ON THE STABILITY OF DUAL SCATTERING CHANNEL SCHEMES

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ABSTRACT. Dual scattering channel (DSC) schemes generalize JOHNS' TLM algorithm in replacing transmission lines with abstract scattering channels in terms of paired distributions. A well known merit of TLM schemes is unconditional stability, a property that is commonly drawn upon the passivity of linear transmission line networks. So the question arises, if DSC algorithms remain stable in a neat sense. It is shown that a large class of α-*passive* processes are in fact unconditionally stable. The analysis applies to TLM and DSC schemes alike and includes non-linear situations.

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1. INTRODUCTION

Dual scattering channel (DSC) schemes result from an incisive revision of the transmission line matrix (TLM) numerical method. The latter has originally been introduced by P.B.JOHNS and coworkers in the early 1970s [JoBe] and has since been subject to assiduous study and publication [Tlm1-3]. The TLM method is today commonplace in scientific computing and largely applied to the numerical solution of MAXWELL's equations [Hoe] but also to manifold wave propagation, transport, and diffusion phenomena; we can here refer to the monographs of CHRISTOPOULOS [Ch] and DE COGAN [dC]. Also, REBEL [Re] gives a fairly complete survey over the state of the art of TLM by the year 2000.

DSC schemes are generalized TLM methods in arising from a twofold abstraction [He1]. Firstly, the scattering channel concept underlying TLM is redefined in terms of paired distributions. (Characteristic impedances are thus neither needed, nor in general defined, e.g.) In the second place, non trivial cell interface scattering is admitted during the connection step of iteration, thus taking advantage of the intrinsic duality in the connection-reflection cycle of the algorithm.

The extended framework bypasses a set of modeling limitations induced by transmission lines (and discussed in more detail in [He1, section2]) while it preserves the main advantages of the TLM method. In particular, the convolution type updating scheme and Johns' two step connection-reflection cycle are essentially retained. The question arises, however, if DSC schemes remain *unconditionally stable* in a well-defined sense.

TLM schemes are unconditionally stable in that they are equivalent to passive linear transmission line network models [Jo2]. A concept that applies to DSC and TLM algorithms alike is α-*passivity* which paraphrases contraction properties with respect to a non-negative (de)limiting functional;

cf. section 2. In the TLM context, α is essentially the sum over the squared incident and outgoing transmission line voltages. The sum is contracted at each scattering event due to energy conservation (or loss).

In a like manner α-passivity characterizes the reflection and connection maps of an unconditionally stable DSC algorithm, but the 'energy' functional α (which may in fact measure *any* conserved quantity) needs not to be a quadratic form. For instance, energy may be linearly related to temperature and quadratically to particle velocity, within the same algorithm. The following approach is sufficiently general to apply to such and other, even non-linear situations.

2. Stability

Algorithm stability prevents the computational process from piling up to infinity (it does not yet imply convergence, or *consistence* of the algorithm, of course). TLM models are *unconditionally* stable in that they are equivalent to passive linear transmission line networks [Jo2]. DSC schemes, in not using lines, need a more general characterization which here is given in terms of α-passive causal functions.

Let $\mathcal{L}$ be a real or complex linear space and I a totally ordered set (e.g. $I \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}\}$; intervals are then naturally defined in I by the order relation. We commonly think with I of a discrete or continuous time domain.) Also, let $\mathcal{E} \subset \mathcal{L}^I$ be a set of functions such that $f \in \mathcal{E}$ implies $\chi_{s \leqslant t}(s) f(s) \in \mathcal{E}$, for every $t \in I$, where $\chi_P(s)$ denotes the characteristic function of property P (which is 1 if s shares that property and 0 else).

Definition 2.1. A function $F : \mathcal{E} \to \mathcal{E}$ is called *causal*, iff for every $f \in \mathcal{E}$ and $t \in I$
$$F f(t) = F[\chi_{s \leqslant t} f(s)](t) \quad .$$

Such functions are also called (causal) *propagators*.

Remark. In some respect, causal functions generalize lower triangular matrices or integral operators such as
$$F f(t) = \int_{-\infty}^{t} K(t - s) f(s) \, ds$$

with a Green's function kernel K, e.g. Note that in general $F : f \mapsto Ff$ needs not to be linear.

Typically f represents a state evolving in time (i.e. a *process*). Then causality of F means that $F f(t)$ depends on the history of f only up to *present* time t.

The proof of the following is easy and left to the reader.

Proposition.
For every $t \in I$; $f, g \in \mathcal{E}$ and causal functions $F, G : \mathcal{E} \to \mathcal{E}$

    (i) $Ff(s) = F[\chi_{r \leqslant t}(r) f(r)](s)$, *for every $s \leqslant t$* .

    (ii) *If $f(s) = g(s)$ for* every $s \leqslant t$,

              then $Ff(s) = Fg(s)$ *for every $s \leqslant t$* .

(iii) *The product of causal functions*
$$FG : \mathcal{E} \longrightarrow \mathcal{E}$$
$$f \longmapsto FGf := F[G[f]]$$

is again causal. In fact, if $\mathcal{E}$ is a linear space, then the causal functions over $\mathcal{E}$ form an algebra.

Let $\|...\|$ be a norm on $\mathcal{L}$ and $\alpha \in \mathbb{R}^{\mathcal{L}}$ a continuous non-negative real functional on $\mathcal{L}$.

Definition 2.2.

(i) We call a process $g : I \to \mathcal{L}$ *stable*, iff g is uniformly bounded on I, i.e. iff there exists $b \in \mathbb{R}_+$ such that $\| g(t) \| < b$ for every $t \in I$.

(ii) The functional $\alpha : \mathcal{L} \to \mathbb{R}$ is named a *(de)limiting functional*, iff there exist any non-negative real constants a, b, c such that

(1) $$\| z \| \leqslant a + b(\alpha(z))^c ,$$

for every $z \in \mathcal{L}$. Then obviously $b, c > 0$, and we say also that α is *minimal increasing (in any order) not lesser than* $1/c$.

Let μ be a measure on I such that intervals are μ-measurable sets. Functions on I are henceforth read *modulo* μ (viz. as equivalence classes of functions that differ at most on sets of μ-measure zero). Also, let $\alpha \in \mathbb{R}^{\mathcal{L}}$ be a delimiting functional on $\mathcal{L}$ (i.e. one that is increasing not lesser than any positive order $1/c$), and assume that $\alpha \circ f$ is μ−summable over finite intervals in I for every $f \in \mathcal{E}$. The latter is for instance the case if $\alpha(z) = \| z \|^p$ for any real $p \geqslant 1$ and $\mathcal{E} \subset L^p(I, \mathcal{L})$, which is the metric completion of

$$\{ f \in \mathcal{L}^I \mid (\int_I \| f \|^p \, d\mu)^{\frac{1}{p}} \leqslant \infty \} ,$$

i.e. $\mathcal{E}$ is a subset of the Banach space with norm $\| f \|_p := (\int_I \| f \|^p \, d\mu)^{\frac{1}{p}}$.

Definition 2.3. A causal function $F : \mathcal{E} \to \mathcal{E}$ is called α-*passive*, iff

(2) $$\int_{s<t} \alpha(Ff(s)) \, d\mu(s) \leqslant \int_{s<t} \alpha(f(s)) \, d\mu(s) ,$$

for every $f \in \mathcal{E}$ and $t \in I$.

Remark. If $\alpha = \|...\|^p$ for any real $p \geqslant 1$ and $\| f \|^p$ is μ-summable over I, i.e. $f \in L^p(I, \mathcal{L})$, then (2) clearly implies $Ff \in L^p(I, \mathcal{L})$ and

$$\| Ff \|_p \leqslant \| f \|_p .$$

Hence, every $\|...\|^p$-passive causal function F defines a *contraction* operator on $\mathcal{E} \cap L^p(I, \mathcal{L})$.

Assume, furthermore, that $\tau \in \mathbb{R}_+$ and let on $I := \mathbb{R}$ the measure μ be concentrated in $\{ k\tau \mid k \in \mathbb{Z} \}$ with uniform weight $\mu(\{k\tau\}) = \tau$, $k \in \mathbb{Z}$. Alternatively, let $I := \{ k\tau \mid k \in \mathbb{Z} \}$ with 'the same' measure μ. (Virtually we deal of course with that discrete situation, even in working on the real axis with functions that are constant over intervals $[k\tau, (k+1)\tau)$; $k \in \mathbb{Z}$. Indeed we retain the integral formalism for simplicity, and the reader may optionally re-write the following integrals as sums.) For every $f \in \mathcal{E}$ let $f(t - \tau) \in \mathcal{E}$, i.e. $\mathcal{E}$ is closed under time shifts by negative integer multiples

of τ. Then, for arbitrary $N \in \mathbb{N}_+$ and *exciting function* $e \in \mathcal{E} \subset \mathcal{L}^I$ with support on $[0, N\tau) \subset I$, the following holds

Theorem 2.1. [Stability of the iterated passive causal process]
For any α-passive causal function $F : \mathcal{E} \to \mathcal{E}$ and $e \in \mathcal{E}$ as stated, if $g \in \mathcal{L}^I$ is a process such that $g(t) \equiv 0$ for $t \leqslant 0$ and recursively for $t = n\tau; n \in \mathbb{N}$

$$(3) \qquad\qquad g(t + \tau) = F[e + g](t) \quad,$$

then g is uniquely defined (modulo μ) and for every $t \geqslant N\tau$ holds

$$(4) \quad \| g(t) \| \leqslant a + \left(\frac{b}{\tau} \int_{[0, N\tau)} \alpha(e + g) - \alpha(g) \ d\mu \right)^c$$

with every constants a, b, c that satisfy (1). Hence g is stable.

Remark. A process $g \in \mathcal{L}^I$ which is recursively generated according to (3) by iteration of an α-passive causal function F is called an α-*passive process*.

The theorem ensures thus that for any excitation of finite duration (and with no further restrictions) the α-passive process is necessarily stable and in this sense *unconditionally* stable.

Note that *existence* of such a process g is not apriori guaranteed, since this obviously depends on the condition that with $s_0 := e \in \mathcal{E}$ recursively also the functions

$$(5) \qquad\qquad s_n(t) := e(t) + F[s_{n-1}](t - \tau) \in \mathcal{E},$$

for $0 < n < N$ - which eventually has to be checked.
Clearly, conditions (5) are always true (and hence g exists) if $\mathcal{E}$ is a linear space. We do not universally premise this, in order to potentially apply the theorem to non-linear situations, where conditions (5) may only be satisfied for sufficiently small excitations e.

Corollary.

(i) *In the special case $\alpha = \|...\|$ estimates (1) holds with $a = 0$, $b = 1$, $c = 1$. Then the triangle inequality applies to the integrand of (4) and validates the bound*

$$\| g(t) \| \leqslant \frac{1}{\tau} \int_{[0, N\tau)} \| e \| \ d\mu \quad .$$

(ii) *If $N = 1$, i.e. $e(t)$ is a DIRAC excitation concentrated on $[0, \tau)$ (where $g = 0$), then (4) reads simply*

$$\| g(t) \| \leqslant a + \left(\frac{b}{\tau} \int_{[0, \tau)} \alpha(e) \ d\mu \right)^c \quad,$$

provided that $\alpha(0) = 0$ (which is the normal case).

Proof. Clearly, $g \in \mathcal{L}^I$ is uniquely defined by the given recurrence relations, since $e + g$ (and hence g) at the right hand side of (3) is evaluated only up to time $t = n\tau$, in virtue of the causality of F.
Furthermore, if $N \leqslant n$ and $a \leqslant \| g(n\tau) \|$ with any a satisfying (1), then

with pertinent b, c that satisfy (1)

$$0 \leqslant \left(\frac{1}{b} \left(\| g \left(n\tau \right) \| - a \right) \right)^{1/c} \leqslant \alpha \left(g \left(n\tau \right) \right)$$

$$= \frac{1}{\tau} \left(\int_{s < (n+1)\tau} \alpha \left(g \right) d\mu(s) - \int_{s < n\tau} \alpha \left(g \right) d\mu(s) \right)$$

$$= \frac{1}{\tau} \left(\int_{s < n\tau} \alpha \left(F[e + g] \right) d\mu(s) - \int_{s < n\tau} \alpha \left(g \right) d\mu(s) \right)$$

$\updownarrow$ recursion formula (3)

$$\leqslant \frac{1}{\tau} \left(\int_{s < n\tau} \alpha \left(e + g \right) d\mu(s) - \int_{s < n\tau} \alpha \left(g \right) d\mu(s) \right)$$

$\updownarrow$ since F is passive

$$= \frac{1}{\tau} \int_{[0, N\tau)} \alpha \left(e + g \right) - \alpha \left(g \right) \quad d\mu(s)$$

$\updownarrow$ since $e(t) \equiv 0$ if $t \notin [0, N\tau)$.

Thus, estimates (4) holds true in the case $a \leqslant \| g \left(n\tau \right) \|$ and trivially otherwise. It follows that $\| g \|$ is uniformly bounded on $I \smallsetminus [0, N\tau)$, hence also on I, that is to say g is stable. $\square$

3. DSC Processes

In this section DSC schemes are represented as paired α-passive processes such as are dealt with in Theorem 2.1 and which hence are stable.

Just as the TLM algorithm, DSC schemes operate on a space $\mathcal{P}$ of *propagating fields*, which is a product of (real or complex) normed linear spaces

$$(6) \qquad\qquad \mathcal{P} = \mathcal{P}_{in} \times \mathcal{P}_{out} \quad ,$$

cf. [He1]. The two factors are named (somewhat off-hand) the *incident* and *outgoing* subspace of $\mathcal{P}$. They are isomorphic in that there is canonical involutary isomorphism of normed linear space

$$(7) \qquad\qquad \begin{aligned} nb &: \mathcal{P} \to \mathcal{P} \\ z = \left(z_{in}, z_{out} \right) &\mapsto \left(z_{out}, z_{in} \right) =: nb \left(z \right), \end{aligned}$$

which is commonly called the *node-boundary* map. Hence, there exists a space $\left(\mathcal{L}, \| ... \| \right)$ such that

$$(8) \qquad\qquad \mathcal{P}_{in} \cong \mathcal{P}_{out} \cong \left(\mathcal{L}, \| ... \| \right)$$

in the sense of isomorpy of normed spaces and

$$(9) \qquad \mathcal{P} \cong \left(\mathcal{L}^2, \| ... \|^{\sim} \right) \quad ,$$

$$\text{e.g. with norm} \quad \| \left(a, b \right) \|^{\sim} := \sqrt{\|a\|^2 + \|b\|^2} \quad ; \quad a, b \in \mathcal{L}$$

(or any equivalent norm).

As is well known, DSC and TLM algorithms follow a two-step iteration cycle in working with alternate application of a *connection* and *reflection* map

$$\mathcal{C} : I \times \mathcal{P}_{out}^I \to \mathcal{P}_{in} \qquad \text{and} \qquad \mathcal{R} : J \times \mathcal{P}_{in}^J \to \mathcal{P}_{out} \quad ,$$

which respectively update the propagating fields at even and odd integer multiples of half the time step, i.e. on $I := \{\, k\tau \mid k \in \mathbb{Z}\,\}$ and $J := \{\,(2k+1)\tau/2 \mid k \in \mathbb{Z}\,\}$. To these maps the following functions $F_C\,, F_R$ are associated in a one-to-one correspondence

$$(10) \qquad \begin{aligned} F_C \;:\; \mathcal{L}^I &\to \mathcal{L}^I \\ f &\mapsto F_C\,f \quad \text{with} \quad F_C\,f\,(\,t\,) \; := \; nb \circ \mathcal{C}\,(\,t\,,\,f\,) \end{aligned}$$

and

$$(11) \qquad \begin{aligned} F_R \;:\; \mathcal{L}^J &\to \mathcal{L}^J \\ g &\mapsto F_R\,g \quad \text{with} \quad F_R\,g\,(\,t\,) \; := \; nb \circ \mathcal{R}\,(\,t\,,\,g\,) \quad . \end{aligned}$$

Proposition 3.1.
    (i) F_C and F_R are causal on $\mathcal{L}^I$ and $\mathcal{L}^J$, respectively.
    (ii) For $r,s \in J$ and $T_s : f(t) \mapsto f(t+s)$ the shift operator on $\mathcal{L}^{I \cup J}$,

$$(\quad r,s \leqslant 0 \quad \text{and} \quad \begin{cases} F_C \\ F_R \end{cases} \text{is } \alpha\text{-passive on} \quad \begin{cases} \mathcal{L}^I \\ \mathcal{L}^J \end{cases}) \implies$$

$$\implies (\quad \begin{cases} T_r \circ F_C \circ T_s \\ T_r \circ F_R \circ T_s \end{cases} \text{is } \alpha\text{-passive on} \quad \begin{cases} \mathcal{L}^J \\ \mathcal{L}^I \end{cases}) \quad .$$

Indeed, the first statement is only a trivial consequence of the definition of the DSC propagators $\mathcal{R}$ and $\mathcal{C}$ as functions on back-in-time running sequences of incident and outgoing states, cf. [He1], section 3. Then (ii) follows from Definition 2.3.   □

Definition 3.1.

The $\begin{cases} \text{connection map } \mathcal{C} \\ \text{reflection map } \mathcal{R} \end{cases}$ is called α-passive, iff $\begin{cases} F_C \\ F_R \end{cases}$ is α-passive

in the sense of Definition 2.3.

We now claim the main statement:

Theorem 3.1. *With every time limited excitation, the DSC process generated by α-passive reflection and connection maps is uniformly bounded, hence stable.*

Proof. It is sufficient to show that every finitely excited DSC process that is generated by α-passive $\mathcal{R}$ and $\mathcal{C}$ can be written as a pair of processes, either of which satisfies Theorem 2.1.

    With $H := I \cup J = \{kt/2 \mid k \in \mathbb{Z}\}$ and the measure on H inherited from I and J jointly, the space of all DSC processes is

$$\mathcal{E} \; := \; \{\, h = (\,f,g\,) \in (\mathcal{L}^2)^H \;\mid\; f((2k+1)\,\tau/2) \,=\, f(\,k\tau\,)$$
$$\text{and} \quad g(\,k\tau\,) \,=\, g((2k-1)\,\tau/2)\,, \quad \text{for every } k \in \mathbb{Z}\,\}\,,$$

i.e. the functions $f : H \to \mathcal{L}$ and $g : H \to \mathcal{L}$ in $h = (\,f\,,g\,) \in \mathcal{E}$ 'switch' at even and odd integer multiples of $\tau/2$, respectively. So, there is a natural bijection

$$\begin{aligned} \mathcal{E} &\to \mathcal{L}^I \times \mathcal{L}^J \\ h = (\,f\,,g\,) &\mapsto (\,f \restriction I\,,\,g \restriction J\,) \quad , \end{aligned}$$

in virtue of which the first and second components in $\mathcal{E}$ can be naturally identified with $\mathcal{L}^I$ and $\mathcal{L}^J$, respectively.

For any incident function $e \in \mathcal{L}^I \times \{0\} \subset \mathcal{E}$ supported on a finite interval $[0, N\tau) \subset I$, the DSC process excited with e and generated by $\mathcal{C}$ and $\mathcal{R}$ is the well-defined function $h \in (\mathcal{L}^2)^H$ such that $h(t) = 0$ for $t \leqslant 0$ and recursively for $0 < t \in H$

$$(12) \quad h\left(t + \frac{\tau}{2}\right) = \begin{cases} (h_1(t), \, T_{-\frac{\tau}{2}} F_R T_{-\frac{\tau}{2}}[e + h_1](t)) & \text{if } t \in I \\ (T_{-\frac{\tau}{2}} F_C T_{-\frac{\tau}{2}}[h_2](t), \, h_2(t)) & \text{if } t \in J . \end{cases}$$

Actually h is well and uniquely defined by relations (12) which provide separate recurrence relations for the two processes h_1, h_2; for instance for h_1 and $t \in I$

$$\begin{aligned}
h_1(t + \tau) &= \quad h_1\left(\tilde{t} + \frac{\tau}{2}\right) \quad \text{with} \quad \tilde{t} := t + \frac{\tau}{2} \in J \\
&= \quad T_{\frac{-\tau}{2}} F_C T_{\frac{-\tau}{2}}[h_2 \,|_{\tilde{t}}](\tilde{t}) \quad\quad\quad\quad \text{by (12)} \\
(13) \quad &= \quad F_C T_{\frac{-\tau}{2}}[\underbrace{h_2 \,|_{\tilde{t}}}_{= \, T_{\frac{\tau}{2}} h_2|_{\tilde{t}} \, , \, \text{cf. (12)}}]\left(\tilde{t} - \frac{\tau}{2}\right) \\
&= \quad F_C T_{\frac{-\tau}{2}} F_R T_{\frac{-\tau}{2}}[h_1 + e](t) \quad .
\end{aligned}$$

In virtue of the causality of $F_C T_{\frac{-\tau}{2}} F_R T_{\frac{-\tau}{2}}$ function h_1 enters the last expression in (13) only up to argument t. Moreover, since products of α-passive operators are obviously α-passive, $F_C T_{-\tau/2} F_R T_{-\tau/2}$ is α-passive due to Proposition 3.1 (ii). Hence Theorem 2.1 applies to h_1, which thus is stable, just as then also is $h_2 = F_R T_{-\tau/2}[e + h_1]$. $\qquad\square$

4. CONCLUSIONS

It has been demonstrated that a class of α-passive processes, characterised by simple contraction properties with respect to a *limiting* functional α, are unconditionally stable. Finitely excited DSC processes, generated by α-passive reflection and connection maps, are then necessarily stable. Stability analysis reduces so in pursuant cases to finding *any* limiting functional α in relation to which the reflection and connection maps are α-passive. In reverse: If the connection and reflection maps are *per se* so designed as to conserve (or admit loss of) any quantity 'measured' by a limiting functional, then the DSC process generated by those maps is essentially stable.

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