$\varphi\bigl(-2n(u,v)1 + u * v\bigr) = \varphi(u) \cdot \varphi(v). \tag{5.13}$$

But

$$\begin{aligned} &\varphi\bigl(-2n(u,v)1 + u * v\bigr) = -2n(u,v)e - 2e(u * v), \\ &\varphi(u) \cdot \varphi(v) = 4(eu) \cdot (ev) = -2\frac{n(eu, ev)}{n(e)}e + \frac{1}{n(e)}[e, (eu)(ev)] \end{aligned} \tag{5.14}$$

and $n(eu, ev) = n(e)n(u,v)$ by the composition property of the norm of C, while (5.10) and (5.11) give

$$\begin{aligned} (eu)(ev) &= -\sigma(eu, ev) + (eu) * (ev) \\ &= -e\bar{e}\sigma(u,v) + \bar{e}^2(u * v) \\ &= -n(e)\bigl(\sigma(u,v) + u * v\bigr). \end{aligned}$$

(Note that $\bar{e} = -e$ and $e\bar{e} = n(e)1$.) Since $[e, K] = 0$ and $eu + ue = -n(e,u)1 = 0$, for any $u, v \in V$:

$$[e, (eu)(ev)] = -n(e)[e, u * v] = -2n(e)e(u * v).$$

This, together with (5.14) proves the validity of (5.13). □

Therefore, the quadratic algebras in $\mathcal{V}$ associated to the $(-1,-1)$-BFKTS's of G-type are precisely the algebras $B^{[-2]}$, where B is a color algebra.

5.(vi) F-type:

Here the characteristic of F will be assumed to be $\neq 2, 3$. Let S be an eight dimensional vector space over F endowed with a nondegenerate symmetric bilinear form $\langle .|. \rangle$, an element $e \in s$ with $\langle e|e \rangle = 1$ and a 3-fold vector cross product X of type I associated to $\langle .|. \rangle$ (see [Eld96], [Oku95, Ch. 8] and the references therein). Then S is a $(-1,-1)$-BFKTS with the triple product

$$xyz = \frac{1}{3}X(x,y,z) + \langle z|x\rangle y - \langle z|y\rangle x + \langle x|y\rangle z$$

for any $x, y, z \in S$.

Then S has the structure of a Cayley algebra, denoted by C, with unity element e, norm $n(x) = \langle x|x\rangle$ and standard involution $x \mapsto \bar{x}$, such that

$$X(x,y,z) = (x\bar{y})z + \langle x|z\rangle y - \langle y|z\rangle x - \langle x|y\rangle z,$$

so the triple product above becomes

$$xyz = \frac{1}{3}\left((x\bar{y})z + 4\langle x|z\rangle y - 4\langle y|z\rangle x + 2\langle x|y\rangle z\right),$$

for any $x, y, z \in S$, while (5.1) becomes

$$\begin{aligned} x \cdot y &= \frac{1}{3}\Big(\bar{x}y + 4\langle e|y\rangle x + 2\langle e|x\rangle y - 4\langle x|y\rangle e\Big) \\ &= \frac{1}{3}\Big(-xy + 4\langle e|y\rangle x + 4\langle e|x\rangle y - 4\langle x|y\rangle e\Big) \end{aligned}$$

for any $x, y \in S$, since $x + \bar{x} = 2\langle e|x\rangle e$.

Now, $(S, \cdot) = Fe \oplus V$ with $V = (Fe)^{\perp}$ (orthogonal relative to $\langle .|. \rangle$ and for any $u, v \in V$, $uv = -\langle u|v\rangle e + \frac{1}{2}[u,v]$. Hence, for any $\alpha, \beta \in F$ and $u, v \in V$

$$(\alpha e + u)\cdot(\beta e + v) = (\alpha\beta - \langle u|v\rangle)e - \frac{1}{3}\left(\frac{1}{2}[u,v]\right),$$

so that $(S, \cdot) = C^{(-\frac{1}{3})}$ (scalar mutation), which is isomorphic to $C^{[9]}$.

Therefore, the quadratic algebras in $\mathcal{V}$ associated to the $(-1,-1)$-BFKTS's of F-type are precisely the algebras $C^{[9]}$, where C is a Cayley algebra.

6. Simple quadratic $\mathcal{V}$-algebras

The classification of the simple finite dimensional $(-1,-1)$-BFKTS's in [EKO03] over fields of characteristic 0, together with the previous sections, does almost all the work needed to prove our last result:

Theorem 6.1. *Let $(Q, \cdot)$ be a finite dimensional simple quadratic algebra in the variety $\mathcal{V}$ over a field F of characteristic 0. Then, up to isomorphism, either:*

(i) *$(Q, \cdot)$ is the Jordan algebra of a nondegenerate quadratic form, with the exception of $(Q, \cdot) \cong F \times F$.*

(ii) *There exists a quadratic étale algebra K over F such that Q is a free K-module of rank ≥ 3, endowed with a nondegenerate hermitian form $h : Q \times Q \to K$ such that $h(1,1) = 1$ and for any $x, y \in Q$,*

$$x \cdot y = h(y,1)x - h(y,x)1 + h(1,x)y. \tag{6.2}$$

In this case, $(Q, \cdot)$ is the structurable algebra of the restriction of the nondegenerate hermitian form $-h$ to $\{x \in Q : h(1,x) = 0\}$.

(iii) *There exists a quaternion algebra H over F such that Q is a free left Q-module of rank ≥ 2, endowed with a hermitian form $h : Q \times Q \to H$ such that* (6.2) *holds.*

(iv) *There exists a quaternion algebra H over F and a nonzero scalar $\mu \in F$ such that $(Q, \cdot) = H^{[\mu]}$.*

(v) *There exists a color algebra B over F such that $(Q, \cdot) = B^{[-2]}$.*

(vi) *There exists a Cayley algebra C over F such that $(Q, \cdot) = C^{[9]}$.*

Moreover, two algebras in different items above are not isomorphic and:

(i') *Two algebras of type* (i) *are isomorphic if and only if their quadratic forms are isometric.*

(ii') *Two algebras Q_1 and Q_2 in item* (ii) *with associated étale algebras K_1 and K_2 and hermitian forms h_1 and h_2 are isomorphic if and only if the hermitian pairs (Q_1, h_1) and (Q_2, h_2) are isomorphic. (That is, there is an isomorphism of F-algebras $\sigma : K_1 \to K_2$ and an F-linear bijection $\varphi : Q_1 \to Q_2$ such that $h_2\big(\varphi(x), \varphi(y)\big) = \sigma\big(h_1(x,y)\big)$ for any $x, y \in Q_1$.)*

(iii') *Two algebras Q_1 and Q_2 in item* (iii) *with associated quaternion algebras H_1 and H_2 and hermitian forms h_1 and h_2 are isomorphic if and only if the hermitian pairs (Q_1, h_1) and (Q_2, h_2) are isomorphic.*

(iv') *Two algebras $H_1^{[\mu_1]}$ and $H_2^{[\mu_2]}$ in item* (iv) *are isomorphic if and only if so are the quaternion algebras H_1 and H_2 and $\mu_1 = \mu_2$.*

(v') *Two algebras $B_1^{[-2]}$ and $B_2^{[-2]}$ in item* (v) *are isomorphic if and only if so are the color algebras B_1 and B_2.*

(vi') *Two algebras $C_1^{[9]}$ and $C_2^{[9]}$ in item* (vi) *are isomorphic if and only if so are the Cayley algebras C_1 and C_2.*

Proof. That the finite dimensional simple quadratic algebras in $\mathcal{V}$ over F are precisely the algebras in the assertion of the Theorem follows directly from Theorem 4.4, Theorem 4.8 and the classification of the simple $(-1,-1)$-BFKTS's in [EKO03, Theorem 4.3].

For the isomorphism problem, notice that any isomorphism φ between two flexible quadratic algebras satisfies $\varphi(\bar{x}) = \overline{\varphi(x)}$ for any x, where $x \mapsto \bar{x}$ denotes the standard involution) and hence extends to an isomorphism between the corresponding $(-1,-1)$-BFKTS's, because of (4.6). Conversely, any isomorphism between the corresponding $(-1,-1)$-BFKTS's that matches the unity elements of the quadratic algebras is indeed an isomorphism of the quadratic algebras.

Now, the assertions in (i')–(iii') would follow from the corresponding assertions in [EKO03, Theorem 4.3] if it could be proved that if the hermitian or quadratic pairs (Q_1, h_1) and (Q_2, h_2) are isomorphic, the isomorphism can be taken to match the unity elements. This is a direct consequence of Witt's Theorem [Sc85] in case (i') and in cases (ii') and (iii') with K and H being division algebras. For the split cases in (ii') ($K = F \times F$) and (iii') ($H = \mathrm{Mat}_2(F)$) an extra argument is needed. First, if $K = F \times F$ and Q is a free K-module of rank ≥ 3 endowed with a nondegenerate hermitian form h, then (see [EKO03, p. 358], up to isomorphism, $Q = W \times W^*$ for a vector space W (W^* being its dual) and $h\big((u,f),(v,g)\big) = \big(g(u), f(v)\big) \in F \times F = K$. Now, given any two elements (u,f) and (u',f') with $h\big((u,f),(u,f)\big) = 1 = h\big((u',f'),(u',f')\big)$ (that is, $f(u) = 1 = f'(u')$), there is a linear bijection $\varphi : W \to W$ such that $\varphi(u) = u'$ and $f \circ \varphi^{-1} = f'$ (just complete $\{u,f\}$ and $\{u',f'\}$ to a couple of dual bases of W and W^*). Then the linear map $\psi : Q \to Q$ given by $\psi\big((v,g)\big) = \big(\varphi(v), g \circ \varphi^{-1}\big)$ is an automorphism of the hermitian pair (Q,h) that carries (u,f) to (u',f'). This finishes the proof of (ii'). Also, if $H = \mathrm{Mat}_2(F)$ and Q is a free left H-module of rank ≥ 2 endowed with a nondegenerate hermitian form h, then ([EKO03, p. 359]), up to isomorphism, $Q = U \otimes_F W$, where U is the irreducible (two dimensional) left H-module and W is a vector space, and $h(u_1 \otimes w_1, u_2 \otimes w_2) = \psi(w_1,w_2)\varphi(-,u_2)u_1 \in \mathrm{End}_F(U) = H$, where $\varphi : U \times U \to F$ and $\psi : W \times W \to F$ are nondegenerate skew symmetric bilinear forms. Let $\{a_1, a_2\}$ be a basis of U with $\varphi(a_1,a_2) = 1$. Then for any element $x = a_1 \otimes w_1 + a_2 \otimes w_2 \in Q$,

$$\begin{aligned} h(x,x) &= \psi(w_1,w_2)\varphi(-,a_2)a_1 + \psi(w_2,w_1)\varphi(-,a_1)a_2 \\ &= \psi(w_1,w_2)\Big(\varphi(-,a_2)a_1 - \varphi(-,a_1)a_2\Big) \\ &= \psi(w_1,w_2)1. \end{aligned}$$

Thus $h(x,x) = 1$ if and only if $\psi(w_1,w_2) = 1$. But if $x = a_1 \otimes w_1 + a_2 \otimes w_2$ and $y = a_1 \otimes w_1' + a_2 \otimes w_2'$ are two elements of Q with $h(x,x) = 1 = h(y,y)$, that is, $\psi(w_1,w_2) = 1 = \psi(w_1',w_2')$, there is an element in the symplectic group $\Phi \in Sp(W,\psi)$ such that $\Phi(w_1) = w_1'$ and $\Phi(w_2) = w_2'$. Then $1 \otimes \Phi$ is an automorphism of the hermitian pair (Q,h) that carries x to y. This completes the proof of (iii').

For (iv')–(vi') notice that any isomorphism of quadratic algebras $\varphi : Q\Big(V_1, (.|.)_1, \times_1\Big) \to Q\Big(V_2, (.|.)_2, \times_2\Big)$ restricts to an isomorphism of anticommutative algebras $\varphi|_{V_1} : (V_1, \times_1) \to (V_2, \times_2)$, which is also an isometry $\big(V_1, (.|.)_1\big) \to \big(V_2, (.|.)_2\big)$. But for a quaternion or Cayley algebra $Q\Big(V, (.|.), \times\Big)$, the bilinear form $(.|.)$ is determined by $\times$ due to the identity:

$$(x \times y) \times y = (x|y)y - (y|y)x, \tag{6.3}$$

and the same happens for color algebras due to the identity:

$$((x \times y) \times y) \times y = \frac{1}{2}(y|y)x \times y. \tag{6.4}$$

(See [Eld88].) Thus, for instance, if

$$\varphi : H_1^{[\mu_1]} = Q\Big(V_1, \mu_1(.|.)_1, \times_1\Big) \to H_2^{[\mu_2]} = Q\Big(V_2, \mu_2(.|.)_2, \times_2\Big)$$

is an isomorphism of algebras in item (iv), where V_i is the set of vectors of the quaternion algebra $H_i = Q\Big(V_i, (.|.)_i, \times_i\Big)$, $i = 1, 2$, then $\psi = \varphi|_{V_1}$:

$(V_1, \times_1) \to (V_2, \times_2)$ is an isomorphism of anticommutative algebras and also an isometry $\psi : \bigl(V_1, \mu_1(.|.)_1\bigr) \to \bigl(V_2, \mu_2(.|.)_2\bigr)$. By (6.3), ψ is an isometry too between $\bigl(V_1, (.|.)_1\bigr)$ and $\bigl(V_2, (.|.)_2\bigr)$, so that $\mu_1 = \mu_2$ and φ is an isomorphism between the quaternion algebras H_1 and H_2. The cases (v') and (vi') follow using the same arguments (but with (6.4) instead of (6.3) for case (v')). □

There appears the natural open question of studying the quadratic simple algebras in $\mathcal{V}$ over arbitrary fields of characteristic $\neq 2$ and check if some other kind of algebras appear. This would lead to a classification of the simple $(-1,-1)$-BFKTS's over these fields. The known classification in characteristic 0 depends heavily on the classification of the simple Lie superalgebras in characteristic 0, and the restriction on the characteristic there is essential.

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