

**THE MOMENTS OF THE GENERATING OPERATOR OF**

$$L(F_2) *_{L(F_1)} L(F_2)$$

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ABSTRACT. In this paper, we will consider an example of a (scalar-valued) moment series, under the compatibility. Suppose that we have an amalgamated free product of free group algebras,  $L(F_2) *_{L(F_1)} L(F_2) = L(\langle a, b \rangle) *_{L(\langle h \rangle)} L(\langle c, d \rangle)$

We will provide the method how to find the moment series of  $a + b + a^{-1} + b^{-1} + c + d + c^{-1} + d^{-1}$ . Amalgamated freeness of  $a + b + a^{-1} + b^{-1}$  and  $c + d + c^{-1} + d^{-1}$  over  $L(F_1)$  is used and some combinatorial functions (to explain the recurrence relations) are used to figure out the  $n$ -th moment of this element.

Voiculescu developed Free Probability Theory. Here, the classical concept of Independence in Probability theory is replaced by a noncommutative analogue called Freeness (See [9]). There are two approaches to study Free Probability Theory. One of them is the original analytic approach of Voiculescu and the other one is the combinatorial approach of Speicher and Nica (See [23], [1] and [24]).

Speicher defined the free cumulants which are the main objects in Combinatorial approach of Free Probability Theory. And he developed free probability theory by using Combinatorics and Lattice theory on collections of noncrossing partitions (See [24]). Also, Speicher considered the operator-valued free probability theory, which is also defined and observed analytically by Voiculescu, when  $\mathbb{C}$  is replaced to an arbitrary algebra  $B$  (See [23]). Nica defined R-transforms of several random variables (See [1]). He defined these R-transforms as multivariable formal series in noncommutative several indeterminants. To observe the R-transform, the Möbius Inversion under the embedding of lattices plays a key role.

In [16], we observed the amalgamated R-transform calculus. Actually, amalgamated R-transforms are defined originally by Voiculescu and are characterized combinatorially by Speicher (See [23]). In [16], we defined amalgamated R-transforms slightly differently from those defined in [23] and [13]. We defined them as  $B$ -formal series and tried to characterize, like in [1] and [24].

In [15], we observed the compatibility of a noncommutative probability space and an amalgamated noncommutative probability space over an unital algebra. In this paper, we have a nice compatibility of  $(L(F_2) *_{L(F_1)} L(F_2), \varphi)$  and  $(L(F_2) *_{L(F_1)} L(F_2), E)$ ,

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where  $tr : L(F_2) *_{L(F_1)} L(F_2) \rightarrow \mathbb{C}$  is the canonical trace and  $F$  is the free product of conditional expectations  $E : L(F_2) \rightarrow L(F_1)$ .

In this paper, we will compute the  $n$ -th (scalar-valued) moment

$$a + b + a^{-1} + b^{-1} + c + d + c^{-1} + d^{-1} \in L(F_2) *_{L(F_1)} L(F_2),$$

where  $\langle a, b \rangle = F_2 = \langle c, d \rangle$ .

## 1. PRELIMINARIES

### 1.1. Amalgamated Free Probability.

In this section, we will summarize and introduced the basic results from [23] and [16]. Throughout this section, let  $B$  be a unital algebra. The algebraic pair  $(A, \varphi)$  is said to be a noncommutative probability space over  $B$  (shortly, NCPSpace over  $B$ ) if  $A$  is an algebra over  $B$  (i.e  $1_B = 1_A \in B \subset A$ ) and  $\varphi : A \rightarrow B$  is a  $B$ -functional (or a conditional expectation) ;  $\varphi$  satisfies

$$\varphi(b) = b, \text{ for all } b \in B$$

and

$$\varphi(bxb') = b\varphi(x)b', \text{ for all } b, b' \in B \text{ and } x \in A.$$

Let  $(A, \varphi)$  be a NCPSpace over  $B$ . Then, for the given  $B$ -functional, we can determine a moment multiplicative function  $\widehat{\varphi} = (\varphi^{(n)})_{n=1}^{\infty} \in I(A, B)$ , where

$$\varphi^{(n)}(a_1 \otimes \dots \otimes a_n) = \varphi(a_1 \dots a_n),$$

for all  $a_1 \otimes \dots \otimes a_n \in A^{\otimes B^n}$ ,  $\forall n \in \mathbb{N}$ .

We will denote noncrossing partitions over  $\{1, \dots, n\}$  ( $n \in \mathbb{N}$ ) by  $NC(n)$ . Define an ordering on  $NC(n)$  ;

$\theta = \{V_1, \dots, V_k\} \leq \pi = \{W_1, \dots, W_l\} \stackrel{def}{\iff}$  For each block  $V_j \in \theta$ , there exists only one block  $W_p \in \pi$  such that  $V_j \subset W_p$ , for  $j = 1, \dots, k$  and  $p = 1, \dots, l$ .

Then  $(NC(n), \leq)$  is a complete lattice with its minimal element  $0_n = \{(1), \dots, (n)\}$  and its maximal element  $1_n = \{(1, \dots, n)\}$ . We define the incidence algebra  $I_2$  by a set of all complex-valued functions  $\eta$  on  $\cup_{n=1}^{\infty} (NC(n) \times NC(n))$  satisfying  $\eta(\theta, \pi) = 0$ , whenever  $\theta \not\leq \pi$ . Then, under the convolution

$$* : I_2 \times I_2 \rightarrow \mathbb{C}$$

defined by

$$\eta_1 * \eta_2(\theta, \pi) = \sum_{\theta \leq \sigma \leq \pi} \eta_1(\theta, \sigma) \cdot \eta_2(\sigma, \pi),$$

$I_2$  is indeed an algebra of complex-valued functions. Denote zeta, Möbius and delta functions in the incidence algebra  $I_2$  by  $\zeta$ ,  $\mu$  and  $\delta$ , respectively. i.e

$$\zeta(\theta, \pi) = \begin{cases} 1 & \theta \leq \pi \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta(\theta, \pi) = \begin{cases} 1 & \theta = \pi \\ 0 & \text{otherwise,} \end{cases}$$

and  $\mu$  is the  $(*)$ -inverse of  $\zeta$ . Notice that  $\delta$  is the  $(*)$ -identity of  $I_2$ . By using the same notation  $(*)$ , we can define a convolution between  $I(A, B)$  and  $I_2$  by

$$\widehat{f} * \eta(a_1, \dots, a_n; \pi) = \sum_{\pi \in NC(n)} \widehat{f}(\pi)(a_1 \otimes \dots \otimes a_n) \eta(\pi, 1_n),$$

where  $\widehat{f} \in I(A, B)$ ,  $\eta \in I_1$ ,  $\pi \in NC(n)$  and  $a_j \in A$  ( $j = 1, \dots, n$ ), for all  $n \in \mathbb{N}$ . Notice that  $\widehat{f} * \eta \in I(A, B)$ , too. Let  $\widehat{\varphi}$  be a moment multiplicative function in  $I(A, B)$  which we determined before. Then we can naturally define a cumulant multiplicative function  $\widehat{c} = (c^{(n)})_{n=1}^{\infty} \in I(A, B)$  by

$$\widehat{c} = \widehat{\varphi} * \mu \quad \text{or} \quad \widehat{\varphi} = \widehat{c} * \zeta.$$

This says that if we have a moment multiplicative function, then we always get a cumulant multiplicative function and vice versa, by  $(*)$ . This relation is so-called "Möbius Inversion". More precisely, we have

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \varphi^{(n)}(a_1 \otimes \dots \otimes a_n) \\ &= \sum_{\pi \in NC(n)} \widehat{c}(\pi)(a_1 \otimes \dots \otimes a_n) \zeta(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \widehat{c}(\pi)(a_1 \otimes \dots \otimes a_n), \end{aligned}$$

for all  $a_j \in A$  and  $n \in \mathbb{N}$ . Or equivalently,

$$c^{(n)}(a_1 \otimes \dots \otimes a_n) = \sum_{\pi \in NC(n)} \widehat{\varphi}(\pi)(a_1 \otimes \dots \otimes a_n) \mu(\pi, 1_n).$$

Now, let  $(A_i, \varphi_i)$  be NCPSpaces over  $B$ , for all  $i \in I$ . Then we can define a amalgamated free product of  $A_i$ 's and amalgamated free product of  $\varphi_i$ 's by

$$A \equiv *_B A_i \quad \text{and} \quad \varphi \equiv *_i \varphi_i,$$

respectively. Then, by Voiculescu,  $(A, \varphi)$  is again a NCPSpace over  $B$  and, as a vector space,  $A$  can be represented by

$$A = B \oplus \left( \bigoplus_{n=1}^{\infty} \left( \bigoplus_{i_1 \neq \dots \neq i_n} (A_{i_1} \ominus B) \otimes \dots \otimes (A_{i_n} \ominus B) \right) \right),$$

where  $A_{i_j} \ominus B = \ker \varphi_{i_j}$ . We will use Speicher's combinatorial definition of amalgamated free product of  $B$ -functionals ;

**Definition 1.1.** Let  $(A_i, \varphi_i)$  be NCPSpaces over  $B$ , for all  $i \in I$ . Then  $\varphi = *_i \varphi_i$  is the amalgamated free product of  $B$ -functionals  $\varphi_i$ 's on  $A = *_B A_i$  if the cumulant multiplicative function  $\widehat{c} = \widehat{\varphi} * \mu \in I(A, B)$  has its restriction to  $\bigcup_{i \in I} A_i, \bigoplus_{i \in I} \widehat{c}_i$ , where  $\widehat{c}_i$  is the cumulant multiplicative function induced by  $\varphi_i$ , for all  $i \in I$  and, for each  $n \in \mathbb{N}$ ,

$$c^{(n)}(a_1 \otimes \dots \otimes a_n) = \begin{cases} c_i^{(n)}(a_1 \otimes \dots \otimes a_n) & \text{if } \forall a_j \in A_i \\ 0_B & \text{otherwise.} \end{cases}$$

Now, we will observe the freeness over  $B$  ;

**Definition 1.2.** Let  $(A, \varphi)$  be a NCPSpace over  $B$ .

(1) Subalgebras containing  $B$ ,  $A_i \subset A$  ( $i \in I$ ) are free (over  $B$ ) if we let  $\varphi_i = \varphi|_{A_i}$ , for all  $i \in I$ , then  $*_i \varphi_i$  has its cumulant multiplicative function  $\widehat{c}$  such that its restriction to  $\bigcup_{i \in I} A_i$  is  $\bigoplus_{i \in I} \widehat{c}_i$ , where  $\widehat{c}_i$  is the cumulant multiplicative function induced by each  $\varphi_i$ , for all  $i \in I$ .

(2) Subsets  $X_i$  ( $i \in I$ ) are free (over  $B$ ) if subalgebras  $A_i$ 's generated by  $B$  and  $X_i$ 's are free in the sense of (1). i.e If we let  $A_i = \text{Alg}(X_i, B)$ , for all  $i \in I$ , then  $A_i$ 's are free over  $B$ .

In [23], Speicher showed that the above combinatorial freeness with amalgamation can be used alternatively with respect to Voiculescu's original freeness with amalgamation.

Let  $(A, \varphi)$  be a NCPSpace over  $B$  and let  $x_1, \dots, x_s$  be  $B$ -valued random variables ( $s \in \mathbb{N}$ ). Define  $(i_1, \dots, i_n)$ -th moment of  $x_1, \dots, x_s$  by

$$\varphi(x_{i_1} b_{i_2} x_{i_2} \dots b_{i_n} x_{i_n}),$$

for arbitrary  $b_{i_2}, \dots, b_{i_n} \in B$ , where  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n, \forall n \in \mathbb{N}$ . Similarly, define a symmetric  $(i_1, \dots, i_n)$ -th moment by the fixed  $b_0 \in B$  by

$$\varphi(x_{i_1} b_0 x_{i_2} \dots b_0 x_{i_n}).$$

If  $b_0 = 1_B$ , then we call this symmetric moments, trivial moments.

Cumulants defined below are main tool of combinatorial free probability theory ; in [16], we defined the  $(i_1, \dots, i_n)$ -th cumulant of  $x_1, \dots, x_s$  by

$$k_n(x_{i_1}, \dots, x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n}),$$

for  $b_{i_2}, \dots, b_{i_n} \in B$ , arbitrary, and  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $\forall n \in \mathbb{N}$ , where  $\widehat{c} = (c^{(n)})_{n=1}^\infty$  is the cumulant multiplicative function induced by  $\varphi$ . Notice that, by Möbius inversion, we can always take such  $B$ -value whenever we have  $(i_1, \dots, i_n)$ -th moment of  $x_1, \dots, x_s$ . And, vice versa, if we have cumulants, then we can always take moments. Hence we can define a symmetric  $(i_1, \dots, i_n)$ -th cumulant by  $b_0 \in B$  of  $x_1, \dots, x_s$  by

$$k_n^{\text{symm}(b_0)}(x_{i_1}, \dots, x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_0 x_{i_2} \otimes \dots \otimes b_0 x_{i_n}).$$

If  $b_0 = 1_B$ , then it is said to be trivial cumulants of  $x_1, \dots, x_s$ .

By Speicher, it is shown that subalgebras  $A_i$  ( $i \in I$ ) are free over  $B$  if and only if all mixed cumulants vanish.

**Proposition 1.1.** (See [23] and [16]) *Let  $(A, \varphi)$  be a NCPSpace over  $B$  and let  $x_1, \dots, x_s \in (A, \varphi)$  be  $B$ -valued random variables ( $s \in \mathbb{N}$ ). Then  $x_1, \dots, x_s$  are free if and only if all their mixed cumulants vanish.  $\square$*

**Remark 1.1.** *The above noncommutative probability space with amalgamation can be replaced by  $W^*$ -probability space with amalgamation and later, we will use the  $W^*$ -probability framework.*

## 1.2. Amalgamated R-transform Theory.

In this section, we will define an R-transform of several  $B$ -valued random variables. Note that to study R-transforms is to study operator-valued distributions. R-transforms with single variable is defined by Voiculescu (over  $B$ , in particular,  $B = \mathbb{C}$ . See [9] and [13]). Over  $\mathbb{C}$ , Nica defined multi-variable R-transforms in [1]. In [16], we extended his concepts, over  $B$ . R-transforms of  $B$ -valued random variables can be defined as  $B$ -formal series with its  $(i_1, \dots, i_n)$ -th coefficients,  $(i_1, \dots, i_n)$ -th cumulants of  $B$ -valued random variables, where  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $\forall n \in \mathbb{N}$ .

**Definition 1.3.** *Let  $(A, \varphi)$  be a NCPSpace over  $B$  and let  $x_1, \dots, x_s \in (A, \varphi)$  be  $B$ -valued random variables ( $s \in \mathbb{N}$ ). Let  $z_1, \dots, z_s$  be noncommutative indeterminants. Define a moment series of  $x_1, \dots, x_s$ , as a  $B$ -formal series, by*

$$M_{x_1, \dots, x_s}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} \varphi(x_{i_1} b_{i_2} x_{i_2} \dots b_{i_n} x_{i_n}) z_{i_1} \dots z_{i_n},$$

where  $b_{i_2}, \dots, b_{i_n} \in B$  are arbitrary for all  $(i_2, \dots, i_n) \in \{1, \dots, s\}^{n-1}$ ,  $\forall n \in \mathbb{N}$ .

Define an  $R$ -transform of  $x_1, \dots, x_s$ , as a  $B$ -formal series, by

$$R_{x_1, \dots, x_s}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} k_n(x_{i_1}, \dots, x_{i_n}) z_{i_1} \dots z_{i_n},$$

with

$$k_n(x_{i_1}, \dots, x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n}),$$

where  $b_{i_2}, \dots, b_{i_n} \in B$  are arbitrary for all  $(i_2, \dots, i_n) \in \{1, \dots, s\}^{n-1}$ ,  $\forall n \in \mathbb{N}$ . Here,  $\hat{c} = (c^{(n)})_{n=1}^{\infty}$  is a cumulant multiplicative function induced by  $\varphi$  in  $I(A, B)$ .

Denote a set of all  $B$ -formal series with  $s$ -noncommutative indeterminants ( $s \in \mathbb{N}$ ), by  $\Theta_B^s$ . i.e if  $g \in \Theta_B^s$ , then

$$g(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} b_{i_1, \dots, i_n} z_{i_1} \dots z_{i_n},$$

where  $b_{i_1, \dots, i_n} \in B$ , for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $\forall n \in \mathbb{N}$ . Trivially, by definition,  $M_{x_1, \dots, x_s}, R_{x_1, \dots, x_s} \in \Theta_B^s$ . By  $\mathcal{R}_B^s$ , we denote a set of all  $R$ -transforms of  $s$ - $B$ -valued random variables. Recall that, set-theoretically,

$$\Theta_B^s = \mathcal{R}_B^s, \text{ for all } s \in \mathbb{N}.$$

We can also define symmetric moment series and symmetric  $R$ -transform by  $b_0 \in B$ , by

$$M_{x_1, \dots, x_s}^{\text{symm}(b_0)}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} \varphi(x_{i_1} b_0 x_{i_2} \dots b_0 x_{i_n}) z_{i_1} \dots z_{i_n}$$

and

$$R_{x_1, \dots, x_s}^{\text{symm}(b_0)}(z_1, \dots, z_s) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n \in \{1, \dots, s\}} k_n^{\text{symm}(b_0)}(x_{i_1}, \dots, x_{i_n}) z_{i_1} \dots z_{i_n},$$

with

$$k_n^{\text{symm}(b_0)}(x_{i_1}, \dots, x_{i_n}) = c^{(n)}(x_{i_1} \otimes b_0 x_{i_2} \otimes \dots \otimes b_0 x_{i_n}),$$

for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $\forall n \in \mathbb{N}$ .

If  $b_0 = 1_B$ , then we have trivial moment series and trivial  $R$ -transform of  $x_1, \dots, x_s$  denoted by  $M_{x_1, \dots, x_s}^t$  and  $R_{x_1, \dots, x_s}^t$ , respectively. By definition, for the fixed random variables  $x_1, \dots, x_s \in (A, \varphi)$ , there are infinitely many  $R$ -transforms of them (resp. moment series of them). Symmetric and trivial  $R$ -transforms of them are special examples. Let

$$C = \bigcup_{(i_1, \dots, i_n) \in \mathbb{N}^n} \{(1_B, b_{i_2}, \dots, b_{i_n}) : b_{i_j} \in B\}.$$

Suppose that we have

$$\text{coef}_{i_1, \dots, i_n}(R_{x_1, \dots, x_s}) = c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n}),$$

where  $(1_B, b_{i_2}, \dots, b_{i_n}) \in C$ , for all  $(i_1, \dots, i_n) \in \mathbb{N}^n$ . Then we can rewrite the  $R$ -transform of  $x_1, \dots, x_s$ ,  $R_{x_1, \dots, x_s}$  by  $R_{x_1, \dots, x_s}^C$ . If  $C_1$  and  $C_2$  are such collections,

then in general  $R_{x_1, \dots, x_s}^{C_1} \neq R_{x_1, \dots, x_s}^{C_2}$  (resp.  $M_{x_1, \dots, x_s}^{C_1} \neq M_{x_1, \dots, x_s}^{C_2}$ ). From now, for the random variables  $x_1, \dots, x_s, y_1, \dots, y_s$ , if we write  $R_{x_1, \dots, x_s}$  and  $R_{y_1, \dots, y_s}$ , then it means that  $R_{x_1, \dots, x_s}^C = R_{y_1, \dots, y_s}^C$ , for the same collection  $C$ . If there's no confusion, we will omit to write such collection.

The followings are known in [23] and [16] ;

**Proposition 1.2.** *Let  $(A, \varphi)$  be a NCPSpace over  $B$  and let  $x_1, \dots, x_s, y_1, \dots, y_p \in (A, \varphi)$  be  $B$ -valued random variables, where  $s, p \in \mathbb{N}$ . Suppose that  $\{x_1, \dots, x_s\}$  and  $\{y_1, \dots, y_p\}$  are free in  $(A, \varphi)$ . Then*

$$(1) R_{x_1, \dots, x_s, y_1, \dots, y_p}(z_1, \dots, z_{s+p}) = R_{x_1, \dots, x_s}(z_1, \dots, z_s) + R_{y_1, \dots, y_p}(z_1, \dots, z_p).$$

$$(2) \text{ If } s = p, \text{ then } R_{x_1+y_1, \dots, x_s+y_s}(z_1, \dots, z_s) = (R_{x_1, \dots, x_s} + R_{y_1, \dots, y_s})(z_1, \dots, z_s).$$

□

The above proposition is proved by the characterization of freeness with respect to cumulants. i.e  $\{x_1, \dots, x_s\}$  and  $\{y_1, \dots, y_p\}$  are free in  $(A, \varphi)$  if and only if their mixed cumulants vanish. Thus we have

$$\begin{aligned} k_n(p_{i_1}, \dots, p_{i_n}) &= c^{(n)}(p_{i_1} \otimes b_{i_2} p_{i_2} \otimes \dots \otimes b_{i_n} p_{i_n}) \\ &= (\widehat{c}_x \oplus \widehat{c}_y)^{(n)}(p_{i_1} \otimes b_{i_2} p_{i_2} \otimes \dots \otimes b_{i_n} p_{i_n}) \\ &= \begin{cases} k_n(x_{i_1}, \dots, x_{i_n}) & \text{or} \\ k_n(y_{i_1}, \dots, y_{i_n}) \end{cases} \end{aligned}$$

and if  $s = p$ , then

$$\begin{aligned} k_n(x_{i_1} + y_{i_1}, \dots, x_{i_n} + y_{i_n}) &= c^{(n)}((x_{i_1} + y_{i_1}) \otimes b_{i_2}(x_{i_2} + y_{i_2}) \otimes \dots \otimes b_{i_n}(x_{i_n} + y_{i_n})) \\ &= c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n}) + c^{(n)}(y_{i_1} \otimes b_{i_2} y_{i_2} \otimes \dots \otimes b_{i_n} y_{i_n}) + [Mixed] \end{aligned}$$

where  $[Mixed]$  is the sum of mixed cumulants of  $x_j$ 's and  $y_i$ 's, by the bimodule map property of  $c^{(n)}$

$$= k_n(x_{i_1}, \dots, x_{i_n}) + k_n(y_{i_1}, \dots, y_{i_n}) + 0_B.$$

Note that if  $f, g \in \Theta_B^s$ , then we can always choose free  $\{x_1, \dots, x_s\}$  and  $\{y_1, \dots, y_s\}$  in (some) NCPSpace over  $B$ ,  $(A, \varphi)$ , such that

$$f = R_{x_1, \dots, x_s} \quad \text{and} \quad g = R_{y_1, \dots, y_s}.$$

**Definition 1.4.** (1) Let  $s \in \mathbb{N}$ . Let  $(f, g) \in \Theta_B^s \times \Theta_B^s$ . Define  $\boxtimes: \Theta_B^s \times \Theta_B^s \rightarrow \Theta_B^s$  by

$$(f, g) = (R_{x_1, \dots, x_s}^{C_1}, R_{y_1, \dots, y_s}^{C_2}) \longmapsto R_{x_1, \dots, x_s}^{C_1} \boxtimes R_{y_1, \dots, y_s}^{C_2}.$$

Here,  $\{x_1, \dots, x_s\}$  and  $\{y_1, \dots, y_s\}$  are free in  $(A, \varphi)$ . Suppose that

$$\text{coef}_{i_1, \dots, i_n} (R_{x_1, \dots, x_s}^{C_1}) = c^{(n)}(x_{i_1} \otimes b_{i_2} x_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n})$$

and

$$\text{coef}_{i_1, \dots, i_n} (R_{y_1, \dots, y_s}^{C_2}) = c^{(n)}(y_{i_1} \otimes b'_{i_2} y_{i_2} \otimes \dots \otimes b'_{i_n} y_{i_n}),$$

for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $n \in \mathbb{N}$ , where  $b_{i_j}, b'_{i_n} \in B$  arbitrary. Then

$$\begin{aligned} & \text{coef}_{i_1, \dots, i_n} (R_{x_1, \dots, x_s}^{C_1} \boxtimes R_{y_1, \dots, y_s}^{C_2}) \\ &= \sum_{\pi \in NC(n)} (\widehat{c}_x \oplus \widehat{c}_y) (\pi \cup Kr(\pi))(x_{i_1} \otimes y_{i_1} \otimes b_{i_2} x_{i_2} \otimes b'_{i_2} y_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n} \otimes b'_{i_n} y_{i_n}) \\ &\stackrel{\text{denote}}{=} \sum_{\pi \in NC(n)} \left( k_{\pi}^{C_1} \oplus k_{Kr(\pi)}^{C_2} \right) (x_{i_1}, y_{i_1}, \dots, x_{i_n}, y_{i_n}), \end{aligned}$$

where  $\widehat{c}_x \oplus \widehat{c}_y = \widehat{c} |_{A_x * B A_y}$ ,  $A_x = \text{Alg}(\{x_i\}_{i=1}^s, B)$  and  $A_y = \text{Alg}(\{y_i\}_{i=1}^s, B)$  and where  $\pi \cup Kr(\pi)$  is an alternating union of partitions in  $NC(2n)$

**Proposition 1.3.** (See [16]) Let  $(A, \varphi)$  be a NCPSpace over  $B$  and let  $x_1, \dots, x_s, y_1, \dots, y_s \in (A, \varphi)$  be  $B$ -valued random variables ( $s \in \mathbb{N}$ ). If  $\{x_1, \dots, x_s\}$  and  $\{y_1, \dots, y_s\}$  are free in  $(A, \varphi)$ , then we have

$$\begin{aligned} & k_n(x_{i_1} y_{i_1}, \dots, x_{i_n} y_{i_n}) \\ &= \sum_{\pi \in NC(n)} (\widehat{c}_x \oplus \widehat{c}_y) (\pi \cup Kr(\pi))(x_{i_1} \otimes y_{i_1} \otimes b_{i_2} x_{i_2} \otimes y_{i_2} \otimes \dots \otimes b_{i_n} x_{i_n} \otimes y_{i_n}) \\ &\stackrel{\text{denote}}{=} \sum_{\pi \in NC(n)} \left( k_{\pi} \oplus k_{Kr(\pi)}^{\text{symm}(1_B)} \right) (x_{i_1}, y_{i_1}, \dots, x_{i_n}, y_{i_n}), \end{aligned}$$

for all  $(i_1, \dots, i_n) \in \{1, \dots, s\}^n$ ,  $\forall n \in \mathbb{N}$ ,  $b_{i_2}, \dots, b_{i_n} \in B$ , arbitrary, where  $\widehat{c}_x \oplus \widehat{c}_y = \widehat{c} |_{A_x * B A_y}$ ,  $A_x = \text{Alg}(\{x_i\}_{i=1}^s, B)$  and  $A_y = \text{Alg}(\{y_i\}_{i=1}^s, B)$ .  $\square$

This shows that ;

**Corollary 1.4.** (See [16]) Under the same condition with the previous proposition,

$$R_{x_1, \dots, x_s} \boxtimes R_{y_1, \dots, y_s}^t = R_{x_1 y_1, \dots, x_s y_s}.$$

$\square$

### 1.3. $B$ -valued Even Random Variables.



In this section, we will consider the  $B$ -evenness. Let  $(A, \varphi)$  be a NCPSpace over  $B$ .

**Definition 1.5.** Let  $a \in (A, \varphi)$  be a  $B$ -valued random variable. We say that this random variable  $a$  is  $B$ -even if

$$\varphi(ab_2a\dots b_ma) = 0_B, \text{ whenever } m \text{ is odd,}$$

where  $b_2, \dots, b_m \in B$  are arbitrary. In particular, if  $a$  is  $B$ -even, then  $\varphi(a^m) = 0_B$ , whenever  $m$  is odd. But the converse is not true, in general.

Recall that in the  $*$ -probability space model, the  $B$ -evenness guarantees the self-adjointness (See [16]). But the above definition is more general. By using the Möbius inversion, we have the following characterization ;

**Proposition 1.5.** Let  $a \in (A, \varphi)$  be a  $B$ -valued random variable. Then  $a$  is  $B$ -even if and only if

$$k_m \left( \underbrace{a, \dots, a}_{m\text{-times}} \right) = 0_B, \text{ whenever } m \text{ is odd.}$$

The above proposition says that  $B$ -evenness is easy to verify when we are dealing with either  $B$ -moments or  $B$ -cumulants. Now, define a subset  $NC^{(even)}(2k)$  of  $NC(2k)$ , for any  $k \in \mathbb{N}$  ;

$$NC^{(even)}(2k) = \{\pi \in NC(2k) : \pi \text{ does not contain odd blocks}\}.$$

We have that ;

**Proposition 1.6.** Let  $k \in \mathbb{N}$  and let  $a \in (A, \varphi)$  be  $B$ -even. Then

$$k_{2k} \left( \underbrace{a, \dots, a}_{2k\text{-times}} \right) = \sum_{\pi \in NC^{(even)}(2k)} \widehat{\varphi}(\pi) (a \otimes b_2a \otimes \dots \otimes b_{2k}a) \mu(\pi, 1_{2k})$$

equivalently,

$$\varphi(ab_2a\dots b_{2k}a) = \sum_{\pi \in NC^{(even)}(2k)} \widehat{c}(\pi) (a \otimes b_2a \otimes \dots \otimes b_{2k}a).$$

*Proof.* By the previous proposition, it is enough to show one of the above two formulæ. Fix  $k \in \mathbb{N}$ . Then

$$\begin{aligned} k_{2k}(a, \dots, a) &= c^{(2k)}(a \otimes b_2 a \otimes \dots \otimes b_{2k} a) \\ &= \sum_{\pi \in NC(2k)} \widehat{\varphi}(\pi)(a \otimes b_2 a \otimes \dots \otimes b_{2k} a) \mu(\pi.1_{2k}). \end{aligned}$$

Now, suppose that  $\theta \in NC(2k)$  and  $\theta$  contains its odd block  $V_o \in \pi(o) \cup \pi(i)$ . Then

$$(2.2.1) \quad \widehat{\varphi}(\theta)(a \otimes b_2 a \otimes \dots \otimes b_{2k} a) = 0_B.$$

Define

$$NC^{(odd)}(2k) = \{\pi \in NC(2k) : \pi \text{ contains at least one odd block}\}.$$

Then, for any  $\theta \in NC^{(odd)}(2k)$ , the formulæ (2.2.1) holds. So,

$$k_{2k}(a, \dots, a) = \sum_{\pi \in NC(2k) \setminus NC^{(odd)}(2k)} \widehat{\varphi}(\pi)(a \otimes b_2 a \otimes \dots \otimes b_{2k} a) \mu(\pi, 1_{2k}).$$

It is easy to see that, by definition,

$$NC^{(even)}(2k) = NC(2k) \setminus NC^{(odd)}(2k).$$

■

**Proposition 1.7.** *Let  $a_1$  and  $a_2$  be  $B$ -even elements in  $(A, \varphi)$ . If  $a_1$  and  $a_2$  are free over  $B$ , then  $a_1 + a_2 \in (A, \varphi)$  is  $B$ -even, again.*

## 2. FREE PROBABILITY THEORY ON $L(F_2) *_{L(F_1)} L(F_2)$ OVER $L(F_1)$

In this chapter, we will consider the free group  $W^*$ -algebras  $B = L(F_1)$  and  $A = L(F_2)$ , where  $F_N$  is a free group with  $N$ -generators ( $N \in \mathbb{N}$ ). i.e

$$B = \left\{ \sum_{h \in F_1} t_h h : t_h \in \mathbb{C} \right\}$$

and

$$A = \left\{ \sum_{g \in F_2} t_g g : t_g \in \mathbb{C} \right\}.$$

Recall that there is a map  $E : A \rightarrow B$  defined by

$$E \left( \sum_{g \in F_2} t_g g \right) = \sum_{h \in F_1} t_h h.$$

Notice that  $E : A \rightarrow B$  is a conditional expectation ( $B$ -functional) and hence  $(A, E)$  is a NCPSpace over  $B$ .

Now, for any  $N \in \mathbb{N}$ , define the canonical trace  $\varphi$  on  $L(F_N)$  ;

$$\varphi \left( \sum_{g \in F_N} t_g g \right) = t_{e_{F_N}},$$

for all  $\sum_{g \in F_N} t_g g \in L(F_N)$ , where  $e_{F_N}$  is the identity of  $F_N$ . For the convenience of using notation, we will denote  $e_{F_N}$  by  $e$ .

In this paper, we will concentrate on finding scalar-valued moments of  $a + b + a^{-1} + b^{-1} + c + d + c^{-1} + d^{-1} \in L(F_2) *_{L(F_1)} L(F_2)$ ,

$$\tau \left( (a + b + a^{-1} + b^{-1} + c + d + c^{-1} + d^{-1})^n \right),$$

where  $L(\langle a, b \rangle) = L(F_2) = L(\langle c, d \rangle)$  and  $\tau : L(F_2) *_{L(F_1)} L(F_2) \rightarrow \mathbb{C}$  is the trace, for all  $n \in \mathbb{N}$ , defined by

$$\tau(y_1 \dots y_n) = t_e, \text{ for all } y_j \in L(F_2), j = 1, \dots, n.$$

Remark that, since each  $y_j$  has the form,  $y_j = \sum_{g \in F_2} t_g^{(j)} g$ , in  $L(F_2)$ , we can find the coefficient of  $e = 1_B = 1_A$  in  $y_1 \dots y_n \in L(F_2) *_{L(F_1)} L(F_2)$ . So, to find moments of an element in  $L(F_2) *_{L(F_1)} L(F_2)$  is to find  $e$ -terms of the element in  $L(F_2) *_{L(F_1)} L(F_2)$ . To directly compute this moments is very complicated. So, later, we will use the compatibility and  $B$ -freeness. Also, later, we will denote this linear functional  $\tau$  by  $\varphi$ , because

$$\tau(x) = \varphi(E * E(x)), \text{ for all } x \in L(F_2) *_{L(F_1)} L(F_2).$$

### 3. COMPATIBILITY OF $(L(F_2) *_{L(F_1)} L(F_2), \varphi)$ AND $(L(F_2) *_{L(F_1)} L(F_2), F)$

From now on, by  $A$  and  $B$ , we will denote  $L(F_2)$  and  $L(F_1)$ ,  $A$  and  $B$ , respectively. By the very definitions of  $E : A \rightarrow B$  and  $\tau : A \rightarrow \mathbb{C}$ ,

$$E \left( \sum_{g \in F_2} t_g g \right) = \sum_{h \in F_1} t_h h \quad \text{and} \quad \tau \left( \sum_{g \in F_2} t_g g \right) = t_e,$$

a NCPSpace  $(A, \varphi)$  and an amalgamated NCPSpace over  $B$ ,  $(A, E)$  are compatible. In this section, we will show that

$$\varphi(x) = \varphi \circ E(x), \text{ for all } x \in A.$$

We can regard  $e$  as the identity element in  $B$  and  $A$ . i.e

$$1_A = e = 1_B.$$

**Lemma 3.1.** *Let  $B \subset A$ ,  $\varphi$  and  $E$  be given as before. Then a NCPSpace  $(A, \varphi)$  and a NCPSpace over  $B$ ,  $(A, E)$  are compatible.*

Note that the trace  $\varphi$  on  $A_1 *_B A_2$  and  $\varphi \circ (E * E)$  coincide. So we have that ;

**Theorem 3.2.**  *$(A_1 *_B A_2, \varphi)$  and  $(A_1 *_B A_2, E * E)$  are compatible.*

**3.1.  $B$ -Evenness and  $B$ -identically distributedness of  $x$  and  $y$  in  $(A_1 *_B A_2, E * E)$ .**

By  $F$ , we will denote the  $B$ -functional  $E * E : A_1 *_B A_2 \rightarrow B$ . And in the rest of this paper, we will let

$$x = a + b + a^{-1} + b^{-1} \quad \text{and} \quad y = c + d + c^{-1} + d^{-1},$$

in  $A_1 *_B A_2$ .

**Lemma 3.3.** *As a  $B$ -valued random variable,  $x = a + b + a^{-1} + b^{-1} \in (A_1 *_B A_2, F)$  is  $B$ -even.  $\square$*

The above lemma is proved by a straightforward observation. Next section, we will observe the  $B$ -evenness of  $x$ , in detail.

**Proposition 3.4.** *Let  $x$  and  $y$  be given as before, as  $B$ -valued random variables in  $(A_1 *_B A_2, F)$ . Then  $\{x\}$  and  $\{y\}$  are free over  $B$ , in  $(A_1 *_B A_2, F)$  and they are identically distributed. i.e*

$$R_x = R_y \quad \text{or} \quad M_x = M_y.$$

*Proof.* Since  $x \in A_1$  and  $y \in A_2$  in  $A_1 *_B A_2$ , they are free over  $B$ , in  $(A_1 *_B A_2, F)$ . By the generating property of  $\{a, b\}$  and  $\{c, d\}$  (i.e they generate same group  $F_2$ ), they are identically distributed. Equivalently,

$$R_x(z) = R_y(z)$$

$\Leftrightarrow$

$$\begin{aligned} K_n \left( \underbrace{x \otimes \dots \otimes x}_{n\text{-times}} \right) &= C^{(n)}(x \otimes b_2 x \otimes \dots \otimes b_n x) \\ &= C^{(n)}(y \otimes b_2 y \otimes \dots \otimes b_n y) \\ &= K_n \left( \underbrace{y, \dots, y}_{n\text{-times}} \right), \end{aligned}$$

for all  $n \in \mathbb{N}$ . By the Möbius inversion,

$$M_x(z) = M_y(z),$$

as  $B$ -formal series. ■

**Corollary 3.5.** *Let  $x$  and  $y$  be given as before, in  $(A_1 *_B A_2, F)$ . Then*

$$R_{x+y}(z) = (R_x + R_y)(z) = 2R_x(z).$$

□

**Corollary 3.6.** *Let  $x$  and  $y$  be given as before. Then  $x + y$  is  $B$ -even, too.*

*Proof.* By the previous lemma,  $x$  is  $B$ -even. Since  $y$  is identically distributed with  $x$ , their  $R$ -transforms are same and hence  $y$  is  $B$ -even, too. In [16], we showed that if two  $B$ -even  $B$ -valued random variables are  $B$ -free, then the sum of them is also  $B$ -even. Since our  $B$ -valued random variables are  $B$ -even,  $x + y$  is also  $B$ -even. ■

### 3.2. Computation of $B$ -valued moments of $x$ , $E(x^n)$ .

To compute  $E(x^n)$ , we will use some results in [15]. We have that  $x \in (A_1 *_B A_2, F)$  is  $B$ -even. Thus

$$F(x^n) = 0_B, \text{ whenever } n \text{ is odd.}$$

So, we concentrate on finding  $B$ -valued  $2n$ -th moments of  $x = a + b + a^{-1} + b^{-1}$ , in  $(A_1 *_B A_2, F)$ . It is known that if we denote

$$X_n = \sum_{|w|=n} w \in \mathbb{C}[F_N] \quad (N \in \mathbb{N}),$$

then

$$X_1 X_1 = X_2 + 2N \cdot e$$

and

$$X_1 X_n = X_{n+1} + (2N - 1)X_{n-1},$$

where  $e = e_{F_N}$ , for all  $n \in \mathbb{N} \setminus \{1\}$ .

In our case, we can regard our  $x = a + b + a^{-1} + b^{-1}$  as  $X_1$  in  $\mathbb{C}[F_2] = A_1$ . Thus we have that

$$X_1 X_1 = X_2 + 4e$$

and

$$X_1 X_n = X_{n+1} + 3X_{n-1} \quad (n = 2, 3, \dots).$$

By using those two results, we can express  $x^n$  in terms of  $X_k$ 's ; For example,

$$x^2 = x \cdot x = X_1 X_1 = X_2 + 4e,$$

$$x^3 = x \cdot x^2 = X_1 (X_2 + 4e) = X_1 X_2 + 4X_1 = X_3 + 3X_1 + 4X_1 = X_3 + (3 + 4)X_1,$$

continuing

$$\begin{aligned} x^4 &= X_4 + (3 + 3 + 4)X_2 + 4(3 + 4)e, \\ x^5 &= X_5 + (3 + 3 + 3 + 4)X_3 + (3(3 + 3 + 4) + 4(3 + 4))X_1, \\ x^6 &= X_6 + (3 + 3 + 3 + 3 + 4)X_4 \\ &\quad + (3(3 + 3 + 3 + 4) + 3(3 + 3 + 4) + 4(3 + 4))X_2 \\ &\quad + 4(3(3 + 3 + 4) + 4(3 + 4))e, \end{aligned}$$

etc.

So, we can find a recurrence relation to get  $x^n$  ( $n \in \mathbb{N}$ ) with respect to  $X_k$ 's ( $k \leq n$ ). Inductively, we have that  $x^{2k-1}$  and  $x^{2k}$  have their representations in terms of  $X_j$ 's as follows ;

$$x^{2k-1} = X_1^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1} X_{2k-3} + q_{2k-5}^{2k-1} X_{2k-5} + \dots + q_3^{2k-1} X_3 + q_1^{2k-1} X_1$$

and

$$x^{2k} = X_1^{2k} = X_{2k} + p_{2k-2}^{2k} X_{2k-2} + p_{2k-4}^{2k} X_{2k-4} + \dots + p_2^{2k} X_2 + p_0^{2k} e,$$

where  $k \geq 2$ . Also, we have the following recurrence relation ;

**Proposition 3.7.** *Let's fix  $k \in \mathbb{N} \setminus \{1\}$ . Let  $q_i^{2k-1}$  and  $p_j^{2k}$  ( $i = 1, 3, 5, \dots, 2k-1, \dots$  and  $j = 0, 2, 4, \dots, 2k, \dots$ ) be given as before. If  $p_0^2 = 4$  and  $q_1^3 = 3 + p_0^2$ , then we have the following recurrence relations ;*

(1) Let

$$x^{2k-1} = X_{2k-1} + q_{2k-3}^{2k-1} X_{2k-3} + \dots + q_3^{2k-1} X_3 + q_1^{2k-1} X_1.$$

Then

$$\begin{aligned} x^{2k} = X_{2k} &+ (3 + q_{2k-3}^{2k-1}) X_{2k-2} + (3q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}) X_{2k-4} \\ &+ (3q_{2k-5}^{2k-1} + q_{2k-7}^{2k-1}) X_{2k-6} + \\ &+ \dots + (3q_3^{2k-1} + q_1^{2k-1}) X_2 + 4q_1^{2k-1} e. \end{aligned}$$

i.e.,

$$p_{2k-2}^{2k} = 3 + q_{2k-3}^{2k-1}, \quad p_{2k-4}^{2k} = 3q_{2k-3}^{2k-1} + q_{2k-5}^{2k-1}, \dots, \quad p_2^{2k} = 3q_3^{2k-1} + q_1^{2k-1} \quad \text{and} \\ p_0^{2k} = 4q_1^{2k-1}.$$

(2) Let

$$x^{2k} = X_{2k} + p_{2k-2}^{2k} X_{2k-2} + \dots + p_2^{2k} X_2 + p_0^{2k} e.$$

Then

$$\begin{aligned} x^{2k+1} = X_{2k+1} &+ (3 + p_{2k-2}^{2k}) X_{2k-1} + (3p_{2k-2}^{2k} + p_{2k-4}^{2k}) X_{2k-3} \\ &+ (3p_{2k-4}^{2k} + p_{2k-6}^{2k}) X_{2k-5} + \\ &+ \dots + (3p_4^{2k} + p_2^{2k}) X_3 + (3p_2^{2k} + p_0^{2k}) X_1. \end{aligned}$$

i.e.,

$$q_{2k-1}^{2k+1} = 3 + p_{2k-2}^{2k}, \quad q_{2k-3}^{2k+1} = 3p_{2k-2}^{2k} + p_{2k-4}^{2k}, \dots, \quad q_3^{2k+1} = 3p_4^{2k} + p_2^{2k} \quad \text{and} \quad q_1^{2k+1} = 3p_2^{2k} + p_0^{2k}. \quad \square$$

The above recurrence relations give us an algorithm, how to find the  $2n$ -th  $B$ -valued moments of  $x^{2n} \in (A_1 *_{B} A_2, F)$ .

**Example 3.1.** *Let  $p_0^2 = 4$  and  $q_1^3 = 3 + p_0^2 = 3 + 4 = 7$ . Put*

$$x^8 = X_8 + p_6^8 X_6 + p_4^8 X_4 + p_2^8 X_2 + p_0^8 e.$$

Then, by the previous proposition, we have that

$$p_6^8 = 3 + q_5^7, \quad p_4^8 = 3q_5^7 + q_3^7, \quad p_2^8 = 3q_3^7 + q_1^7 \quad \text{and} \quad p_0^8 = 4q_1^7.$$

Similarly, by the previous proposition,

$$\begin{aligned} q_5^7 &= 3 + p_4^6, & q_3^7 &= 3p_4^6 + p_2^6 & \text{and} & & q_1^7 &= 3p_2^6 + p_0^6, \\ p_4^6 &= 3 + q_3^5, & p_2^6 &= 3q_3^5 + q_1^5 & \text{and} & & p_0^6 &= 4q_1^5, \end{aligned}$$

$$q_3^5 = 3 + p_2^4 \quad \text{and} \quad q_1^5 = 3p_2^4 + p_0^4,$$

$$p_2^4 = 3 + q_1^3 \quad \text{and} \quad p_0^4 = 4q_1^3,$$

and

$$q_1^3 = 3 + p_0^2 = 7.$$

Therefore, combining all information,

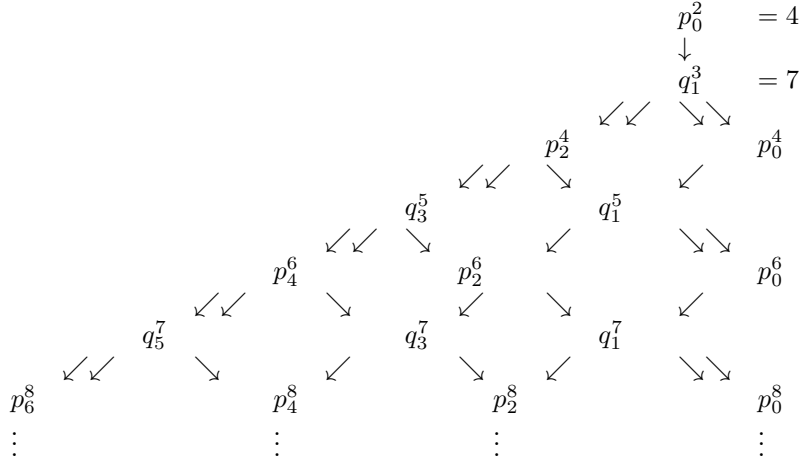
$$x^8 = X_8 + 22 X_6 + 202 X_4 + 744 X_2 + 1316 e.$$

We have the following diagram with arrows which mean that

$$\begin{aligned} \swarrow \swarrow &: 3 + [\text{former term}] \\ \searrow &: 3 \cdot [\text{former term}] \\ \swarrow &: \cdot + [\text{former term}] \end{aligned}$$

and

$$\searrow \searrow : 4 \cdot [\text{former term}].$$



Now, recall that  $h = aba^{-1}b^{-1}$  (or  $h = cdc^{-1}d^{-1}$ ) in  $(A_1 *_B A_2, F)$ , where  $F_1 = \langle h \rangle$ . So, since  $F_1 = \langle h \rangle$  is a cyclic group, WLOG, we denote  $\sum_{g \in F_1} t_g g \in$



$B \hookrightarrow A_1 *_B A_2$  by  $\sum_{n=-\infty}^{\infty} t_n h^n \in B$ , where  $t_g, t_n \in \mathbb{C}$ , with  $t_0 = t_e$ . Therefore, we can let

$$\varphi \left( \sum_{n=-\infty}^{\infty} t_n h^n \right) = t_0 \equiv t_e$$

And hence, to find a  $B$ -value moment of  $x$  is to find  $h^n$ -terms, where  $n \in \mathbb{Z}$ . Note that  $h$  and  $h^{-1}$  are words with their length 4. Therefore,  $X_{4k}$  contains  $h^k$ -terms and  $h^{-k}$ -terms, for  $k \in \mathbb{N}$  !

**Theorem 3.8.** Fix  $k \in \mathbb{N}$ . Let  $h = aba^{-1}b^{-1} \in A_1 *_B A_2$  with  $h^0 = e$ .

$$(1) E(x^{4k}) = (h^k + h^{-k}) + \sum_{j=1}^{k-1} p_{4k-4j}^{4k} (h^{k-j} + h^{-(k-j)}) + p_0^{4k} h^0,$$

where  $p_0^4 = 28$ .

(2) If  $4 \nmid 2k$  and if there are  $X_{4l_1}, \dots, X_{4l_p}$  terms in  $x^{2k}$ , then

$$E(x^{2k}) = \sum_{j=1}^{k-1} p_{(2k-2)-4j}^{2k} \left( h^{\frac{k-1}{2}-2j} + h^{-(\frac{k-1}{2}-2j)} \right) + p_0^{2k} h^0,$$

where  $p_0^2 = 4$ .

*Proof.* (1) By the straightforward computation using the previous proposition, we have that

$$\begin{aligned} E(x^{4k}) &= E(X_{4k} + p_{4k-2}^{4k} X_{4k-2} + p_{4k-4}^{4k} X_{4k-4} + \dots + p_4^{4k} X_4 + p_2^{4k} X_2 + p_0^{4k} h^0) \\ &= E(X_{4k}) + p_{4k-2}^{4k} E(X_{4k-2}) + p_{4k-4}^{4k} E(X_{4k-4}) + \\ &\quad \dots + p_4^{4k} E(X_4) + p_2^{4k} E(X_2) + p_0^{4k} h^0. \end{aligned}$$

Since  $h^p$  and  $h^{-p}$  terms are in  $X_{4p}$ , for any  $p \in \mathbb{N}$ , we can continue the above computation ;

$$\begin{aligned} &= E(X_{4k}) + p_{4k-4}^{4k} E(X_{4k-4}) + \dots + p_4^{4k} E(X_4) + p_0^{4k} h^0 \\ &= (h^k + h^{-k}) + p_{4k-4}^{4k} (h^{k-1} + h^{-(k-1)}) + \dots + p_4^{4k} (h + h^{-1}) + p_0^{4k} h^0. \end{aligned}$$

(2) If  $4 \nmid 2k$ , then  $k = 1, 3, 5, \dots$ . If  $k = 1$ , then the above formula holds true ;

$$E(x^2) = E(X_2 + 4h^0) = 4h^0.$$

If  $k \neq 1$  is odd, then

$$\begin{aligned}
& E(x^{2k}) \\
&= E(X_{2k} + p_{2k-2}^{2k} X_{2k-2} + p_{2k-4}^{2k} X_{2k-4} + p_{2k-6}^{2k} X_{2k-6} + \\
&\quad \dots + p_4^{2k} X_4 + p_2^{2k} X_2 + p_0^{2k} h^0) \\
&= E(X_{2k}) + p_{2k-2}^{2k} E(X_{2k-2}) + p_{2k-4}^{2k} E(X_{2k-4}) + p_{2k-6}^{2k} E(X_{2k-6}) + \\
&\quad \dots + p_4^{2k} E(X_4) + p_2^{2k} E(X_2) + p_0^{2k} h^0 \\
&= 0_B + p_{2k-2}^{2k} (h^{k-1} + h^{-(k-1)}) + 0_B + p_{2k-6}^{2k} (h^{k-3} + h^{-(k-3)}) + \\
&\quad \dots + p_4^{2k} (h + h^{-1}) + 0_B + p_0^{2k} h^0,
\end{aligned}$$

since  $X_{2k-2}, X_{2k-6}, \dots, X_4$  contain  $h^p$ -terms and  $h^{-p}$ -terms. ■

#### 4. THE AMALGAMATED R-TRANSFORM OF $x + y$

Throughout this section, we will use the same notations used in the previous sections. To compute  $F((x + y)^n)$ , we will consider the  $R_{x+y}^t$ . Since  $x$  and  $y$  are free over  $B$ , we have that

$$R_{x+y}^t = R_x^t + R_y^t.$$

And since  $x$  and  $y$  are identically distributed,

$$R_{x+y}^t = R_x^t + R_y^t = 2R_x^t \quad \text{or} \quad 2R_y^t.$$

The above paragraph shows that why we need to observe  $R_{x+y}^t$ , to get a  $n$ -th coefficients of  $M_{x+y}$ . By the  $B$ -freeness of  $x$  and  $y$ , we can compute  $n$ -th coefficients of  $R_{x+y}^t = 2R_x^t$ , relatively easier than to compute  $n$ -th coefficients of  $M_{x+y}^t$ , directly. Moreover, since we have the recurrence relation for  $F(x^n) = E(x^n)$ ,  $n \in \mathbb{N}$ , we have a tool for computing  $n$ -th coefficients of  $2R_x^t$ . But there is a difficulty to compute them, because of the insertion property of noncrossing partitions (different from the scalar-valued case). Hence we need to find other recurrence relation related to this insertion property. By the  $B$ -evenness of  $x \in (A_1 *_B A_2, F)$ , we have that

$$\text{coef}_n(R_x^t) = k_n^t \left( \underbrace{x, \dots, x}_{n\text{-times}} \right) = 0_B, \text{ whenever } n \text{ is odd.}$$

So, we will only consider the even coefficient of  $R_x^t$ .

**Lemma 4.1.** For  $k \in \mathbb{N}$ ,

$$\begin{aligned} \text{coef}_{2k}(R_x^t) &= \sum_{l_1, \dots, l_p \in \{2, 4, \dots, 2k\}, l_1 + l_2 + \dots + l_p = 2k} \\ &\quad \sum_{\theta \in NC_{l_1, \dots, l_p}(2k)} \widehat{F}(\theta)(x \otimes \dots \otimes x) \mu(\theta, 1_{2k}). \end{aligned}$$

The above lemma shows that we need to construct a recurrence relation for

$$F(x^{m_1 k_1} x^{m_2 k_2} h^{k_2} x^{m_3} \dots h^{k_n} x^{m_n}) = E(x^{m_1} h^{k_1} x^{m_2} h^{k_2} x^{m_3} \dots h^{k_n} x^{m_n}),$$

where  $m_j \in \mathbb{N}$  and  $k_j \in \mathbb{Z}$ ,  $j = 1, \dots, n$ , for all  $n \in \mathbb{N}$ . This recurrence relation can explain the computation of partition-dependent  $B$ -valued moments with respect to  $E$ . By the observation of Section 3.2.2, it suffices to find the recurrence relation for

$$F(X_{q_1} h^{k_1} X_{q_2} h^{k_2} X_{q_3} \dots h^{k_n} X_{q_n}) = E(X_{q_1} h^{k_1} X_{q_2} h^{k_2} X_{q_3} \dots h^{k_n} X_{q_n}),$$

where  $q_1, \dots, q_n \in \mathbb{N}$  and  $X_N = \sum_{|w|=N} w$ , where  $w$  is a word in  $\{a, b, a^{-1}, b^{-1}\}$ , for  $N \in \mathbb{N}$ .

#### 4.1. Recurrence Relation For $E(X_m h^k X_n)$ .

In this section, we will consider the recurrence relation for  $E(X_m h X_n)$ . Then we can generalize this case to  $E(X_m h^k X_n)$ , where  $k \in \mathbb{Z}$  and  $m, n \in \mathbb{N}$ . We have that

$$E(X_1 h X_3) = e = E(X_3 h X_1) \quad \text{and} \quad E(X_2 h X_2) = e$$

and

**Proposition 4.2.** Let  $m, n \in \mathbb{N}$  and  $k \in \mathbb{Z}$ . Then

$$(1) \quad E(h^k X_n) = \begin{cases} h^k h^{\frac{n}{4}} = h^{k + \frac{n}{4}} & \text{if } 4 \mid n \\ 0_B & \text{otherwise.} \end{cases}$$

(2)

$$E(X_m h^k) = \begin{cases} h^{\frac{m}{4}} h^k = h^{\frac{m}{4}+k} & \text{if } 4 \mid m \\ 0_B & \text{otherwise.} \end{cases}$$

*Proof.* Since  $E : A_j \rightarrow B$  is a conditional expectation ( $j = 1, 2$ ), we have that

(1)

$$E(h^k X_n) = h^k E(X_n) = \begin{cases} h^k h^{\frac{n}{4}} = h^{k+\frac{n}{4}} & \text{if } 4 \mid n \\ 0_B & \text{otherwise.} \end{cases}$$

(2)

$$E(X_m h^k) = E(X_m) h^k = \begin{cases} h^{\frac{m}{4}} h^k = h^{\frac{m}{4}+k} & \text{if } 4 \mid m \\ 0_B & \text{otherwise.} \end{cases}$$

■

Now fix the sufficiently big numbers  $m$  and  $n$  in  $\mathbb{N}$ . Then we can have that (3.3.1.1)

$$X_m h X_n = \left( \sum_{|w|=m} w \right) h \left( \sum_{|w'|=n} w' \right),$$

where  $w$  and  $w'$  are words in  $\{a, b, a^{-1}, b^{-1}\}$ . Recall that  $h = aba^{-1}b^{-1}$  is a word with length 4. Hence, by the possible cancellation, we can rewrite (3.3.1.1) as

$$X_m h X_n$$

$$= W_{m+n+4} + W_{m+n+2} + W_{m+n} + W_{m+n-2} + \left( \sum_{|w|=m-4, \text{End}(w) \neq b} w \right) (X_n) + X_m \left( \sum_{|w|=n-4, \text{Init}(w) \neq a} w \right) \quad (3.3.1.2)$$

$$+ \left( \sum_{|w|=m-3, \text{End}(w) \neq a^{-1}} w \right) \left( \sum_{|w|=n-1, \text{Init}(w) \neq b^{-1}} w \right) \quad (3.3.1.3)$$

$$+ \left( \sum_{|w|=m-1, \text{End}(w) \neq a} w \right) \left( \sum_{|w|=n-3, \text{Init}(w) \neq b} w \right) \quad (3.3.1.4)$$

$$+ \left( \sum_{|w|=m-2, \text{End}(w) \neq a} w \right) \left( \sum_{|w|=n-2, \text{Init}(w) \neq a^{-1}} w \right), \quad (3.3.1.5)$$

where  $W_{m+n+k}$  is the subsum of words with length  $m+n+k$ . In the above formula, (3.3.1.2) is gotten from the full cancellation of  $X_m$  and  $h$ , and the full cancellation of  $h$  and  $X_n$ . (3.3.1.3) (resp. (3.3.1.4)) is gotten from the 3-letter-cancellation of  $X_m$  and  $h$  (resp. 1-letter-cancellation of  $X_m$  and  $h$ ) and the 1-letter-cancellation of  $X_n$  and  $h$  (resp. 3-letter-cancellation of  $X_n$  and  $h$ ). Similarly, (3.3.1.5) is gotten from the 2-letter-cancellation from the left and right of  $h$ . Similar to the full  $h$ -cancellation, (3.3.1.2)  $\sim$  (3.3.1.5), we can rewrite that

(3.3.1.6)

$$W_{m+n+2} = \left( \sum_{|w|=m-1, \text{End}(w) \neq a} w \right) (ba^{-1}b^{-1})X_n + X_m(aba^{-1}) \left( \sum_{|w|=n-1, \text{Init}(w) \neq b^{-1}} w \right),$$

(3.3.1.7)

$$W_{m+n} = \left( \sum_{|w|=m-2, \text{End}(w) \neq b} w \right) (a^{-1}b^{-1})X_n + X_m(ab) \left( \sum_{|w|=n-2, \text{Init}(w) \neq a^{-1}} w \right) \\ + \left( \sum_{|w|=m-1, \text{End}(w) \neq a} w \right) ba^{-1} \left( \sum_{|w|=n-1, \text{Init}(w) \neq b^{-1}} w \right)$$

and

(3.3.1.8)

$$W_{m+n-2} = \left( \sum_{|w|=m-3, \text{End}(w) \neq a^{-1}} w \right) b^{-1}X_n + X_m a \left( \sum_{|w|=n-3, \text{Init}(w) \neq b} w \right).$$

Now, we will define a function  $F_{pq}$  from  $\mathbb{N} \times \mathbb{N}$  to  $B$ .

**Definition 4.1.** Define a function  $F : \mathbb{N} \times \mathbb{N} \rightarrow B$  by

$$F_{pq}(k, l) = E \left( \left( \sum_{|w|=k, \text{End}(w)=p} w \right) \left( \sum_{|w'|=l, \text{Init}(w')=q} w' \right) \right),$$

where  $p, q \in \{a, b, a^{-1}, b^{-1}\}$ , where  $\text{End}(w)$  and  $\text{Init}(w)$  mean the end letter of the word  $w$  and initial letter of the word  $w$ , respectively.

**Definition 4.2.** Let  $p, q \in \{a, a^{-1}, b, b^{-1}\}$  and let  $w = p_1 \dots p_k$  be a word with length  $k$  in  $\{a, a^{-1}, b, b^{-1}\}$ . We define the following relation denoted by " $\triangleleft$ ";

$$pq \triangleleft w = p_1 \dots p_k \stackrel{\text{def}}{\iff} \exists j \in \{1, \dots, k-1\} \text{ s.t } pq = p_j p_{j+1} \text{ and } pq \neq e$$

For example,

$$pq \triangleleft h = aba^{-1}b^{-1}$$

if and only if

$$pq = ab \text{ or } pq = ba^{-1} \text{ or } pq = a^{-1}b^{-1}.$$

Again recall that, for all  $n \in \mathbb{N}$ ,

$$E(X_n) = \begin{cases} h^{\frac{n}{4}} + h^{-\frac{n}{4}} & \text{if } 4 \mid n \\ 0_B & \text{otherwise.} \end{cases}$$

Then we have the following lemmas ;

**Lemma 4.3.** *Let  $p, q \in \{a, b, a^{-1}, b^{-1}\}$  and  $k, l \in \mathbb{N}$  (sufficiently big). Then*

$$F_{pq}(k, l) = 0_B, \text{ whenever } pq \not\triangleleft h = aba^{-1}b^{-1}.$$

*Proof.* Suppose that  $pq \not\triangleleft aba^{-1}b^{-1}$ . Then

$$F_{pq}(k, l) = E \left( \left( \sum_{\substack{|w|=k-1 \\ \text{End}(w) \neq p^{-1}}} w \right) pq \left( \sum_{\substack{|w'|=l-1 \\ \text{Init}(w') \neq q^{-1}}} w' \right) \right) = 0_B,$$

since every word  $W = l_{i_1} \dots l_{i_{k-1}} p q l_{j_1} \dots l_{j_{l-1}}$  cannot be  $h^{\frac{k+l}{4}}$ , where  $l_{i_1} \dots l_{i_{k-1}}$  is the word with length  $k-1$  such that  $l_{i_{k-1}} \neq p^{-1}$  and  $l_{j_1} \dots l_{j_{l-1}}$  is the word with length  $l-1$  such that  $l_{j_1} \neq q^{-1}$ , we can get the above equality. ■

**Lemma 4.4.** *We have the following equalities ;*

(1)

$$F_{ab}(k, l) = \begin{cases} h^{\frac{k+l}{4}} + h^{-\frac{k+l}{4}} & \text{if } 4 \mid (k-1) \text{ and } 4 \mid k+l \\ 0_B & \text{otherwise.} \end{cases}$$

(2)

$$F_{ba^{-1}}(k, l) = \begin{cases} h^{\frac{k+l}{4}} + h^{-\frac{k+l}{4}} & \text{if } 4 \mid k \text{ and } 4 \mid k+l \\ 0_B & \text{otherwise.} \end{cases}$$

(3)

$$F_{a^{-1}b^{-1}}(k, l) = \begin{cases} h^{\frac{k+l}{4}} + h^{-\frac{k+l}{4}} & \text{if } 4 \mid (l-1) \text{ and } 4 \mid k+l \\ 0_B & \text{otherwise.} \end{cases}$$

**Lemma 4.5.** *We have the following recurrence relation for  $F_{p,q}(k, l)$ , where  $p, q \in \{a, b, a^{-1}, b^{-1}\}$  and  $k, l \in \mathbb{N}$  (sufficiently large) ;*

$$(4) F_{aa^{-1}}(k, l) = F_{aa^{-1}}(k-1, l-1) + F_{ab}(k-1, l-1) \\ + F_{ba^{-1}}(k-1, l-1) + F_{bb^{-1}}(k-1, l-1) + F_{b^{-1}b}(k-1, l-1).$$

$$(5) F_{bb^{-1}}(k, l) = F_{bb^{-1}}(k-1, l-1) + F_{aa^{-1}}(k-1, l-1) \\ + F_{a^{-1}a}(k-1, l-1) + F_{ba^{-1}}(k-1, l-1) + F_{a^{-1}b^{-1}}(k-1, l-1).$$

$$(6) F_{a^{-1}a}(k, l) = F_{bb^{-1}}(k-1, l-1) + F_{a^{-1}a}(k-1, l-1) \\ + F_{b^{-1}b}(k-1, l-1) + F_{a^{-1}b^{-1}}(k-1, l-1).$$

$$(7) F_{b^{-1}b}(k, l) = F_{aa^{-1}}(k-1, l-1) + F_{a^{-1}a}(k-1, l-1) \\ + F_{b^{-1}b}(k-1, l-1) + F_{ab}(k-1, l-1).$$

By the previous lemmas, we can get three equalities (1), (2) and (3) and four recurrence relations (4) ~ (7). Again, by the previous lemmas, we can conclude that

**Theorem 4.6.** *Let  $k, l \in \mathbb{N}$  be suffieintly big and let  $p, q \in \{a, b, a^{-1}, b^{-1}\}$ . Then*

(i) *If  $pq \not\triangleleft h = aba^{-1}b^{-1}$ , then  $F_{pq}(k, l) = 0_B$ .*

(ii) *If  $pq \triangleleft h = aba^{-1}b^{-1}$ , then*

$$F_{pq}(k, l) = \begin{cases} h^{\frac{k+l}{4}}, & \text{if } pq = ab, 4 \mid (k-1) \text{ \& } 4 \mid (k+l) \\ h^{\frac{k+l}{4}}, & \text{if } pq = ba^{-1}, 4 \mid k \text{ \& } 4 \mid k+l \\ h^{\frac{k+l}{4}}, & \text{if } pq = a^{-1}b^{-1}, 4 \mid (l-1) \text{ \& } 4 \mid (k+l) \\ 0_B & \text{otherwise.} \end{cases}$$

(iii) *If  $pq = e$ , then we have the following recurrence relation ;*

$$F_{pq}(k, l) = \sum_{r, s \in \{a, b, a^{-1}, b^{-1}\}, (r, s) \neq (q, p)} F_{rs}(k-1, l-1).$$

*Proof.* (i) is proved by Lemma 3.12 and (ii) is proved by Lemma 3.13. Now, by using the results (i) and (ii), we can characterize Lemma 3.14. The characterization is the statement (iii) ;

Suppose that  $pq = e = aa^{-1} = a^{-1}a = bb^{-1} = b^{-1}b$ . Then

$$\begin{aligned}
F_{pq}(k, l) &= E \left( \left( \sum_{|w|=k, \text{End}(w)=p} w \right) \left( \sum_{|w'|=l, \text{Init}(w')=q} w' \right) \right) \\
&= E \left( \left( \sum_{|w|=k-1, \text{End}(w) \neq p^{-1}} w \right) \left( \sum_{|w'|=l-1, \text{Init}(w')=q^{-1}} w' \right) \right) \\
&= \sum_{r, s \in \{a, b, a^{-1}, b^{-1}\}, r \neq p^{-1}, s \neq q^{-1}} E \left( \left( \sum_{|w|=k-1, \text{End}(w)=r} w \right) \left( \sum_{|w'|=l-1, \text{Init}(w')=s} w' \right) \right) \\
(3.15.1) \quad & \\
&= \sum_{r, s \in \{a, b, a^{-1}, b^{-1}\}, r \neq p^{-1}, s \neq q^{-1}} F_{rs}(k-1, l-1).
\end{aligned}$$

Since  $pq = e$ ,  $q = p^{-1}$  and hence

$$r \neq p^{-1} \text{ and } s \neq q^{-1} \iff r \neq q \text{ and } s \neq p.$$

Therefore, the formula (3.15.1) is equivalent to

$$(3.15.2) \quad \sum_{r, s \in \{a, b, a^{-1}, b^{-1}\}, (r, s) \neq (q, p)} F_{rs}(k-1, l-1).$$

■

Now, we will consider the case  $X_m h^{-1} X_n$ . But this case will be very similar to the previous case.

**Theorem 4.7.** *Let  $k, l \in \mathbb{N}$  be sufficiently big and let  $p, q \in \{a, b, a^{-1}, b^{-1}\}$ . Then*

(i) *If  $pq \not\triangleleft h^{-1} = bab^{-1}a^{-1}$ , then  $F_{pq}(k, l) = 0_B$ .*

(ii) *If  $pq \triangleleft h^{-1} = bab^{-1}a^{-1}$ , then*

$$F_{pq}(k, l) = \begin{cases} h^{-\frac{k+l}{4}}, & \text{if } pq = ba, \ 4 \mid (k-1) \ \& \ 4 \mid (k+l) \\ h^{-\frac{k+l}{4}}, & \text{if } pq = ab^{-1}, \ 4 \mid k \ \& \ 4 \mid k+l \\ h^{-\frac{k+l}{4}}, & \text{if } pq = b^{-1}a^{-1}, \ 4 \mid (l-1) \ \& \ 4 \mid (k+l) \\ 0_B & \text{otherwise.} \end{cases}$$



(iii) If  $pq = e$ , then we have the following recurrence relation ;

$$F_{pq}(k, l) = \sum_{r, s \in \{a, b, a^{-1}, b^{-1}\}, (r, s) \neq (q, p)} F_{rs}(k-1, l-1).$$

□

Now, by using the above equalities and recurrence relation, we can compute the  $E(X_m h X_n)$ . We need the following definition ; Recall that

$X_m h X_n$

$$\begin{aligned} &= W_{m+n+4} + W_{m+n+2} + W_{m+n} + W_{m+n-2} \\ &\quad + \left( \sum_{|w|=m-4, \text{End}(w) \neq b} w \right) (X_n) + X_m \left( \sum_{|w|=n-4, \text{Init}(w) \neq a} w \right) \end{aligned} \quad (3.3.1.2)$$

$$+ \left( \sum_{|w|=m-3, \text{End}(w) \neq a^{-1}} w \right) \left( \sum_{|w|=n-1, \text{Init}(w) \neq b^{-1}} w \right) \quad (3.3.1.3)$$

$$+ \left( \sum_{|w|=m-1, \text{End}(w) \neq a} w \right) \left( \sum_{|w|=n-3, \text{Init}(w) \neq b} w \right) \quad (3.3.1.4)$$

$$+ \left( \sum_{|w|=m-2, \text{End}(w) \neq a} w \right) \left( \sum_{|w|=n-2, \text{Init}(w) \neq a^{-1}} w \right), \quad (3.3.1.5)$$

where

(3.3.1.6)

$$W_{m+n+2} = \left( \sum_{|w|=m-1, \text{End}(w) \neq a} w \right) (ba^{-1}b^{-1})X_n + X_m(aba^{-1}) \left( \sum_{|w|=n-1, \text{Init}(w) \neq b^{-1}} w \right),$$

(3.3.1.7)

$$\begin{aligned} W_{m+n} &= \left( \sum_{|w|=m-2, \text{End}(w) \neq b} w \right) (a^{-1}b^{-1})X_n + X_m(ab) \left( \sum_{|w|=n-2, \text{Init}(w) \neq a^{-1}} w \right) \\ &\quad + \left( \sum_{|w|=m-1, \text{End}(w) \neq a} w \right) ba^{-1} \left( \sum_{|w|=n-1, \text{Init}(w) \neq b^{-1}} w \right) \end{aligned}$$

and

(3.3.1.8)

$$W_{m+n-2} = \left( \sum_{|w|=m-3, \text{End}(w) \neq a^{-1}} w \right) b^{-1} X_n + X_m a \left( \sum_{|w|=n-3, \text{Init}(w) \neq b} w \right).$$

Now, define a certain generalization of  $F_{pq}(k, l)$  ;

**Definition 4.3.** Let  $p_1, \dots, p_N \in \{a, b, a^{-1}, b^{-1}\}$  and let  $d, N < M \in \mathbb{N}$ . Define the relation  $\triangleleft$  by

$$p_1 \dots p_N \triangleleft l_1 \dots l_M \stackrel{\text{def}}{\iff} \exists j \in \{1, \dots, M - N - 1\} \text{ s.t. } p_1 \dots p_N = l_{j+1} \dots l_{j+N}.$$

Also, define a map  $F_{p_1 \dots p_j] \text{--} [p_{j+1} \dots p_N} : \mathbb{N} \times \mathbb{N} \rightarrow B$ , for all  $j = 1, \dots, N - 1$  by

$$F_{p_1 \dots p_j] \text{--} [p_{j+1} \dots p_N}(k, l) = E \left( \left( \sum_{|w|=k, \text{End}(w) = p_1 \dots p_j} w \right) \left( \sum_{|w'|=l, \text{Init}(w') = p_{j+1} \dots p_N} w' \right) \right).$$

For example,

$$aba^{-1} \triangleleft h \text{ or } bab^{-1} \triangleleft h$$

and

$$a^2b \not\triangleleft h$$

etc.

By using the above new definition, we can rewrite that

$$F_{pq}(k, l) = F_{p] \text{--} [q}(k, l),$$

for all  $p, q \in \{a, b, a^{-1}, b^{-1}\}$ , where  $k$  and  $l$  are sufficiently large natural numbers.

**Proposition 4.8.** Suppose that  $k, l$  and  $d > N \geq 3$  in  $\mathbb{N}$ . Then

$$F_{p_1 \dots p_j] \text{--} [p_{j+1} \dots p_N}(k, l) = \begin{cases} \left\{ \begin{array}{l} h^{\frac{k+l}{4}} & \text{if } p_1 = a, \ 4 \mid (k-j) \ \& \ 4 \mid (k+l) \\ h^{\frac{k+l}{4}} & \text{if } p_1 = b, \ 4 \mid (k-j+1) \ \& \ 4 \mid (k+l) \\ h^{\frac{k+l}{4}} & \text{if } p_1 = a^{-1}, \ 4 \mid (k-j+2) \ \& \ 4 \mid (k+l) \\ h^{\frac{k+l}{4}} & \text{if } p_1 = b^{-1}, \ 4 \mid (k-j+3) \ \& \ 4 \mid (k+l). \\ 0_B & \text{otherwise.} \end{array} \right\} & \text{if } p_1 \dots p_N \triangleleft h^d \\ 0_B & \text{if } p_1 \dots p_N \not\triangleleft h^d \end{cases}$$

where  $j \in \{1, \dots, N - 1\}$ . Moreover, if  $4 \mid (k - j + i)$  and  $4 \mid (k + l)$ , for the fixed  $i \in \{0, 1, \dots, 3\}$ , then,  $4 \nmid (k - j + i')$ , whenever  $i' \neq i$ . (i.e, if  $4 \mid (k - j)$ , then  $4 \nmid (k - j + i)$ , for all  $i = 1, 2, 3$ .)

**Corollary 4.9.** *Suppose that  $d > N \geq 3$  in  $\mathbb{N}$ . Then*

$$F_{p_1 \dots p_j]_{-[p_{j+1} \dots p_N]}(k, l) = \begin{cases} \begin{cases} h^{-\frac{k+l}{4}} & \text{if } p_1 = b, 4 \mid (k-j) \text{ \& } 4 \mid (k+l) \\ h^{-\frac{k+l}{4}} & \text{if } p_1 = a, 4 \mid (k-j+1) \text{ \& } 4 \mid (k+l) \\ h^{-\frac{k+l}{4}} & \text{if } p_1 = b^{-1}, 4 \mid (k-j+2) \text{ \& } 4 \mid (k+l) \\ h^{-\frac{k+l}{4}} & \text{if } p_1 = a^{-1}, 4 \mid (k-j+3) \text{ \& } 4 \mid (k+l). \\ 0_B & \text{otherwise.} \end{cases} & \text{if } p_1 \dots p_N \triangleleft h^{-d} \\ 0_B & \text{if } p_1 \dots p_N \not\triangleleft h^{-d} \end{cases}$$

where  $j \in \{1, \dots, N-1\}$ .  $\square$

**Corollary 4.10.** *Let  $p, q \in \{a, b, a^{-1}, b^{-1}\}$  and  $k, l \in \mathbb{N}$ , sufficiently large. If  $pq \neq e$ , then*

$$F_{pq}(k, l) = F_{p]_{-[q]}(k, l) = \begin{cases} h^{\frac{k+l}{4}} & \text{if } p = a, 4 \mid (k-1) \text{ \& } 4 \mid (k+l) \\ h^{\frac{k+l}{4}} & \text{if } p = b, 4 \mid k \text{ \& } 4 \mid (k+l) \\ h^{\frac{k+l}{4}} & \text{if } p = a^{-1}, 4 \mid (k+1) \text{ \& } 4 \mid (k+l) \\ h^{\frac{k+l}{4}} & \text{if } p = b^{-1}, 4 \mid (k+2) \text{ \& } 4 \mid (k+l). \\ 0_B & \text{otherwise.} \end{cases}$$

and

$$F_{pq}(k, l) = F_{p]_{-[q]}(k, l) = \begin{cases} h^{-\frac{k+l}{4}} & \text{if } p = b, 4 \mid (k-1) \text{ \& } 4 \mid (k+l) \\ h^{-\frac{k+l}{4}} & \text{if } p = a, 4 \mid k \text{ \& } 4 \mid (k+l) \\ h^{-\frac{k+l}{4}} & \text{if } p = b^{-1}, 4 \mid (k+1) \text{ \& } 4 \mid (k+l) \\ h^{-\frac{k+l}{4}} & \text{if } p = a^{-1}, 4 \mid (k+2) \text{ \& } 4 \mid (k+l). \\ 0_B & \text{otherwise.} \end{cases}$$

In particular, this is the generalization of (ii) of the Theorem 3.6.  $\square$

Now, consider the case of  $F_{p_1^{-1} \dots p_N}^{-1}(k, l)$  and  $F_{p_1 \dots p_N p_N^{-1}}^{-1}(k, l)$ . Of course, we will restrict our interests to the case when  $p_1 \dots p_N \triangleleft h^d$ .

**Proposition 4.11.** *Let  $k, l$  and  $d > N > 2$  be in  $\mathbb{N}$ . Assume that  $p_1 \dots p_N \triangleleft h^d$ . Then*

$$F_{p_1^{-1} \dots p_N}^{-1}(k, l) = F_{p_2^{-1} \dots p_N}^{-1}(k-1, l-1)$$

and

$$F_{p_1 \dots p_N}^{-1}(k, l) = F_{p_1 \dots p_{N-1}}^{-1}(k-1, l-1).$$

Now, we need the following new function from  $\mathbb{N} \times \mathbb{N}$  to  $B$  ;

**Definition 4.4.** *Let  $p_1, \dots, p_M \in \{a, b, a^{-1}, b^{-1}\}$  and assume that  $p_1 \dots p_M \triangleleft h^d$ . Let  $p^1, \dots, p^i, p^j, \dots, p^N \in \{a, b, a^{-1}, b^{-1}\}$  and let  $p^1 \dots p^i$  and  $p^j \dots p^N$  be words in  $\{a, b, a^{-1}, b^{-1}\}$ . Always assume that  $|p^1 \dots p^i p^j \dots p^N| < |p_1 \dots p_M|$ . Define a map*

$$F_{p^1 \dots p^i \dots p^j \dots p^N} : \mathbb{N} \times \mathbb{N} \rightarrow B$$

by

$$\begin{aligned} & F_{p^1 \dots p^i \dots p^j \dots p^N}(k, M, l) \\ &= E \left( \left( \sum_{|w|=k, \text{End}(w)=p^1 \dots p^i} w \right) (p_1 \dots p_M) \left( \sum_{|w'|=l, \text{Init}(w')=p^j \dots p^N} w' \right) \right), \end{aligned}$$

for all  $k, l \in \mathbb{N}$ .

For example,

$$F_{p^1 \dots p^i \dots p^j \dots p^N}(k, 0, l) = F_{p^1 \dots p^i \dots p^j \dots p^N}(k, l).$$

**Proposition 4.12.** *By using the same notations in the previous definition, we have that*

(1) *If  $p^1 \dots p^i p_1 \dots p_M p^j \dots p^N \not\triangleleft h^d$ , then*

$$F_{p^1 \dots p^i \dots p^j \dots p^N}(k, M, l) = 0_B.$$

*e.) (Remaind that  $|p^1 \dots p^i p^j \dots p^N| < |p_1 \dots p_M|$  and hence  $p^1 \dots p^i p_1 \dots p_M p^j \dots p^N \neq$*

*If  $p^1 \dots p^i p_1 \dots p_M p^j \dots p^N \triangleleft h^d$ , then*

$$F_{p^1 \dots p^i \dots p^j \dots p^N}(k, M, l)$$

$$= \begin{cases} h^{\frac{k+M+l}{4}} & \text{if } p_1 = a, 4 \mid (k-i) \text{ \& } 4 \mid (k+M+l) \\ h^{\frac{k+M+l}{4}} & \text{if } p_1 = b, 4 \mid (k-j+1) \text{ \& } 4 \mid (k+M+l) \\ h^{\frac{k+M+l}{4}} & \text{if } p_1 = a^{-1}, 4 \mid (k-j+2) \text{ \& } 4 \mid (k+M+l) \\ h^{\frac{k+M+l}{4}} & \text{if } p_1 = b^{-1}, 4 \mid (k-j+3) \text{ \& } 4 \mid (k+M+l) \\ 0_B & \text{otherwise.} \end{cases}$$

(2) Assume that  $p^i = p_1^{-1}$ . Then

$$F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) = F_{p_1 \dots p^{i-1}] \llcorner \langle p_2 \dots p_M \rangle \llbracket [p^j \dots p^N (k-1, M-1, l).$$

In particular, if  $p^1 \dots p^i p_1 \dots p_M p^j \dots p^N \triangleleft h^d$ , then  $F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) = 0_B$ .

(3) Assume that  $p^j = p_M^{-1}$ . Then

$$F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) = F_{p^1 \dots p^i \llcorner p_1 \dots p_{M-1} \llbracket [p^{j+1} \dots p^N (k, M-1, l-1).$$

In particular, if  $p^1 \dots p^i p_1 \dots p_M p^j \dots p^N \triangleleft h^d$ , then  $F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) = 0_B$ .

(4) Let  $p^i = p_1^{-1}$  and  $p^j = p_M^{-1}$ . Then

$$F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) = F_{p^1 \dots p^{i-1}] \llcorner \langle p_2 \dots p_{M-1} \rangle \llbracket [p^{j+1} \dots p^N (k-1, M-1, l-1).$$

In particular, if  $p^1 \dots p^i p_1 \dots p_M p^j \dots p^N \triangleleft h^d$ , then  $F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) = 0_B$ .

**Proposition 4.13.** Let  $p_1 \dots p_i p_{i+1} \dots p_M \triangleleft h^d$ . Then

$$(1) F_{p_M^{-1} \dots p_1^{-1}] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p^j \dots p^N (k, M, l) \\ = \sum_{p_0 \in \{a, b, a^{-1}, b^{-1}\}, p_0 \neq p_M, p_0 p^j \dots p^N \triangleleft h^d} F_{p_0] \llbracket [p^j \dots p^N (k-M, l).$$

where  $p_0 \neq p_M$  and  $p_0 p^j \dots p^N \triangleleft h^d$ .

$$(2) F_{p^1 \dots p^i] \llcorner \langle p_1 \dots p_M \rangle \llbracket [p_M^{-1} \dots p_1^{-1} (k, M, l) \\ = \sum_{p_0 \in \{a, b, a^{-1}, b^{-1}\}, p_0 \neq p_1, p^1 \dots p^i p_0 \triangleleft h^d} F_{p_0] \llbracket [p^j \dots p^N (k-M, l).$$

$$\begin{aligned}
(3) \quad & F_{p_i^{-1} \dots p_1^{-1}]_{<p_1 \dots p_i p_{i+1} \dots p_M >}_{-} [p_M^{-1} \dots p_{i+1}^{-1}]_{<k, M, l >} \\
&= \sum_{p \neq p_i, q \neq p_{i+1}} F_{p]}_{-} [q]_{<k-i, l-(M-i) >}.
\end{aligned}$$

Now, we will compute  $E(X_m h X_n)$  ;

**Lemma 4.14.** *We have that*

$$\begin{aligned}
(1) \quad & E(W_{m+n+4}) = \begin{cases} h^{m+n+1} & \text{if } 4 \mid m \text{ and } 4 \mid n \\ 0_B & \text{otherwise.} \end{cases} \\
(2) \quad & E(W_{m+n+2}) = F_{b^{-1}]_{-} [b a^{-1} b^{-1} (m-1, n+3) + F_{a b a^{-1}]_{-} [a (m+3, n-1). \\
(3) \quad & E(W_{m+n}) = \sum_{p \neq b, q \neq a^{-1}} F_{p]}_{-} [q]_{<m-2, n-2 >}. \\
(4) \quad & E(W_{m+n-2}) = \sum_{p \neq a^{-1}} F_{p]}_{-} [b^{-1} (m-3, n+1) + \sum_{p \neq b} F_{a]}_{-} [p (m+1, n-3) \\
& + \sum_{p \neq b, q \neq b^{-1}} F_{p]}_{-} [a >]_{-} [q]_{<m-2, 1, n-1 >} + \sum_{p \neq a, q \neq a^{-1}} F_{p]}_{-} [b >]_{-} [q]_{<m-1, 1, n-2 >}.
\end{aligned}$$

Now, we will consider the general case  $E(X_m h^d X_n)$ , where  $d \in \mathbb{N}$ .

**Lemma 4.15.** *Let  $m, n, d \in \mathbb{N}$ . Then*

$$E(W_{m+n+4d}) = \begin{cases} h^{\frac{m+n+4d}{4}} & \text{if } 4 \mid m \text{ and } 4 \mid n \\ 0_B & \text{otherwise.} \end{cases}$$

The above case is the NO-cancellation case of  $X_m h^d X_n$ .

**Lemma 4.16.** *Let  $m, n \in \mathbb{N}$  be sufficiently big and  $d \in \mathbb{N}$ . Then*

$$\begin{aligned}
E(W_{m+n+(4d-2j)}) &= \sum_{p \neq p_{2j}} F_{p]}_{-} [p_{2j+1} \dots p_{4d} (m-2j, n+4d) \\
&+ \sum_{i, j \in \{1, \dots, 2d-1\}, (m-i)+(n+j)=m+n+(4d-2j)} \\
&\quad \sum_{p \neq p_i, p' \neq p_j} F_{p]}_{-} [p_{i+1} \dots p_{j-1} >]_{-} [p']_{<m-i, j-1-i, n+j >} \\
&+ \sum_{p \neq p_{4d-(2j+1)}} F_{p_1 \dots p_{4d-2j}]_{-} [p (m+4d, n-2j).
\end{aligned}$$

We will call the above cancellation case a  $(i + j)$ -cancellation from the left and from the right.

**Lemma 4.17.** *Let  $m, n, d \in \mathbb{N}$ . Then*

$$\begin{aligned} E(W_{m+n-4d}) &= \sum_{p \in \{a, b, a^{-1}\}, p' \in \{a, b, a^{-1}, b^{-1}\}} F_{p]_{-} [p'}(m - 4d, n) \\ &\quad + \sum_{p \in \{a, b, a^{-1}, b^{-1}\}, p' \in \{b, a^{-1}, b^{-1}\}} F_{p]_{-} [p'}(m, n - 4d) \\ &\quad + \sum_{i=1}^{4d} \sum_{p \neq p_i, p' \neq p_{i+1}} F_{p]_{-} [p'}(m - i, n - (4d - i)). \end{aligned}$$

**Lemma 4.18.** *Let  $m, n, N \in \mathbb{N}$ . If  $p_{i_1} \dots p_{i_N} \triangleleft h^d$ , then*

$$\begin{aligned} \sum_{j=0}^{2d-1} E(W_{m+n-2j}) &= \sum_{j=0}^{2d-1} \sum_{k+N+l=m+n-2j} \sum_{i_1 \neq i_2 \neq \dots \neq i_N \in \{1, \dots, 4d\}} \\ &\quad \sum_{p, q \in \{a, b, a^{-1}, b^{-1}\}, pp_{i_1} \dots p_{i_N} \triangleleft h^d} F_{p]_{-} [p_{i_1} \dots p_{i_N}}(k, N, l). \end{aligned}$$

□

Finally, we can get the  $B$ -functional value  $E(X_m h^d X_n)$  ;

**Theorem 4.19.** *Let  $m, n, d \in \mathbb{N}$ . We have that*

$$\begin{aligned} E(X_m h^d X_n) &= E(W_{m+n+4d}) + \sum_{j=1}^{2d-1} E(W_{m+n+(4d-2j)}) \\ &\quad + \sum_{j=0}^{2d-1} E(W_{m+n-2j}) + E(W_{m+n-4d}), \end{aligned}$$

where, by putting  $h^d = p_1 \dots p_{4d}$ ,

$$E(W_{m+n+4d}) = \begin{cases} h^{\frac{m+n+4d}{4}} & \text{if } 4 \mid m \text{ and } 4 \mid n \\ 0_B & \text{otherwise,} \end{cases}$$

$$\sum_{j=1}^{2d-1} E(W_{m+n+(4d-2j)}) = \sum_{j=1}^{2d-1} \left( \sum_{p \neq p_{2j}} F_{p]_{-} [p_{2j+1} \dots p_{4d}}(m - 2j, n + 4d) \right.$$

$$\begin{aligned} &+ \sum_{i, j \in \{1, \dots, 2d-1\}, (m-i)+(n+j)=m+n+(4d-2j)} \\ &\quad \sum_{p \neq p_i, p' \neq p_j} F_{p]_{-} \langle p_{i+1} \dots p_{j-1} \rangle_{-} [p'}(m - i, j - 1 - i, n + j) \end{aligned}$$

$$+ \sum_{p \neq p_{4d-(2j+1)}} F_{p_1 \dots p_{4d-2j}]_{-} [p}(m + 4d, n - 2j) \quad )$$

$$\begin{aligned} \sum_{j=0}^{2d-1} E(W_{m+n-2j}) &= \sum_{j=0}^{2d-1} \sum_{k+N+l=m+n-2j} \sum_{i_1 \neq i_2 \neq \dots \neq i_N \in \{1, \dots, 4d\}} \\ &\quad \sum_{p, q \in \{a, b, a^{-1}, b^{-1}\}, pp_{i_1} \dots p_{i_N} \triangleleft h^d} F_{p]_{-[p_{i_1} \dots p_{i_N}}(k, N, l). \end{aligned}$$

and

$$\begin{aligned} E(W_{m+n-4d}) &= \sum_{p \in \{a, b, a^{-1}\}, p' \in \{a, b, a^{-1}, b^{-1}\}} F_{p]_{-[p'}(m-4d, n) \\ &\quad + \sum_{p \in \{a, b, a^{-1}, b^{-1}\}, p' \in \{b, a^{-1}, b^{-1}\}} F_{p]_{-[p'}(m, n-4d) \\ &\quad + \sum_{i=1}^{4d} \sum_{p \neq p_i, p' \neq p_{i+1}, p_i p_{i+1} \triangleleft h^d} F_{p]_{-[p'}(m-i, n-(4d-i)). \end{aligned}$$

□

For considering  $h^{-d}$  ( $d \in \mathbb{N}$ ), we have the following result, like the former theorem ;

**Theorem 4.20.** *Let  $m, n, d \in \mathbb{N}$ . We have that*

$$\begin{aligned} E(X_m h^{-d} X_n) &= E(W_{m+n+4d}) + \sum_{j=1}^{2d-1} E(W_{m+n+(4d-2j)}) \\ &\quad + \sum_{j=0}^{2d-1} E(W_{m+n-2j}) + E(W_{m+n-4d}), \end{aligned}$$

where, by putting  $h^{-d} = p_1 \dots p_{4d}$ ,

$$E(W_{m+n+4d}) = \begin{cases} h^{-\frac{m+n+4d}{4}} & \text{if } 4 \mid m \text{ and } 4 \mid n \\ 0_B & \text{otherwise,} \end{cases}$$

$$\sum_{j=1}^{2d-1} E(W_{m+n+(4d-2j)}) = \sum_{j=1}^{2d-1} \left( \sum_{p \neq p_{2j}} F_{p]_{-[p_{2j+1} \dots p_{4d}}(m-2j, n+4d) \right.$$

$$\begin{aligned} &+ \sum_{i, j \in \{1, \dots, 2d-1\}, (m-i)+(n+j)=m+n+(4d-2j)} \\ &\quad \sum_{p \neq p_i, p' \neq p_j} F_{p]_{-<p_{i+1} \dots p_{j-1}>_{-[p'}(m-i, j-1-i, n+j) \end{aligned}$$

$$+ \sum_{p \neq p_{4d-(2j+1)}} F_{p_1 \dots p_{4d-2j}]_{-[p}(m+4d, n-2j) \quad )$$

$$\sum_{j=0}^{2d-1} E(W_{m+n-2j}) = \sum_{j=0}^{2d-1} \sum_{k+N+l=m+n-2j} \sum_{i_1 \neq i_2 \neq \dots \neq i_N \in \{1, \dots, 4d\}}$$

$$\sum_{p, q \in \{a, b, a^{-1}, b^{-1}\}, pp_{i_1} \dots p_{i_N} \triangleleft h^d} F_{p]_{-[p_{i_1} \dots p_{i_N}}(k, N, l).$$



and

$$\begin{aligned}
E(W_{m+n-4d}) &= \sum_{p \in \{a, a^{-1}, b^{-1}\}, p' \in \{a, b, a^{-1}, b^{-1}\}} F_{p]_{-}[p'}(m-4d, n) \\
&+ \sum_{p \in \{a, b, a^{-1}, b^{-1}\}, p' \in \{a, a^{-1}, b^{-1}\}} F_{p]_{-}[p'}(m, n-4d) \\
&+ \sum_{i=1}^{4d} \sum_{p \neq p_i, p' \neq p_{i+1}, p_i p_{i+1} < h^d} F_{p]_{-}[p'}(m-i, n-(4d-i)).
\end{aligned}$$

□

#### 4.2. Recurrence Relation For $E(X_{m_1} h^{d_2} X_{m_2} h^{d_3} \dots h^{d_n} X_{m_n})$ .

In this section, we will compute more general form

$$E(X_{m_1} h^{d_2} X_{m_2} h^{d_3} \dots h^{d_n} X_{m_n}),$$

where  $m_1, \dots, m_n \in \mathbb{N}$  and  $d_2, \dots, d_n \in \mathbb{Z}$ . First, we will consider the most simple form, among them,

$$E(X_{m_1} h^{d_2} X_{m_2} h^{d_3} X_{m_3}),$$

for  $m_1, m_2, m_3 \in \mathbb{N}$  and  $d_2, d_3 \in \mathbb{Z}$ . Notice that, by the evenness of  $x = a + b + a^{-1} + b^{-1}$ , we have the following trivial condition ;

$$m_1 + \dots + m_n \in N \text{ should be even.}$$

Throughout this paper, we will assume that  $m_1 + \dots + m_n$  are even! (Notice that each  $m_j$  need not be even. For instance, we can have that  $m_1 = 1, m_2 = 3$  and  $m_3 = 2$ .) Recall that, by the previous section, we have that, for  $m_1, m_2 \in \mathbb{N}$  and  $d_2 \in \mathbb{N}$ ,

$$\begin{aligned}
E(X_{m_1} h^{d_2} X_{m_2}) &= E(W_{m_1+m_2+4d_2}) + \sum_{j=1}^{2d_2-1} E(W_{m_1+m_2+(4d_2-2j)}) \\
&+ \sum_{j=0}^{2d_2-1} E(W_{m_1+m_2-2j}) + E(W_{m_1+m_2-4d_2}),
\end{aligned}$$

where each summands are determined recursively.

But, in  $X_{m_1} h^{d_2} X_{m_2} h^{d_3} X_{m_3}$ , there will be much more terms which we have to consider. Also, different from the  $X_{m_1} h^{d_2} X_{m_2}$ , we have to consider the case when both  $d_2$  and  $d_3$  are positive integers or both  $d_2$  and  $d_3$  are negative integers or either  $d_2$  or  $d_3$  is a positive integer and the other is negative. So, we need the following new definition.

**Definition 4.5.** Let  $m_1, m_2, m_3, N_2, N_3 \in \mathbb{N}$ . Define a map  $\Phi_0$  from  $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$  to  $B$

$$F_{p_{m_1}^E] \prec p_{11} \dots p_{1N_2} \succ \lrcorner [p_{m_2}^E \lrcorner p_{m_2}^I] \prec p_{21} \dots p_{2N_3} \succ \lrcorner [p_{m_3}^I] (m_1, N_2, m_2, N_3, m_3),$$

denoted by  $\Phi_0(m_1, N_2, m_2, N_3, m_3)$ , by

$$E \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{11} \dots p_{1N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \cdot (p_{21} \dots p_{2N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right).$$

If  $p_{11} \dots p_{1N_2} = e$  or  $p_{21} \dots p_{2N_3} = e$ , then we have that

$$\begin{aligned} & \Phi_0(m_1, 0, m_2, N_3, m_3) \\ &= F_{p_{m_1}^E] \lrcorner [p_{m_2}^E \lrcorner p_{m_2}^I] \prec p_{21} \dots p_{2N_3} \succ \lrcorner [p_{m_3}^I] (m_1, 0, m_2, N_3, m_3) \end{aligned}$$

or

$$\begin{aligned} & \Phi_0(m_1, N_2, m_2, 0, m_3) \\ &= F_{p_{m_1}^E] \prec p_{11} \dots p_{1N_2} \succ \lrcorner [p_{m_2}^E \lrcorner p_{m_2}^I] \lrcorner [p_{m_3}^I] (m_1, N_2, m_2, 0, m_3). \end{aligned}$$

And if both of them are  $e$ , then we can define

$$\begin{aligned} & \Phi_0(m_1, 0, m_2, 0, m_3) \\ &= F_{p_{m_1}^E] \lrcorner [p_{m_2}^E \lrcorner p_{m_2}^I] \lrcorner [p_{m_3}^I] (m_1, 0, m_2, 0, m_3). \end{aligned}$$

In the above definition of  $\Phi_0(m_1, N_2, m_2, N_3, m_3)$ , the half-open bracket "  $[p$  " and "  $p]$  " mean that words with initial letter  $p$  and the words with ending letter  $p$ . Also the bracket  $\prec p \dots q \succ$  means the word  $p \dots q$  and  $[p^I \lrcorner p^E]$  means  $\sum_{\text{Init}(w)=p^I, \text{End}(w)=p^E} w$ .

Now, we will observe the above function  $\Phi_0(m_1, N_2, m_2, N_3, m_3)$ .

**Lemma 4.21.** Let  $m_1, m_2, m_3, N_2, N_3 \in \mathbb{N}$ . Assume that  $[p_{m_1}^E \neq p_{11}^{-1}$  and  $p_{m_2}^I \neq p_{1N_2}^{-1}]$  or  $[p_{m_2}^E \neq p_{21}^{-1}$  and  $p_{m_3}^I \neq p_{2N_3}^{-2}]$ . Then

$$\Phi_0(m_1, N_2, m_2, N_3, m_3) = 0_B,$$

whenever  $p_{11} \dots p_{1N_2} \not\triangleleft h^{d_2}$  or  $p_{21} \dots p_{2N_3} \not\triangleleft h^{d_3}$ , where  $d_2, d_3 \in \mathbb{Z} \setminus \{0\}$ .

*Proof.* By definition, we have that

$$\begin{aligned} & \Phi_0(m_1, N_2, m_2, N_3, m_3) \\ &= E \left( \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{11} \dots p_{1N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \right. \\ & \quad \left. \cdot (p_{21} \dots p_{2N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right) \right). \end{aligned}$$

Under the hypothesis, since  $p_{11} \dots p_{1N_2} \not\triangleleft h^{d_2}$  or  $p_{21} \dots p_{2N_3} \not\triangleleft h^{d_3}$ , it vanishes.  $\blacksquare$

From now assume that

$$(A) \quad p_{11} \dots p_{1N_2} \triangleleft h^{d_2} \quad \text{and} \quad p_{21} \dots p_{2N_3} \triangleleft h^{d_3},$$

where  $N_2, N_3 \in \mathbb{N}$  and  $d_2, d_3 \in \mathbb{Z}$ . Observe that if (A) is satisfied, then

$$\begin{aligned} & \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{11} \dots p_{1N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \\ & \quad \cdot (p_{21} \dots p_{2N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right) \\ &= (\sum_{j_2=0}^{N_2} \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{11} \dots p_{1j_2}) \\ & \quad \cdot (p_{1(j_2+1)} \dots p_{1N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \\ & \quad \cdot (p_{21} \dots p_{2N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right) \end{aligned}$$

(3.3.2.1)

$$\begin{aligned} &= \sum_{j_2=0}^{N_2-1} \sum_{j_3=0}^{N_3} \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{11} \dots p_{1j_2}) \\ & \quad \cdot (p_{1(j_2+1)} \dots p_{1N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \\ & \quad \cdot (p_{21} \dots p_{2j_3}) (p_{j_3+1} \dots p_{N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right), \end{aligned}$$

where  $p_{10} = e$  and  $p_{20} = e$ . Above, the case  $[j_2 = 0]$  (resp.  $[j_2 = N_2]$ ) means the case when there is no cancellation (resp. full cancellation) for  $\left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right)$

and  $p_{11} \dots p_{1N_2}$ . Similarly, the case  $[j_3 = 0]$  (resp.  $[j_3 = N_3]$ ) means the case when there is no cancellation (resp. full cancellation) for  $p_{21} \dots p_{2N_3}$  and  $\left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right)$ .

Consider the summand,

$$\begin{aligned} S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3} &= \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{11} \dots p_{1j_2}) \\ &\cdot (p_{1(j_2+1)} \dots p_{1N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \\ &\cdot (p_{21} \dots p_{2j_3}) (p_{2(j_3+1)} \dots p_{2N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I} w \right). \end{aligned}$$

**Lemma 4.22.** *If we have the assumption (A), then the formula (3.3.2.1) contains the following  $h$ -terms ;*

$$\begin{aligned} &\left( F_{p_{m_1}^E} ]_{\prec p_{11} \dots p_{1j_2} \succ} (m_1, (j_2 - 1)) \right) \\ &\cdot \left( F_{\prec p_{1(j_2+1)} \dots p_{1N_2} \succ} ]_{\prec p_{m_2}^I \dots p_{m_2}^E \succ} ]_{\prec p_{21}, \dots, p_{2j_3} \succ} ((N_2 - (j_2 + 1)), m_2, (j_3 - 1)) \right) \\ &\cdot \left( F_{\prec p_{2(j_3+1)} \dots p_{2N_3} \succ} ]_{\prec p_{m_3}^I \succ} ((N_3 - (j_3 + 1)), m_3) \right), \end{aligned}$$

where

$$F_{\prec p_1 \dots p_n \succ} ]_{\prec p^I \dashv p^E \succ} ]_{\prec q_1 \dots q_m \succ} (n, k, m)$$

$$\stackrel{\text{def}}{=} E \left( (p_1 \dots p_n) \left( \sum_{|w|=k, \text{Init}(w)=p^I, \text{End}(w)=p^E} w \right) (q_1 \dots q_m) \right).$$

*Proof.* To find the  $h$ -terms in the given summand (3.3.2.1) is equivalent to compute

$$E(S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3}).$$

Then

$$\begin{aligned} &E(S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3}) \\ &= \left( F_{p_{m_1}^E} ]_{\prec p_{11} \dots p_{1j_2} \succ} (m_1, (j_2 - 1)) \right) \\ &\cdot \left( F_{\prec p_{1(j_2+1)} \dots p_{1N_2} \succ} ]_{\prec p_{m_2}^I \dots p_{m_2}^E \succ} ]_{\prec p_{21}, \dots, p_{2j_3} \succ} ((N_2 - (j_2 + 1)), m_2, (j_3 - 1)) \right) \\ &\cdot \left( F_{\prec p_{2(j_3+1)} \dots p_{2N_3} \succ} ]_{\prec p_{m_3}^I \succ} ((N_3 - (j_3 + 1)), m_3) \right). \blacksquare \end{aligned}$$

So, we can conclude that ;

**Theorem 4.23.** *Suppose that (A) is satisfied. Then*

$$\Phi_0(m_1, N_2, m_2, N_3, m_3) = \sum_{j_2=0}^{N_2-1} \sum_{j_3=0}^{N_3-1} E(S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3}).$$

□

Now, we will apply the above theorem to our case ;

**Theorem 4.24.** *Let  $m_1, m_2, m_3 \in \mathbb{N}$  and  $d_2, d_3 \in \mathbb{Z}$ . Then*

$$\begin{aligned} E(X_{m_1} h^{d_2} X_{m_2} h^{d_3} X_{m_3}) &= \sum_{p_1^E, p_2^I \in \{a, b, a^{-1}, b^{-1}\}} \sum_{j_2=0}^{4d_2} \sum_{j_3=0}^{4d_3} \\ &\cdot (F_{p_{m_1}^E}]_{\langle p_{21} \dots p_{2j_2} \rangle} (m_1, (j_2 - 1)) \\ &\cdot F_{\langle p_{2(j_2+1)} \dots p_{2(4d_2)} \rangle} [p_{m_2}^I \dots p_{m_2}^E]_{\langle p_{31} \dots p_{3j_3} \rangle} ((4d_2 - (j_2 + 1)), m_2, (j_3 - 1)) \\ &\cdot F_{\langle p_{3(j_3+1)} \dots p_{3(4d_3)} \rangle} [p_{m_3}^I] ((4d_3 - (j_3 + 1)), m_3), \end{aligned}$$

where

$$h^{d_2} = p_{21} \dots p_{2(4d_2)} \quad \text{and} \quad h^{d_3} = p_{31} \dots p_{3(4d_3)}.$$

Based on the above results, we can extend our interests to the general  $E(X_{m_1} h^{d_2} X_{m_2} \dots h^{d_n} X_{m_n})$ -case.

**Definition 4.6.** *Let  $n \in \mathbb{N}$ ,  $m_1, \dots, m_n \in \mathbb{N}$  and  $N_2, \dots, N_n \in \mathbb{N} \cup \{0\}$ . Define a map  $\Phi$  from  $\underbrace{\mathbb{N} \times \dots \times \mathbb{N}}_{(2n-1)\text{-times}}$  to  $B$  by*

$$\begin{aligned} &\Phi(m_1, N_2, m_2, \dots, N_n, m_n) \\ &:= \sum_{p_{m_1}^E, p_{m_2}^I, p_{m_2}^E, p_{m_3}^I, p_{m_3}^E, \dots, p_{m_{n-1}}^I, p_{m_{n-1}}^E, p_{m_n}^I \in \{a, b, a^{-1}, b^{-1}\}} \\ &\quad F_{p_{m_1}^E}]_{\langle p_{21} \dots p_{2N_2} \rangle} [p_{m_2}^I \dots p_{m_2}^E]_{\langle p_{31} \dots p_{3N_3} \rangle} [p_{m_3}^I \dots p_{m_3}^E]_{\dots} [p_{m_n}^I]_{\langle p_{n1} \dots p_{nN_n} \rangle} \\ &\quad (m_1, N_2, m_2, \dots, N_n, m_n) \\ &= E\left(\left(\sum_{|w|=m_1} w\right) (p_{21} \dots p_{2N_2}) \left(\sum_{|w|=m_2} w\right)\right) \end{aligned}$$

$$\begin{aligned} & \cdot (p_{31} \dots p_{3N_3}) \left( \sum_{|w|=m_3} w \right) (p_{41} \dots p_{4N_4}) \\ & \cdot \dots (p_{n1} \dots p_{nN_n}) \left( \sum_{|w|=m_n} w \right), \end{aligned}$$

where  $p_{ij}, p_{m_k}^I, p_{m_k}^E \in \{a, b, a^{-1}, b^{-1}\}$ .

For the fixed  $p_{m_1}^E, p_{m_n}^I, p_{m_j}^I, p_{m_j}^E \in \{a, b, a^{-1}, b^{-1}\}$ , for  $j = 2, \dots, n-1$ , define

$$\begin{aligned} & S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3, \dots, (j_n, N_n), m_n} \\ &= \left( \sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w \right) (p_{21} \dots p_{2j_2}) \\ & \quad \cdot (p_{2(j_2+1)} \dots p_{2N_2}) \left( \sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w \right) \\ & \quad \cdot (p_{31} \dots p_{3j_3}) (p_{3(j_3+1)} \dots p_{3N_3}) \left( \sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I, \text{End}(w)=p_{m_3}^E} w \right) \\ & \quad \cdot (p_{41} \dots p_{4j_4}) (p_{4(j_4+1)} \dots p_{4N_4}) \left( \sum_{|w|=m_4, \text{Init}(w)=p_{m_4}^I, \text{End}(w)=p_{m_4}^E} w \right) \\ & \quad \dots \\ & \quad \dots \\ & \quad \cdot (p_{n1} \dots p_{nj_n}) (p_{n(j_n+1)} \dots p_{nN_n}) \left( \sum_{|w|=m_n, \text{Init}(w)=p_{m_n}^I} w \right) \end{aligned}$$

Also, similar to the former discussion, we will assume that

$$(AA) \quad p_{k1} \dots p_{kN_k} \triangleleft h^{d_k}, \text{ for all } k = 1, \dots, n,$$

where  $d_k \in \mathbb{Z}$ .

**Theorem 4.25.** *Suppose that the condition (AA) is satisfied. Then*

$$\begin{aligned} & E(S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3, \dots, (j_n, N_n), m_n}) \\ &= \left( F_{p_{m_1}^E} \right]_{\triangleleft p_{21} \dots p_{2j_2}} (m_1, j_2 - 1) \\ & \quad \cdot \left( \prod_{k=2}^{n-1} F_{\triangleleft p_{kj_{k+1}} \dots p_{kN_k}} \right]_{\triangleleft [p_{m_k}^I \dots p_{m_k}^E] \triangleleft p_{(k+1)1} \dots p_{(k+1)j_{k+1}}} (N_k - (j_k + 1), m_k, j_{k+1} - 1) \\ & \quad \cdot \left( F_{\triangleleft p_{n(j_n+1)} \dots p_{nN_n}} \right]_{\triangleleft p_{m_n}^I} (N_n - (j_n + 1), m_n) \right). \end{aligned}$$

*Proof.* If (AA) is satisfied, then

$$\begin{aligned}
& E(S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3, \dots, (j_n, N_n), m_n}) \\
&= E\left(\left(\sum_{|w|=m_1, \text{End}(w)=p_{m_1}^E} w\right) (p_{21} \dots p_{2j_2})\right. \\
&\quad \cdot (p_{2(j_2+1)} \dots p_{2N_2}) \left(\sum_{|w|=m_2, \text{Init}(w)=p_{m_2}^I, \text{End}(w)=p_{m_2}^E} w\right) \\
&\quad \cdot (p_{31} \dots p_{3j_3}) (p_{3(j_3+1)} \dots p_{3N_3}) \left(\sum_{|w|=m_3, \text{Init}(w)=p_{m_3}^I, \text{End}(w)=p_{m_3}^E} w\right) \\
&\quad \cdot (p_{41} \dots p_{4j_4}) (p_{4(j_4+1)} \dots p_{4N_4}) \left(\sum_{|w|=m_4, \text{Init}(w)=p_{m_4}^I, \text{End}(w)=p_{m_4}^E} w\right) \\
&\quad \dots \\
&\quad \cdot (p_{n1} \dots p_{nj_n}) (p_{n(j_n+1)} \dots p_{nN_n}) \left(\sum_{|w|=m_n, \text{Init}(w)=p_{m_n}^I} w\right) \\
&= \left(F_{p_{m_1}^E} \lfloor_{\leftarrow} \langle p_{21} \dots p_{2j_2} \rangle (m_1, j_2 - 1)\right) \\
&\quad \cdot \left(F_{\langle p_{2(j_2+1)} \dots p_{2N_2} \rangle} \lfloor_{\leftarrow} [p_{m_2}^I \dots p_{m_2}^E] \lfloor_{\leftarrow} \langle p_{31} \dots p_{3j_3} \rangle (N_2 - (j_2 + 1), m_2, j_3 - 1)\right) \\
&\quad \cdot \left(F_{\langle p_{3(j_3+1)} \dots p_{3N_3} \rangle} \lfloor_{\leftarrow} [p_{m_3}^I \dots p_{m_3}^E] \lfloor_{\leftarrow} \langle p_{41} \dots p_{4j_4} \rangle (N_2 - (j_2 + 1), m_2, j_3 - 1)\right) \\
&\quad \dots \\
&\quad \cdot \left(F_{\langle p_{(n-1)j_{n-1}+1} \dots p_{(n-1)N_{n-1}} \rangle} \lfloor_{\leftarrow} [p_{m_{n-1}}^I \dots p_{m_{n-1}}^E] \lfloor_{\leftarrow} \langle p_{n1} \dots p_{nj_n} \rangle (N_{n-1} - (j_{n-1} + 1), m_{n-1}, j_n - 1)\right) \\
&\quad \cdot \left(F_{\langle p_{n(j_n+1)} \dots p_{nN_n} \rangle} \lfloor_{\leftarrow} p_{m_n}^I (N_n - (j_n + 1), m_n)\right) \\
&= \left(F_{p_{m_1}^E} \lfloor_{\leftarrow} \langle p_{21} \dots p_{2j_2} \rangle (m_1, j_2 - 1)\right) \\
&\quad \cdot \left(\prod_{k=2}^{n-1} F_{\langle p_{kj_{k+1}} \dots p_{kN_k} \rangle} \lfloor_{\leftarrow} [p_{m_k}^I \dots p_{m_k}^E] \lfloor_{\leftarrow} \langle p_{(k+1)1} \dots p_{(k+1)j_{k+1}} \rangle (N_k - (j_k + 1), m_k, j_{k+1} - 1)\right) \\
&\quad \cdot \left(F_{\langle p_{n(j_n+1)} \dots p_{nN_n} \rangle} \lfloor_{\leftarrow} p_{m_n}^I (N_n - (j_n + 1), m_n)\right) \cdot \blacksquare
\end{aligned}$$

By using the above theorem, we have that ;

**Theorem 4.26.** *Suppose that the condition (AA) is satisfied. Then*

$$\begin{aligned}
& \Phi(m_1, N_2, m_2, \dots, N_n, m_n) \\
&= \sum_{p_{m_1}^E, p_{m_n}^I, p_{m_j}^I, p_{m_j}^E \in \{a, b, a^{-1}, b^{-1}\}, j=2, \dots, n-1} \sum_{j_2=0}^{N_2} \sum_{j_3=0}^{N_3} \dots \sum_{j_n=0}^{N_n}
\end{aligned}$$

$$E \left( S_{m_1, (j_2, N_2), m_2, (j_3, N_3), m_3, \dots, (j_n, N_n), m_n} \right).$$

□

Applying the above two theorems, we have that ;

**Theorem 4.27.** *Let  $n \in \mathbb{N}$ ,  $m_1, \dots, m_n \in \mathbb{N}$  and  $d_2, \dots, d_n \in \mathbb{Z}$ . Then*

$$\begin{aligned} E \left( X_{m_1} h^{d_2} X_{m_2} h^{d_3} X_{m_3} \dots h^{d_n} X_{m_n} \right) \\ = \Phi \left( m_1, 4d_2, m_3, 4d_3, \dots, 4d_n, m_n \right), \end{aligned}$$

by the triangular relation " $\triangleleft$ ". □

**Theorem 4.28.** *Let  $n \in \mathbb{N}$ ,  $m_1, \dots, m_n \in \mathbb{N}$  and  $d_2, \dots, d_n \in \mathbb{N} \cup \{0\}$ . Then*

$$\begin{aligned} E \left( X_{m_1} (h^{d_2} + h^{-d_2}) X_{m_2} (h^{d_3} + h^{-d_3}) X_{m_3} \dots (h^{d_n} + h^{-d_n}) X_{m_n} \right) \\ = \sum_{r_2 \in \{\pm d_2\}, r_3 \in \{\pm d_3\}, \dots, r_n \in \{\pm d_n\}} \Phi \left( m_1, 4r_2, m_3, 4r_3, \dots, 4r_n, m_n \right). \end{aligned}$$

### 4.3. Computing Trivial Cumulants of $x$ .

Let  $k \in \mathbb{N}$ . Consider the  $2k$ -th trivial cumulants of  $x = a + b + a^{-1} + b^{-1}$ ,

$$\begin{aligned} K_{2k}^t \left( \underbrace{x, \dots, x}_{2k\text{-times}} \right) &= \sum_{\pi \in NC^{(even)}(2k)} \widehat{E}(\pi) (x \otimes \dots \otimes x) \mu(\pi, 1_{2k}) \\ &= \sum_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2k} \sum_{\pi \in NC_{l_1, \dots, l_p}(2k)} \widehat{E}(\pi) (x \otimes \dots \otimes x) \mu(\pi, 1_{2k}), \end{aligned}$$

where

$$NC_{l_1, \dots, l_p}(2k) = \{ \pi \in NC^{(even)}(2k) : V \in \pi \Leftrightarrow |V| = l_j, j = 1, \dots, p \}.$$

For example,

$$NC^{(even)}(8) = NC_{2,2,2,2}(8) \cup NC_{2,2,4}(8) \cup NC_{2,6}(8) \cup NC_{4,4}(8) \cup NC_8(8).$$



**Lemma 4.29.** (See Section 3.2.2) Fix  $k \in \mathbb{N}$ . Let  $h = aba^{-1}b^{-1} \in A_1 *_B A_2$  with  $h^0 = e$ .

(1) If  $4 \mid 2k$ , then

$$E(x^{2k}) = \sum_{j=0}^{\frac{k-2}{2}} p_{2k-4j}^{2k} \left( h^{\frac{k}{2}-j} + h^{-(\frac{k}{2}-j)} \right) + p_0^{2k} h^0,$$

where  $p_0^4 = 28$  and  $p_{2k}^{2k} = 1$ .

(2) If  $4 \nmid 2k$  and if there are  $X_{4l_1}, \dots, X_{4l_p}$  terms in  $x^{2k}$ , then

$$E(x^{2k}) = \sum_{j=0}^{\frac{k-3}{2}} p_{(2k-2)-4j}^{2k} \left( h^{\frac{k-1}{2}-2j} + h^{-(\frac{k-1}{2}-2j)} \right) + p_0^{2k} h^0,$$

where  $p_0^2 = 4$ .  $\square$

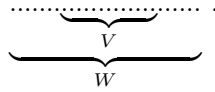
**Definition 4.7.** Let  $n \in \mathbb{N}$  and let  $\pi \in NC(n)$ . Let  $V = (v_1, \dots, v_k), W = (w_1, \dots, w_l) \in \pi$  and assume that there exists  $j \in \{1, \dots, k\}$  such that

$$1 \leq v_1 < \dots < v_j < w_1 < \dots < w_l < v_{j+1} < \dots < v_k \leq n,$$

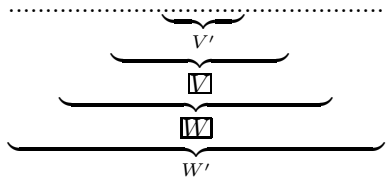
in  $\{1, \dots, n\}$ . Then we say that the block  $W$  is a subblock of the block  $V$ .

Suppose that  $V \in \pi(i)$  is an inner block of a partition  $\pi$  with its outer block  $W \in \pi(o)$ . Then  $V$  is a subblock of  $W$ .

**Definition 4.8.** Let  $\pi \in NC_{l_1, \dots, l_p}(2k)$ . Let  $V^o \in \pi(o)$  and let  $V, W \in \pi$  be subblocks in  $V^o$ . We say that  $V$  is inner in  $W$ , if  $V$  is a subblock of  $W$ . It has the following pictorial expression ;



Notice that it does not mean  $V \in \pi(i)$  with its outer block  $W \in \pi(o)$ . i.e we can have the following pictorial expression of subblocks  $V', V, W, W'$  in  $V^o \in \pi(o)$  ;



where  $V'$  and  $W'$  are other blocks in  $\pi$  which are subblocks of  $V^\circ \in \pi(o)$ . We also say that  $V$  is a **deepest subblock** if there is no subblock  $V'$  which is inner in  $V$ . We define that  $V \in \pi(o)$  containing no inner blocks is also a deepest block. We will express pictorially,

$$\begin{array}{c} \text{.....} \\ \underbrace{\hspace{10em}} \\ V \\ \underbrace{\hspace{10em}} \\ W \end{array} \quad \text{by} \quad \begin{array}{c} \boxed{[L1]} \text{.....} \boxed{[L2]} \\ \underbrace{\hspace{10em}} \\ V \\ \underbrace{\hspace{10em}} \\ W \end{array},$$

where  $m_1 = |W|^{(1)}$  and  $m_2 = |W|^{(2)}$  such that  $|W| = m_1 + m_2$ .

**Definition 4.9.** Suppose that  $\pi \in NC_{l_1, \dots, l_p}(2k)$ . Then  $\pi = \{V_1, \dots, V_p\}$  with  $|V_j| = l_j, \forall j = 1, \dots, p$ . Define a map  $\Lambda$  from

$$\bigcup_{k=1}^{\infty} \left( \bigcup_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2k} NC_{l_1, \dots, l_p}(2k) \right)$$

to

$$\bigcup_{n=1}^{\infty} \left( \underbrace{\mathbb{N} \times \dots \times \mathbb{N}}_{n\text{-times}} \right),$$

by the following rules with respect to subblocks **in each outer block** of  $\pi$  ;

(1) Let  $\pi \in NC_{l_1, \dots, l_p}(2k)$  and  $V \in \pi(o)$  and let  $V_1, \dots, V_k, W$  be subblocks in  $V$ . Let  $V_1, V_2, \dots, V_k$  be deepest subblocks in  $W$ , which is outer of  $V_1, \dots, V_k$ . If

$$\begin{array}{c} m_1 \text{.....} m_2 \text{.....} m_3 \dots m_k \text{.....} m_{k+1} \\ \underbrace{\hspace{10em}} \\ V_1 \quad V_2 \quad V_k \\ \underbrace{\hspace{10em}} \\ W \end{array}$$

with

$$|V_1| = n_1, \dots, |V_k| = n_k$$

then the restriction of  $\Lambda$  to  $(V_1, \dots, V_k, W)$  is

$$\Lambda|_{(V_1, \dots, V_k, W)}(\pi) = (m_1, [n_1], m_2, [n_2], m_3, \dots, m_k, [n_k], m_{k+1}).$$

(2) Let  $V \in \pi(o)$ ,  $W \in \pi(i)$  be given as in the step (1). Suppose that the subblock of  $V$ ,  $W_1 \in \pi$  is outer of  $W$  such that

$$\begin{array}{c} p_1 \text{.....} p_2 \\ \underbrace{\hspace{10em}} \\ W \\ \underbrace{\hspace{10em}} \\ W_1 \end{array} \quad \text{with} \quad |W_1| = p_1 + p_2.$$

Then the restriction of  $\Lambda$  to  $(W, W_1)$  is

$$\Lambda|_{(W, W_1)}(\pi) = (p_1, [\Lambda|_{(V_1, \dots, V_k, W)}(\pi)], p_2).$$

If  $W_2$  is outer of  $W_1$ , then, similarly, we can define  $\Lambda|_{(W_1, W_2)}(\pi)$ , inductively. This is the insertion property of the map  $\Lambda$ . Suppose that there are restrictions of the map  $\Lambda$ ,  $\Lambda|_{(V_1^1, \dots, V_{k_1}^1, W^1)}(\pi)$ , ...,  $\Lambda|_{(V_1^l, \dots, V_{k_l}^l, W^l)}(\pi)$  given as in (1) and assume that  $W^1, \dots, W^l$  are inner in  $W_1$  and  $W_1$  is outer of them. Then similar to (1), we have that

$$\begin{aligned} &\Lambda_{(W^1, \dots, W^l, W_1)}(\pi) \\ &= (q_1, [\Lambda|_{(V_1^1, \dots, V_{k_1}^1, W^1)}(\pi)], q_2, \dots, q_l, [\Lambda|_{(V_1^l, \dots, V_{k_l}^l, W^l)}(\pi)], q_{l+1}) \end{aligned}$$

(3) Let  $V^o \in \pi(o)$  be an outer block having the above insertion property (1) and (2). Then the restriction of the map  $\Lambda$  to  $V^o$ ,  $\Lambda|_{V^o}(\pi)$ , is defined like in (2), inductively.

(4) Let  $\pi(o) = \{V_1^o, \dots, V_t^o\}$ . Define the map  $\Lambda$  for  $\pi \in NC_{l_1, \dots, l_p}(2k)$  by

$$\Lambda(\pi) = \Lambda|_{V_1^o}(\pi) \times \dots \times \Lambda|_{V_t^o}(\pi),$$

where the Cartesian product “ $\times$ ” is set-theoretically determined. Recall that each  $\Lambda|_{V_j^o}(\pi)$  has the insertion property as in (2) and (3).

The map  $\Lambda$ , on even noncrossing partitions is called the “(partition-dependent) Numbering” map. Remark that the image of this numbering map contains the number with rectangular bracket  $[\cdot]$  which represents the length of the deepest blocks in the outer blocks of partitions.

**Example 4.1.** Let  $\pi \in NC_{2,4,4,4,4,6}(24)$  be given as follows ;

$$\pi = \underbrace{\underbrace{\circ \circ \circ \circ \circ \circ \circ \circ}_{V_1} \underbrace{\circ \circ \circ \circ \circ \circ}_{V_2}}_{W_1} \underbrace{\underbrace{\circ \circ \circ \circ \circ \circ}_{V_3} \underbrace{\circ \circ \circ \circ \circ \circ}_{V_4}}_{W_2} \circ \circ.$$

Then we can reexpress it by

$$\pi = \boxed{1} \underbrace{\underbrace{\circ \circ \circ \circ \circ \circ \circ \circ}_{V_1} \underbrace{\circ \circ \circ \circ \circ \circ}_{V_2}}_{W_1} \boxed{3} \boxed{2} \underbrace{\underbrace{\circ \circ \circ \circ \circ \circ}_{V_3} \underbrace{\circ \circ \circ \circ \circ \circ}_{V_4}}_{W_2} \boxed{2}$$

and hence, by the map  $\Lambda$ , we have that

$$\Lambda(\pi) = (1, [4], [4], 3) \times (2, (1, [6], 1), 2).$$

**Definition 4.10.** Fix  $k \in \mathbb{N}$  and  $\pi \in NC_{l_1, \dots, l_p}(2k)$ . Let  $(A, E)$  be a NCPSpace over  $B$  and let  $x_0 \in (A, E)$  be a  $B$ -even random variable. Define a map

$$\Psi_{x_0} : \bigcup_{k=1}^{\infty} \left( \bigcup_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2k} NC_{l_1, \dots, l_p}(2k) \right) \rightarrow B$$

by

$$\Psi_{x_0}(\pi) = \Phi_{x_0} \circ \Lambda(\pi) \stackrel{def}{=} \prod_{V \in \pi(o)} \Phi_{x_0}(\Lambda(V)).$$

Remark that  $\Lambda(V)$  satisfies the insertion property and hence  $\Phi_{x_0}(\Lambda(V))$  is also defined by the insertion property. And hence  $\Psi_{x_0}$  is defined by the insertion property.

**Example 4.2.** Let  $\pi \in NC_{2,4,4,4,4,6}(24)$  be given as follows ;

$$\pi = \underbrace{\underbrace{\circ \circ \circ \circ \circ \circ}_{V_1} \underbrace{\circ \circ \circ \circ}_{V_2}}_{W_1} \underbrace{\underbrace{\circ \circ \circ \circ}_{V_3} \underbrace{\circ \circ \circ \circ}_{V_4}}_{W_2}$$

and hence,

$$\begin{aligned} \Psi_{x_0}(\pi) &= (\Psi_{x_0}(W_1)) (\Psi_{x_0}(W_2)) \\ &= (\Phi_{x_0}(\Lambda(W_1))) (\Phi_{x_0}(\Lambda(W_2))) \\ &= (\Phi_{x_0}(1, [4], [4], 3)) (\Phi_{x_0}(2, (1, [6], 1), 2)) \\ &= E(x_0 E(x_0^4) E(x_0^4) x_0^3) \cdot E(x_0^2 E(x_0 E(x_0^6) x_0) x_0^2). \end{aligned}$$

for the (arbitrary) fixed  $B$ -valued random variable  $x_0$  in (some) NCPSpace over  $B$ ,  $(A, E)$ .

Now, let's go back to our problem and observe the following ;

**Theorem 4.30.** Let  $B$ -valued random variables,  $x = a + b + a^{-1} + b^{-1}$  and  $y = c + d + c^{-1} + d^{-1}$  in  $(A_1 *_B A_2, F := E * E)$  be given as before. Then

(1) If  $R_{x+y}^t$  is the trivial  $B$ -valued  $R$ -transform of  $x + y$ , then

$$\begin{aligned} \text{coef}_{2k}(R_{x+y}^t) &= K_{2k}^t((x+y), \dots, (x+y)) \\ &= 2 \sum_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2k} \sum_{\pi \in NC_{l_1, \dots, l_p}(2k)} \mu_\pi \cdot \Psi_x(\pi), \end{aligned}$$

where  $\mu_\pi = \mu(\pi, 1_{2k}) \in \mathbb{C}$ , for all  $\pi \in NC^{(even)}(2k)$ ,  $k \in \mathbb{N}$ .

(2) If  $M_{x+y}^t$  is the trivial  $B$ -valued moment series of  $x + y$ , then

$$\begin{aligned} \text{coef}_{2k}(M_{x+y}^t) &= E((x+y)^{2k}) \\ &= \sum_{\theta \in NC^{(even)}(2k)} \left( 2^{|\theta|} \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_\pi^\theta \cdot \Psi_x(\pi) \right), \end{aligned}$$

where  $\mu_\pi^\theta = \mu(\pi, \theta)$ , for all  $k \in \mathbb{N}$ .

So, to compute  $B$ -valued  $2k$ -th cumulants of  $x$  (and hence to compute  $B$ -valued moments of  $x$ ), it is sufficeint to compute  $\Psi_x(\pi)$ , for each  $\pi \in NC^{(even)}(2k)$ , for  $k \in \mathbb{N}$ .

**Notation** (1) From now, for the convenience of using notations, we will denote

$$E(x^{2k}) = \sum_{n=0}^{a(2k)} \alpha_{4n}^{2k} (h^n + h^{-n}) \in B,$$

where  $\alpha_0^{2k} = p_0^{2k}$ ,  $\alpha_4^{2k} = p_4^{2k}$ ,  $\alpha_8^{2k} = p_8^{2k}$ , ...,  $\alpha_{4n}^{2k} = p_{4n}^{2k}$ , .... Moreover, we will denote

$$(h^0 + h^{-0}) \stackrel{\text{denote}}{\equiv} e.$$

Remark that the above equality "≡" is just mean the notation, not equality ! Also, we will denote that

$$a(2k) = \begin{cases} \frac{k}{2} & \text{if } 4 \mid 2k \\ \frac{k-1}{2} & \text{if } 4 \nmid 2k. \end{cases}$$

(2) Let  $m \in \mathbb{N}$ . By Section 3.2.2, we know that

$$x^m = \begin{cases} X_m + p_{m-2}^m X_{m-2} + \dots + p_2^m X_2 + p_0^m e & \text{if } m \in 2\mathbb{N} \\ X_m + q_{m-2}^m X_{m-2} + \dots + q_3^m X_3 + q_1^m X_1 & \text{if } m \in 2\mathbb{N} - 1 \end{cases}$$

We will denote  $x^m$ , at once, by

$$x^m = \sum_{n=0}^m \beta_n^m X_n,$$

where

$$\beta_n^m = \begin{cases} p_n^m & \text{if } m \in 2\mathbb{N} \\ q_n^m & \text{if } m \in 2\mathbb{N} - 1 \\ 0 & \text{if } n \neq m - 2j. \end{cases}$$

Notice that if  $n \neq m - 2j$ , then  $\beta_n^m = 0$  in  $\mathbb{C}$ .  $\square$

Recall that, there are two recurrence relations for computing  $E(x^{2k})$ , the first is the case when  $4 \mid 2k$  and the second is the case when  $4 \nmid 2k$ . The above notation is used because we want to consider  $E(x^{2k})$ , at once. i.e we want to avoid the situation where we have to observe case-by-case.

**Proposition 4.31.** *Let  $m_1, \dots, m_n \in \mathbb{N}$  and  $d_2, \dots, d_n \in \mathbb{N} \cup \{0\}$ . Then*

$$\begin{aligned} & E(x^{m_1}(h^{d_2} + h^{-d_2})x^{m_2}(h^{d_3} + h^{-d_3})x^{m_3} \dots (h^{d_n} + h^{-d_n})x^{m_n}) \\ &= \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} \dots \sum_{i_n=0}^{m_n} (\beta_{i_1}^{m_1} \beta_{i_2}^{m_2} \beta_{i_3}^{m_3} \dots \beta_{i_n}^{m_n}) \\ & \quad \left( \sum_{r_2 \in \{\pm d_2\}, r_3 \in \{\pm d_3\}, \dots, r_n \in \{\pm d_n\}} \Phi([m_1, [4r_2], m_3, [4r_3], \dots, [4r_n], m_n]) \right). \end{aligned}$$

**Theorem 4.32.** *Let  $k \in \mathbb{N}$ . Let  $l_1, \dots, l_p \in 2\mathbb{N}$  and  $l_1 + \dots + l_p = 2k$ . Let  $\pi \in NC_{l_1, \dots, l_p}(2k)$  and  $V \in \pi$ . If  $V_1, \dots, V_n$  are deepest subblocks in  $V$  which is outer of  $V_1, \dots, V_n$  and if we have*

$$\Lambda |_{(V_1, \dots, V_n, V)} (\pi) = (m_1, [l_{j_2}], m_2, [l_{j_3}], m_3, \dots, [l_{j_n}], m_n),$$

where  $l_{j_2}, \dots, l_{j_n} \in \{l_1, \dots, l_p\}$  and  $m_1, \dots, m_n \in \mathbb{N} \cup \{0\}$ , then the restriction of  $\Psi_x$  to  $(V_1, \dots, V_n, V)$  is

$$\begin{aligned} \Psi_x |_{(V_1, \dots, V_n, V)} (\pi) &= \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \dots \sum_{n_n=0}^{m_n} \sum_{i_2=0}^{a(l_{j_2})} \sum_{i_3=0}^{a(l_{j_3})} \dots \sum_{i_n=0}^{a(l_{j_n})} \\ & \quad (\beta_{n_1}^{m_1} \beta_{n_2}^{m_2} \dots \beta_{n_n}^{m_n}) \left( \alpha_{4i_2}^{l_{j_2}} \alpha_{4i_3}^{l_{j_3}} \dots \alpha_{4i_n}^{l_{j_n}} \right) \\ & \quad \left( \sum_{r_2 \in \{\pm i_2\}, r_3 \in \{\pm i_3\}, \dots, r_n \in \{\pm i_n\}} \Phi(n_1, 4r_2, n_3, 4r_3, \dots, 4r_n, n_n) \right). \end{aligned}$$

*Proof.* Fix  $\pi \in NC_{l_1, \dots, l_p}(2k)$  and let  $V^o \in \pi(o)$  and assume that  $\Lambda(V^o) = (m_1, [l_{j_2}], m_2, \dots, [l_{j_n}], m_n)$ , where  $1 \leq j_1 < j_2 < \dots < j_n \leq p$ . Then

$$\begin{aligned}
 \Psi_x |_{(V_1, \dots, V_n, V)}(\pi) &= \Phi_x \circ \Lambda |_{(V_1, \dots, V_n, V)}(\pi) \\
 &= E(x^{m_1} E(x^{l_{j_2}}) x^{m_2} E(x^{l_{j_3}}) x^{m_3} \dots E(x^{l_{j_n}}) x^{m_n}) \\
 &= E\left(\left(\sum_{n_1=0}^{m_1} \beta_{n_1}^{m_1} X_{n_1}\right) \left(\sum_{i_2=0}^{a(l_{j_2})} \alpha_{4i_2}^{l_{j_2}} (h^{i_2} + h^{-i_2})\right) \right. \\
 &\quad \left. \left(\sum_{n_2=0}^{m_2} \beta_{n_2}^{m_2} X_{n_2}\right) \left(\sum_{i_3=0}^{a(l_{j_3})} \alpha_{4i_3}^{l_{j_3}} (h^{i_3} + h^{-i_3})\right) \left(\sum_{n_3=0}^{m_3} \beta_{n_3}^{m_3} X_{n_3}\right) \right. \\
 &\quad \left. \dots \left(\sum_{i_n=0}^{a(l_{j_n})} \alpha_{4i_n}^{l_{j_n}} (h^{i_n} + h^{-i_n})\right) \left(\sum_{n_n=0}^{m_n} \beta_{n_n}^{m_n} X_{n_n}\right)\right) \\
 &= \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \dots \sum_{n_n=0}^{m_n} \sum_{i_2=0}^{a(l_{j_2})} \sum_{i_3=0}^{a(l_{j_3})} \dots \sum_{i_n=0}^{a(l_{j_n})} \\
 &\quad \left(\beta_{n_1}^{m_1} \beta_{n_2}^{m_2} \dots \beta_{n_n}^{m_n}\right) \left(\alpha_{4i_2}^{l_{j_2}} \alpha_{4i_3}^{l_{j_3}} \dots \alpha_{4i_n}^{l_{j_n}}\right) \\
 &\quad E(X_{n_1}(h^{i_2} + h^{-i_2}) X_{n_2}(h^{i_3} + h^{-i_3}) X_{n_3} \dots (h^{i_n} + h^{-i_n}) X_{n_n}) \\
 &= \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \dots \sum_{n_n=0}^{m_n} \sum_{i_2=0}^{a(l_{j_2})} \sum_{i_3=0}^{a(l_{j_3})} \dots \sum_{i_n=0}^{a(l_{j_n})} \\
 &\quad \left(\beta_{n_1}^{m_1} \beta_{n_2}^{m_2} \dots \beta_{n_n}^{m_n}\right) \left(\alpha_{4i_2}^{l_{j_2}} \alpha_{4i_3}^{l_{j_3}} \dots \alpha_{4i_n}^{l_{j_n}}\right) \\
 &\quad \left(\sum_{r_2 \in \{\pm i_2\}, r_3 \in \{\pm i_3\}, \dots, r_n \in \{\pm i_n\}} \Phi(n_1, 4r_2, n_3, 4r_3, \dots, 4r_n, n_n)\right) \cdot \blacksquare
 \end{aligned}$$

**Proposition 4.33.** *Let  $k \in \mathbb{N}$  and let  $\pi \in NC^{(even)}(2k)$ . If  $x = a + b + a^{-1} + b^{-1} \in A_1 *_B A_2$  is our  $B$ -valued random variable, then*

$$\widehat{E}(\pi)(x \otimes \dots \otimes x) = \Psi_x(\pi).$$

□

The above proposition is proved by only using the definitions of partition-dependent  $B$ -moments of  $x$  and of  $\Psi_{x_0}$ . Now, we will put all our information ;

**Theorem 4.34.** *Let  $k \in \mathbb{N}$  and let  $x = a + b + a^{-1} + b^{-1}$  and  $y = c + d + c^{-1} + d^{-1}$  be given as before. Then*

$$\begin{aligned}
 &K_{2k}^t((x + y), \dots, (x + y)) \\
 &= 2 \cdot \sum_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2k} \sum_{\pi \in NC_{l_1, \dots, l_p}(2k)} \mu_\pi \cdot \Psi_x(\pi) \\
 &= 2 \cdot \sum_{l_1, \dots, l_p \in 2\mathbb{N}, l_1 + \dots + l_p = 2k} \sum_{\pi \in NC_{l_1, \dots, l_p}(2k)} \mu_\pi \cdot \left( \prod_{V^o \in \pi(o)} (\Psi_{x_0} |_{V^o}(\pi)) \right),
 \end{aligned}$$

by the previous proposition.  $\square$

**Theorem 4.35.** *Let  $k \in \mathbb{N}$  and let  $x = a + b + a^{-1} + b^{-1}$  and  $y = c + d + c^{-1} + d^{-1}$  be given as before. Then*

$$\text{coef}_{2k}(M_{x+y}^t) = \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_\pi^\theta \cdot \Psi_x(\pi).$$

*Proof.* Fix  $k \in \mathbb{N}$ . Then we already observed that

$$E((x+y)^{2k}) = \sum_{\theta \in NC^{(even)}(2k)} \widehat{C}_E(\theta)((x+y) \otimes \dots \otimes (x+y))$$

where  $\widehat{C} = (C^{(n)})_{n=1}^\infty \in I(L(F_2), L(F_1))^c$  is the cumulant multiplicative bimodule map induced by  $E: L(F_2) \rightarrow L(F_1)$

$$= \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \widehat{C}(\theta)(x \otimes x \otimes \dots \otimes x)$$

by the  $B$ -freeness of  $x$  and  $y$  and by the identically  $B$ -distributedness of  $x$  and  $y$

$$= \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \left( \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \widehat{E}(\pi)(x \otimes x \otimes \dots \otimes x) \mu_\pi^\theta \right)$$

where  $\mu_\pi^\theta = \mu(\pi, \theta)$ .  $\blacksquare$

**Remark 4.1.** *We have that*

$$K_{2k}^t((x+y), \dots, (x+y)) = 2 \cdot \sum_{\pi \in NC^{(even)}(2k)} \mu_\pi \cdot \Psi_x(\pi)$$

and

$$E((x+y)^{2k}) = \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_\pi^\theta \cdot \Psi_x(\pi),$$

for all  $k \in \mathbb{N}$ , where  $\Psi_x = \Phi_x^0 \circ \Lambda$  such that

$$\Psi_x(\pi) = \widehat{E}(\pi)(x \otimes \dots \otimes x), \text{ for all } \pi \in NC^{(even)}(2k).$$



**Remark 4.2.** *The above theorems says that the  $B$ -valued moments and cumulants of  $x + y$  are determined by certain recurrence relations, introduced in Section 4.2, which are depending on partitions and images of numbering map of partitions.*

#### 4.4. The Trivial R-transform of $x + y$ and The Trivial Moment Series of $x + y$ .

By the previous section, we can conclude that

**Corollary 4.36.** *Let  $x = a + b + a^{-1} + b^{-1}$  and  $y = c + d + c^{-1} + d^{-1}$  be given as before. Then*

$$(1) R_{x+y}^t(z) = \sum_{k=1}^{\infty} \left( 2 \cdot \sum_{\pi \in NC^{(even)}(2k)} \mu_{\pi} \cdot \Psi_x(\pi) \right) z^{2k}$$

$$(2) M_{x+y}^t(z) = \sum_{k=1}^{\infty} \left( \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \Psi_x(\pi) \right) z^{2k},$$

where, for the fixed  $\pi \in NC_{l_1, \dots, l_p}(2k) \subset NC^{(even)}(2k)$ ,

$$\Psi_x(\pi) = \Phi_x \circ \Lambda(\pi).$$

□

## 5. SCALAR-VALUED MOMENTS OF $x + y$ IN $(L(F_2) *_{L(F_1)} L(F_2),)$

### 5.1. Scalar-Valued Moments of $x + y$ .

In this section, finally, we will compute the scalar-valued moment,

$$\varphi((x + y)^n), \quad \text{for } n \in \mathbb{N}.$$

In Section 2.6, we showed that  $x + y$  is even random variable in  $(A_1 *_B A_2, \varphi)$ . More generally, if  $a \in (A, E)$  is a  $B$ -even and if  $(A, E)$  and  $(A, \varphi)$  are compatible, then  $a$  is (scalar-valued) even in  $(A, \varphi)$ , where  $B$  is a unital algebra and  $A$  is an algebra over  $B$  (See [15]). So, it suffices to consider

$$\varphi((x+y)^{2k}),$$

for all  $k \in \mathbb{N}$ , furthermore, by the compatibility, we have that

$$\varphi((x+y)^{2k}) = \varphi(E((x+y)^{2k})), \text{ for all } k \in \mathbb{N}.$$

In the previous section, we showed that

$$R_{x+y}^t(z) = 2R_x^t(z) = \sum_{k=1}^{\infty} \left( 2 \cdot \sum_{\pi \in NC^{(even)}(2k)} \mu_{\pi} \cdot \Psi_x(\pi) \right) z^{2k}$$

and hence

$$M_{x+y}^t(z) = \sum_{k=1}^{\infty} \left( \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu(\pi, \theta) \cdot \Psi_x(\pi) \right) z^{2k}.$$

By the compatibility of  $(A_1 *_B A_2, E)$  and  $(A_1 *_B A_2, \varphi)$ , we have that

$$\varphi((x+y)^n) = \varphi(E((x+y)^n)), \text{ for all } n \in \mathbb{N}.$$

By the  $B$ -evenness of  $x$  and  $y$ , since  $x$  and  $y$  are  $B$ -even,  $x+y$  is also  $B$ -even. Hence  $x+y$  is (scalar-valued) even, too (See Section 2.6). Thus we have that

$$\varphi((x+y)^n) = 0, \text{ whenever } n \in 2\mathbb{N} - 1.$$

So, we need to observe  $\varphi((x+y)^{2k})$ , for all  $k \in \mathbb{N}$ . Fix  $k \in \mathbb{N}$ . Then

$$\begin{aligned} \varphi((x+y)^{2k}) &= \varphi(E((x+y)^{2k})) \\ &= \varphi \left( \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \Psi_x(\pi) \right) \end{aligned}$$

(5.1)

$$= \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \varphi(\Psi_x(\pi)).$$

**Theorem 5.1.** *Let  $x = a+b+a^{-1}+b^{-1}$  and  $y = c+d+c^{-1}+d^{-1}$  in  $(L(F_2) *_L(F_1) L(F_2), F)$ , where  $F_2 = \langle a, b \rangle = \langle c, d \rangle$  and  $F_1 = \langle h = aba^{-1}b^{-1} = cdc^{-1}d^{-1} \rangle$ . Then*

$$\varphi((x+y)^n) = 0, \text{ whenever } n \in 2\mathbb{N} - 1$$

and

$$\varphi((x+y)^{2k}) = \sum_{\theta \in NC^{(even)}(2k)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(2k), \pi \leq \theta} \mu_\pi^\theta \cdot \varphi(\Psi_x(\pi)).$$

□

## 5.2. Examples.

In this section, we will compute the scalar-valued moments,  $\tau((x+y)^3)$  and  $\tau((x+y)^4)$ .

1.  $\tau((x+y)^6)$  ;

By the previous result, we have that

$$\varphi((x+y)^6) = \sum_{\theta \in NC^{(even)}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_\pi^\theta \cdot \varphi(\Psi_x(\pi)).$$

(Step 1) Observe  $NC^{(even)}(6)$  ;

$$NC^{(even)}(6) = NC_{2,2,2}(6) \cup NC_{2,4}(6) \cup NC_6(6).$$

We have that  $NC_6(6) = \{1_6\}$  and

$$NC_{2,4}(6) = \{(1, 2), (3, 4, 5, 6)\}, \{(1, 4, 5, 6), (2, 3)\} \\ \{(1, 2, 5, 6), (3, 4)\}, \{(1, 2, 3, 6), (4, 5)\} \\ \{(1, 2, 3, 4), (5, 6)\}, \{(1, 6), (2, 3, 4, 5)\}.$$

The entries of the above set,  $NC_{2,4}(6)$ , is gotten from the  $\frac{360^\circ \cdot k}{6}$ -anticlockwise-rotations of the circular expression of the first entry  $\{(1, 2), (3, 4, 5, 6)\}$ , where  $k = 0, 1, \dots, 5$ . We will call this fixed entry,  $\{(1, 2), (3, 4, 5, 6)\}$ , a candidate. (Remark that, for instance,  $\{(1, 4, 5, 6), (2, 3)\}$ , can be the entry of  $NC_{2,4}(6)$ , etc.) This candidate can be determined by the relation,  $1_2 + 1_4$ . We have that

$$\begin{aligned} \tau((x+y)^6) &= \sum_{\theta \in NC^{(even)}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\ &= \sum_{\theta \in NC_{2,2,2}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\ &\quad + \sum_{\theta \in NC_{2,4}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\ &\quad + \sum_{\theta \in NC_6(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\theta \in NC_{2,2,2}(6)} 2^{|\theta|} \cdot \mu_{\theta}^{\theta} \cdot \tau(\Psi_x(\pi)) \\
&\quad + \sum_{\theta \in NC_{2,4}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \tau(\Psi_x(\pi)) \\
&\quad + \sum_{\theta \in NC^{(even)}(6)} 2^{16} \cdot \mu_{\theta} \cdot \tau(\Psi_x(\pi)),
\end{aligned}$$

where  $\mu_{\theta}^{\theta} = \mu(\theta, \theta) = 1$  and  $\mu_{\theta} = \mu(\theta, 1_6)$ . Notice that the last line of the above formular is nothing but the 6-th cumulant of  $x + y$ , because that  $2^{16} = 2$  and  $x$  and  $y$  are  $B$ -valued identically distributed.

$$(\text{Step 2}) \text{ Observe } \sum_{\theta \in NC_{2,2,2}(6)} 2^{|\theta|} \cdot \mu_{\theta}^{\theta} \cdot \tau(\Psi_x(\pi));$$

Fix  $\theta \in NC_{2,2,2}(6)$ . If  $\pi \in NC^{(even)}(6)$  satisfies  $\pi \leq \theta$ , then  $\pi = \theta$ , by the ordering on  $NC(6)$ . Therefore,

$$\begin{aligned}
&\sum_{\theta \in NC_{2,2,2}(6)} 2^{|\theta|} \cdot \mu_{\theta}^{\theta} \cdot \tau(\Psi_x(\pi)) \\
&= \sum_{\theta \in NC_{2,2,2}(6)} 2^3 \cdot \tau(\Psi_x(\theta)) = \sum_{\theta \in NC_{2,2,2}(6)} 8 \cdot \tau(E(x^2)^3),
\end{aligned}$$

since  $|\theta| = 3$ , for all  $\theta \in NC_{2,2,2}(6)$ . So,

$$\sum_{\theta \in NC_{2,2,2}(6)} 8 \cdot \tau(\Psi_x(\theta)) = |NC_{2,2,2}(6)| \cdot (8 \cdot (p_0^2)^3).$$

Therefore, since  $|NC_{2,2,2}(6)| = c_2^3 = 5$ ,

$$\begin{aligned}
&\sum_{\theta \in NC_{2,2,2}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \tau(\Psi_x(\pi)) \\
&= 5 \cdot 8 \cdot (p_0^2)^3 = 2560.
\end{aligned}$$

$$(\text{Step 3}) \text{ Observe } \sum_{\theta \in NC_{2,4}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(6), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \tau(\Psi_x(\pi));$$

We know all entries,  $\theta$ , of  $NC_{2,4}(6)$ , from the (Step 1). It is easy to check that if  $\theta \in NC_{2,4}(6)$  and if  $\pi \in NC^{(even)}(6)$  satisfies  $\pi \leq \theta$ , then  $\pi = \theta$  or  $\pi \in NC_{2,2,2}(6)$ . Moreover, each partition  $\theta \in NC_{2,4}(6)$  contains exactly two partitions  $\pi$  in  $NC_{2,2,2}(6)$  such that  $\pi < \theta$ . For example, if  $\theta = \{(1, 2), (3, 4, 5, 6)\}$ , then we have that  $\pi_1 = \{(1, 2), (3, 4), (5, 6)\}$  and  $\pi_2 = \{(1, 2), (3, 6), (4, 5)\}$  in  $NC_{2,2,2}(6)$ .

Similar to (Step 2), we have that

$$\begin{aligned}
&\sum_{\theta \in NC_{2,4}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC_{2,2,2}(6), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \tau(\Psi_x(\pi)) \\
&= (|NC_{2,4}(6)|) \cdot 2^2 \cdot (\tau(\Psi_x(\theta)) - (\tau(\Psi_x(\pi_1)) + \tau(\Psi_x(\pi_2)))) \\
&= (|NC_{2,4}(6)|) \cdot 4 \cdot (\tau(\Psi_x(\theta)) - 2(\tau(\Psi_x(\pi_1)))) ,
\end{aligned}$$

where  $\pi_1, \pi_2 \in NC_{2,2,2}(6)$  such that  $\pi_i \leq \theta$ ,  $\theta \in NC_{2,4}(6)$  is arbitrarily fixed. We can get the last line of the above formular, since all  $\Psi_x(\pi)$ 's are same, for all  $\pi \in NC_{2,2,2}(6)$ . Also, we have that

$$\begin{aligned}\mu_\pi^\theta &= \mu([0_1, 1_1]^2 \times [0_2, 1_2]) \\ &= \mu(0_1, 1_1)^2 \cdot \mu(0_2, 1_2) = (-1)^{2-1} c_2^{2-1} \\ &= -1,\end{aligned}$$

for all pair  $(\pi_i, \theta) \in NC_{2,2,2}(6) \times NC_{2,4}(6)$ , where  $\pi_i \leq \theta$ . Futhermore, by (Step 1), we know all entries of  $NC_{2,4}(6)$ . Hence we can compute each  $\Psi_x(\theta)$ .

$$\Psi_x(\{(1, 2), (3, 4, 5, 6)\}) = \Phi((2)) \Phi((4)) = p_0^2 ((h + h^{-1}) + p_0^4).$$

$$\Psi_x(\{(1, 4, 5, 6), (2, 3)\}) = \Phi(1, [1], 3) = ((h + h^{-1}) + p_0^4) p_0^2.$$

$$\Psi_x(\{(1, 2, 5, 6), (3, 4)\}) = \Phi((2, [1], 2)) = ((h + h^{-1}) + p_0^4) p_0^2.$$

$$\Psi_x(\{(1, 2, 3, 6), (4, 5)\}) = \Phi((3, [1], 1)) = ((h + h^{-1}) + p_0^4).$$

$$\Psi_x(\{(1, 2, 3, 4), (5, 6)\}) = \Phi((4)) \Phi((2)) = ((h + h^{-1}) + p_0^4) p_0^2.$$

$$\begin{aligned}\Psi_x(\{(1, 6), (2, 3, 4, 5)\}) &= \Phi((1, [4], 1)) \\ &= E(x(h + h^{-1})x) + p_0^4 p_0^2 = 0_B + p_0^4 p_0^2 \\ &= p_0^2 p_0^4.\end{aligned}$$

And, for any  $\pi \in NC^{(even)}(6)$  such that  $\pi < \theta$ , (i.e,  $\pi \in NC_{2,2,2}(6)$  !)

$$\mu_\pi = \mu(\pi, \theta) = \mu([0_1, 1_1]^2 \times [0_2, 1_2]) = -1.$$

Therefore,

$$\begin{aligned}(|NC_{2,4}(6)|) \cdot 4 \cdot (\tau(\Psi_x(\theta)) - 2(\tau(\Psi_x(\pi_1)))) \\ = 6 \cdot 4 \cdot (112 - 128) = 24 \cdot (-16) = -384.\end{aligned}$$

i.e,

$$\sum_{\theta \in NC_{2,4}(6)} 2^{|\theta|} \cdot \sum_{\pi \in NC_{2,2,2}(6), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) = -384.$$

(Step 4) Observe  $\sum_{\pi \in NC^{(even)}(6)} \mu_\pi \cdot \tau(\Psi_x(\pi))$  ;

It is easy to check that

$$\begin{aligned} \sum_{\pi \in NC^{(even)}(6)} 2 \cdot \mu_\pi \cdot \tau(\Psi_x(\pi)) &= \sum_{\pi \in NC_{2,2,2}(6)} 2 \cdot \mu_\pi \cdot \tau(\Psi_x(\pi)) \\ &+ \sum_{\pi \in NC_{2,4}(6)} 2 \cdot \mu_\pi \cdot \tau(\Psi_x(\pi)) + 2 \cdot \tau(\Psi_x(1_6)). \end{aligned}$$

Also, it is easy to see that

$$\sum_{\pi \in NC_{2,2,2}(6)} 2 \cdot \mu_\pi \cdot \varphi(\Psi_x(\pi)) = 2 \cdot 2 \cdot \mu_{\pi_1} \tau(\Psi_x(\pi_1)) + 3 \cdot 2 \cdot \mu_{\pi_2} \tau(\Psi_x(\pi_2))$$

since there are two kinds of block structures in  $NC_{2,2,2}(6)$  ; one kind is

$$\{(1, 2), (3, 4), (5, 6)\} \text{ and its rotations}$$

(there are two such partitions) and another kind is

$$\{(1, 2), (3, 6), (4, 5)\} \text{ and its rotations}$$

(there are three such partitions), so, we have that,

$$= 4 \cdot 2 \cdot (p_0^2)^3 + 6 \cdot 1 \cdot (p_0^2)^3 = 8 \cdot 64 + 6 \cdot 64 = 896.$$

Also,

$$\sum_{\pi \in NC_{2,4}(6)} 2 \cdot \mu_\pi \cdot \tau(\Psi_x(\pi)) = 6 \cdot 2 \cdot (-1) \cdot 112 = -1344$$

and

$$2 \cdot \tau(\Psi_x(1_6)) = 2 \cdot p_0^6 = 2 \cdot 232 = 464.$$

Therefore,

$$\sum_{\pi \in NC^{(even)}(6)} 2 \cdot \mu_\pi \cdot \tau(\Psi_x(\pi)) = 16.$$

(Step 5) Add all information ;

$$\tau((x+y)^6) = 2560 - 384 + 16 = 2192.$$

By (Step 1)  $\sim$  (Step 5), we can get that

$$\tau((x+y)^6) = 2192.$$

**Example 5.1.** Let  $x, y \in (L(F_2) *_{L(F_1)} L(F_2), F)$  be  $L(F_1)$ -valued random variables such that  $x = a + b + a^{-1} + b^{-1}$  and  $y = c + d + c^{-1} + d^{-1}$  and let  $E : L(F_2) *_{L(F_1)} L(F_2) \rightarrow L(F_1)$  and  $\tau : L(F_2) *_{L(F_1)} L(F_2) \rightarrow \mathbb{C}$  be the conditional expectation (finding  $h$ -terms) and the canonical trace, respectively. Then

$$\tau((x+y)^6) = \tau(E((x+y)^6)) = 2192.$$

□

**Remark 5.1.** The above result also gotten from the following way ; First, recall that  $L(F_2) *_{L(F_1)} L(F_2) \simeq L(F_2 *_{F_1} F_2)$ . Also, we can regard the group  $F_2 *_{F_1} F_2$  as a (topological) fundamental group of torus with genus 2,

$$G = \langle a, b, c, d : aba^{-1}b^{-1}d^{-1}c^{-1}dc = e \rangle .$$

(Remember that  $aba^{-1}b^{-1} = cdc^{-1}d^{-1}$  is our  $h$  !) We need to recognize that  $aba^{-1}b^{-1}d^{-1}c^{-1}dc$  is a word with length 8, without considering the relation in the group  $G$ . Again, denote  $a + b + a^{-1} + b^{-1}$  and  $c + d + c^{-1} + d^{-1}$  by  $x$  and  $y$ , respectively. Now, define the following trace

$$\tau_4 : L(F_4) \rightarrow \mathbb{C}$$

by

$$\tau_4 \left( \sum_{g \in F_4} \alpha_g g \right) = \alpha_{e_{F_4}},$$

where  $F_4 = \langle a, b, c, d \rangle$ . Notice that

$$\tau_4((x+y)^6) = \tau((x+y)^6),$$

because, in both cases, we cannot make the words with length 8 in  $(x+y)^6$ . (Of course, in our case, the word  $aba^{-1}b^{-1}d^{-1}c^{-1}dc$  is  $e$ , but this can be come from making the words with length 8 !) Now, let's compute  $\tau_4((x+y)^6)$ . This can be computed by using the method introduced in [35], as follows ; this method is also used in Section 3.3.

$$\begin{aligned} \tau_4((x+y)^6) &= \tau((a+b+a^{-1}+b^{-1}+c+d+c^{-1}+d^{-1})^6) \\ &= \tau_4((X_1)^6), \end{aligned}$$

where  $X_1 =$  the sum of length 1 words in  $L(F_4)$ . We have the following recurrence relations, by [35] ;

$$X_1 X_1 = X_2 + 8e$$

and

$$X_1 X_N = X_{N+1} + 7X_{N-1}, \text{ for all } N \geq 2.$$

So, to get  $\tau_4((x+y)^6) = \tau_4((X_1)^6)$ , we need to compute that

$$\begin{aligned} (x+y)^2 &= (X_1)^2 = X_1X_1 = X_2 + 8e, \\ (x+y)^3 &= (X_1)^3 = X_1(X_2 + 8e) = X_1X_2 + 8X_1 \\ &= (X_3 + 7X_1) + 8X_1 = X_3 + 15X_1, \\ (x+y)^4 &= (X_1)^4 = X_1(X_3 + 15X_1) = X_1X_3 + 15X_1X_1 \\ &= X_4 + 7X_2 + 15(X_2 + 8e) \\ &= X_4 + 22X_2 + 120e, \\ (x+y)^5 &= (X_1)^5 = X_1(X_4 + 22X_2 + 120e) \\ &= X_5 + 29X_3 + 274X_1, \end{aligned}$$

and

$$\begin{aligned} (x+y)^6 &= (X_1)^6 = X_1(X_5 + 29X_3 + 274X_1) \\ &= X_6 + 36X_4 + 203X_2 + 274X_2 + 2192e. \end{aligned}$$

Thus, we have that

$$\begin{aligned} \tau_4((x+y)^6) &= \tau_4(X_6 + 36X_4 + 203X_2 + 274X_2 + 2192e) \\ &= 2192. \end{aligned}$$

Therefore, we can conclude that

$$\tau((x+y)^6) = 2192 = \tau_4((x+y)^6).$$

2.  $\tau((x+y)^8)$ ; by Section 4.4, we have that

$$\tau((x+y)^8) = \sum_{\theta \in NC^{(even)}(8)} 2^{|\theta|} \cdot \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)).$$

By the separation of  $NC^{(even)}(8)$ , we have that

$$NC^{(even)}(8) = NC_{2,2,2,2}(8) \cup NC_{2,2,4}(8) \cup NC_{2,6}(8) \cup NC_{4,4}(8) \cup \{1_8\}.$$

Therefore,

$$\begin{aligned} &\tau((x+y)^8) \\ &= \sum_{\theta \in NC_{2,2,2,2}(8)} 2^4 \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \end{aligned}$$



$$\begin{aligned}
& + \sum_{\theta \in NC_{2,2,4}(8)} 2^3 \cdot \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\
& + \sum_{\theta \in NC_{2,6}(8)} 2^2 \cdot \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\
& + \sum_{\theta \in NC_{4,4}(8)} 2^2 \cdot \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\
& + 2 \cdot K_8^t(x, \dots, x) \\
& = \sum_{\theta \in NC_{2,2,2,2}(8)} (16) \cdot \mu_\theta^\theta \cdot \tau(\Psi_x(\theta))
\end{aligned}$$

(since there is no  $\pi \in NC^{(even)}(8)$  such that  $\pi \not\leq \theta$ , for  $\theta \in NC_{2,2,2,2}(8)$ )

$$+ \sum_{\theta \in NC_{2,2,4}(8)} (8) \cdot (\mu_\theta^\theta \tau(\Psi_x(\theta)) + \mu_{\pi_1}^\theta \tau(\Psi_x(\pi_1)) + \mu_{\pi_2}^\theta \tau(\Psi_x(\pi_2)))$$

(for each given  $\theta \in NC_{2,2,4}(8)$ , we have only two proper partitions  $\pi_1, \pi_2$  such that  $\pi_i \not\leq \theta$ ,  $i = 1, 2$ )

$$\begin{aligned}
& + \sum_{\theta \in NC_{2,6}(8)} (4) \cdot (\mu_\theta^\theta \tau(\Psi_x(\theta)) + 6 \cdot (\mu_\pi^\theta \tau(\Psi_x(\pi))) \\
& \quad + (2 \cdot (\mu_{\pi_1}^\theta \tau(\Psi_x(\pi_1))) + 3 \cdot (\mu_{\pi_2}^\theta \tau(\Psi_x(\pi_2))))))
\end{aligned}$$

(for each given  $\theta \in NC_{2,6}(8)$ , like in the **step 5** in **1.**, we have the proper partitions  $\pi \in NC_{2,2,4}(8)$  and  $\pi_1, \pi_2 \in NC_{2,2,2,2}(8)$ . Notice that  $\pi_1$  and  $\pi_2$  have different type of pairings in  $NC_{2,2,2}(6) \leftrightarrow NC_{2,2,2,2}(8)$ )

$$\begin{aligned}
& + \sum_{\theta \in NC_{4,4}(8)} (4) \cdot \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_\pi^\theta \cdot \tau(\Psi_x(\pi)) \\
& + 2 \cdot \sum_{\theta \in NC^{(even)}(8)} \mu_\theta^{1_8} \cdot \tau(\Psi_x(\theta))
\end{aligned}$$

**(Step 1)** Compute

$$\sum_{\theta \in NC_{2,2,2,2}(8)} (16) \cdot \mu_\theta^\theta \cdot \tau(\Psi_x(\theta)) = \sum_{\theta \in NC_{2,2,2,2}(8)} (16) \cdot \tau(\Psi_x(\theta))$$

since  $\mu_\theta^\theta = 1 \in \mathbb{C}$

$$= (14)(16)\varphi(E(x^2)^4) = (14)(16)(256) = 57344.$$

**(Step 2)** Compute

$$\begin{aligned}
& \sum_{\theta \in NC_{2,2,4}(8)} (8) \cdot (\mu_{\theta}^{\theta} \tau(\Psi_x(\theta)) + \mu_{\pi_1}^{\theta} \tau(\Psi_x(\pi_1)) + \mu_{\pi_2}^{\theta} \tau(\Psi_x(\pi_2))) \\
&= \sum_{\theta \in NC_{2,2,4}(8)} (8) \cdot (\tau(\Psi_x(\theta)) - \tau(\Psi_x(\pi_1)) - \tau(\Psi_x(\pi_2))) \\
&= \sum_{\theta \in NC_{2,2,4}(8)} (8) \cdot ((p_0^2 p_0^2 p_0^4) - (p_0^2)^4 - (p_0^2)^4) \\
&= \sum_{\theta \in NC_{2,2,4}(8)} (8) \cdot (448 - 256 - 256) = \sum_{\theta \in NC_{2,2,4}(8)} (8) \cdot (-64) \\
&= \sum_{\theta \in NC_{2,2,4}(8)} (-512) = (28) \cdot (-512) \\
&= -14336.
\end{aligned}$$

We can get that  $|NC_{2,2,4}(8)| = 28$ .

**(Step 3)** Compute

$$\begin{aligned}
& \sum_{\theta \in NC_{2,6}(8)} (4) \cdot (\mu_{\theta}^{\theta} \tau(\Psi_x(\theta)) + 6 \cdot (\mu_{\pi}^{\theta} \tau(\Psi_x(\pi))) \\
& \quad + (2 \cdot (\mu_{\pi_1}^{\theta} \tau(\Psi_x(\pi_1))) + 3 \cdot (\mu_{\pi_2}^{\theta} \tau(\Psi_x(\pi_2)))))) \\
&= \sum_{\theta \in NC_{2,6}(8)} (4) \cdot (\tau(\Psi_x(\theta)) - 6 \cdot \tau(\Psi_x(\pi)) \\
& \quad + 2 \cdot 2 \cdot \tau(\Psi_x(\pi_1)) + 3 \cdot 1 \cdot \tau(\Psi_x(\pi_2))) \\
&= \sum_{\theta \in NC_{2,6}(8)} (4) \cdot (p_0^2 p_0^6 - 6 \cdot p_0^2 p_0^2 p_0^4 + 4 \cdot (p_0^2)^4 + 3 \cdot (p_0^2)^4) \\
&= \sum_{\theta \in NC_{2,6}(8)} (4) \cdot (928 - 2688 + 1024 + 768) \\
&= \sum_{\theta \in NC_{2,6}(8)} (4) \cdot (32) = \sum_{\theta \in NC_{2,6}(8)} (128) \\
&= 8 \cdot (128) = 1024.
\end{aligned}$$

**(Step 4)** Compute

$$\begin{aligned}
& \sum_{\theta \in NC_{4,4}(8)} (4) \cdot \sum_{\pi \in NC^{(even)}(8), \pi \leq \theta} \mu_{\pi}^{\theta} \cdot \tau(\Psi_x(\pi)) \\
&= \sum_{\theta \in NC_{4,4}(8)} (4) \cdot (\mu_{\theta}^{\theta} \cdot \tau(\Psi_x(\theta)) + \mu_{\pi_1}^{\theta} \cdot \tau(\Psi_x(\pi_1)) \\
& \quad + \mu_{\pi_2}^{\theta} \cdot \tau(\Psi_x(\pi_2)) + \mu_{\pi_3}^{\theta} \cdot \tau(\Psi_x(\pi_3)) \\
& \quad + \mu_{\pi_4}^{\theta} \cdot \tau(\Psi_x(\pi_4)) + \mu_{\pi_5}^{\theta} \cdot \tau(\Psi_x(\pi_5)) \\
& \quad + \mu_{\pi_6}^{\theta} \cdot \tau(\Psi_x(\pi_6)) + \mu_{\pi_7}^{\theta} \cdot \tau(\Psi_x(\pi_7)))
\end{aligned}$$

$$+\mu_{\pi_8}^\theta \cdot \tau(\Psi_x(\pi_6)),$$

since for each  $\theta \in NC_{4,4}(8)$ , there are proper partitions  $\pi_1, \pi_2, \pi_3, \pi_4 \in NC_{2,2,4}(8)$  and  $\pi_5, \pi_6, \pi_7, \pi_8 \in NC_{2,2,2,2}(8)$ .

$$\begin{aligned} &= \sum_{\theta \in NC_{4,4}(8)} (4) \cdot ((p_0^4)^2 + 2) - p_0^2 p_0^2 p_0^4 \\ &\quad - p_0^2 p_0^2 p_0^4 - p_0^2 p_0^2 p_0^4 \\ &\quad - p_0^2 p_0^2 p_0^4 + (p_0^2)^4 \\ &\quad + (p_0^2)^4 + (p_0^2)^4 \\ &\quad + (p_0^2)^4, \\ &= \sum_{\theta \in NC_{4,4}(8)} (4) \cdot (786 + (4)(-448) + 4(256)) \\ &= \sum_{\theta \in NC_{4,4}(8)} (4) \cdot (786 - 1792 + 1024) = \sum_{\theta \in NC_{4,4}(8)} (4)(18) \\ &= \sum_{\theta \in NC_{4,4}(8)} (72) = (4)(72) \\ &= 288. \end{aligned}$$

**(Step 5)** Compute

$$\begin{aligned} &2 \cdot \sum_{\theta \in NC^{(even)}(8)} \mu_\theta^{1s} \cdot \tau(\Psi_x(\theta)) \\ &= 2 \cdot (2 \cdot (p_0^2)^4 \mu(0_4, 1_4) + 8 \cdot (p_0^2)^4 \mu(0_2, 1_2) \mu(0_3, 1_3) \\ &\quad + 4 \cdot (p_0^2)^4 (\mu(0_2, 1_2))^3) \\ &\quad + 2 \cdot (8 \cdot (p_0^2)^2 (p_0^4) \mu(0_3, 1_3) + 4 \cdot (p_0^2)^2 (p_0^4) (\mu(0_2, 1_2))^2 \\ &\quad + 8 \cdot (p_0^2)^2 (p_0^4) (\mu(0_2, 1_2))^2 \\ &\quad + 8 \cdot (p_0^2)^2 (p_0^4) (\mu(0_2, 1_2))^2) \\ &\quad + 2 \cdot (8 \cdot p_0^2 p_0^6 \mu(0_2, 1_2)) \\ &\quad + 2 \cdot \left( \sum_{\theta \in NC_{4,4}(8)} \mu_\theta^{1s} \cdot \varphi(\Psi_x(\theta)) \right) \\ &\quad + 2 \cdot p_0^8 \\ &= 2(-2560 - 4096 - 1024) + 2(7168 + 1792 + 3584 + 3584) \\ &\quad + 2(-7424) + 2\left( \sum_{\theta \in NC_{4,4}(8)} \mu_\theta^{1s} \cdot \varphi(\Psi_x(\theta)) \right) + 2(2092) \\ &= -15360 + 32256 - 14848 \end{aligned}$$

$$\begin{aligned}
& +2 \left( \sum_{\theta \in NC_{4,4}(8)} \mu_{\theta}^{1s} \cdot \tau(\Psi_x(\theta)) \right) + 4184 \\
& = 2 \left( \sum_{\theta \in NC_{4,4}(8)} \mu_{\theta}^{1s} \cdot \tau(\Psi_x(\theta)) \right) + 6232
\end{aligned}$$

Now, let's compute  $\sum_{\theta \in NC_{4,4}(8)} \mu_{\theta}^{1s} \cdot \tau(\Psi_x(\theta))$ ; we have that

$$\begin{aligned}
NC_{4,4}(8) = \{ & \{(1, 2, 3, 4), (5, 6, 7, 8)\}, \{(1, 6, 7, 8), (2, 3, 4, 5)\} \\
& \{(1, 2, 7, 8), (3, 4, 5, 6)\}, \{(1, 2, 3, 8), (4, 5, 6, 7)\}\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Psi_x(\{(1, 2, 3, 4), (5, 6, 7, 8)\}) &= \Psi_x(1, 2, 3, 4) \Psi_x(5, 6, 7, 8) \\
&= E(x^4) \cdot E(x^4) = ((h + h^{-1}) + p_0^4) ((h + h^{-1}) + p_0^4) \\
&= (h + h^{-1})^2 + 2p_0^4(h + h^{-1}) + (p_0^4)^2 \\
&= h^2 + 2e + h^{-2} + 2p_0^4(h + h^{-1}) + 784e \\
&= h^2 + h^{-2} + 2p_0^4(h + h^{-1}) + 786e,
\end{aligned}$$

$$\begin{aligned}
\Psi_x(\{(1, 6, 7, 8), (2, 3, 4, 5)\}) &= \Psi_x(1, [4], 3) \\
&= ((h^2 + h^{-2}) + 2e) + 784e \\
&= (h^2 + h^{-2}) + 786e,
\end{aligned}$$

$$\begin{aligned}
\Psi_x(\{(1, 2, 7, 8), (3, 4, 5, 6)\}) &= \Psi_x(2, [4], 2) \\
&= (h^2 + h^{-2}) + 786e
\end{aligned}$$

and

$$\begin{aligned}
\Psi_x(\{(1, 2, 3, 8), (4, 5, 6, 7)\}) &= \Psi_x(3, [4], 1) \\
&= (h^2 + h^{-2}) + 786e.
\end{aligned}$$

Notice that, for any  $\pi \in NC_{4,4}(8)$ , we have that

$$\mu_{\pi}^{1s} = \mu(0_1, 1_1)^6 \mu(0_2, 1_2) = -1.$$

Thus,

$$\begin{aligned}
\sum_{\theta \in NC_{4,4}(8)} \mu_{\theta}^{1s} \cdot \tau(\Psi_x(\theta)) &= -\tau(h + h^{-1} + 2p_0^4(h + h^{-1}) + 786e) \\
&\quad -3 \cdot \tau((h^2 + h^{-2}) + 786e) \\
&= (-4)(786) = -3144.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& 2 \cdot \sum_{\theta \in NC^{(even)}(8)} \mu_\theta^{1s} \cdot \tau(\Psi_x(\theta)) \\
&= 2 \left( \sum_{\theta \in NC_{4,4}(8)} \mu_\theta^{1s} \cdot \tau(\Psi_x(\theta)) \right) + 6232 \\
&= 2(-3144) + 6232 \\
&= -6288 + 6232 = -56
\end{aligned}$$

(Step 6) By (Step 1)  $\sim$  (Step 5), we can conclude that

$$\begin{aligned}
\tau((x+y)^8) &= \tau(E((x+y)^8)) \\
&= 57344 + (-14336) + 1024 + 288 + (-56) \\
&= 44264.
\end{aligned}$$

**Example 5.2.** Let  $x, y \in L(F_2) *_{L(F_1)} L(F_2)$  be  $L(F_1)$ -valued random variables such that  $x = a + b + a^{-1} + b^{-1}$  and  $y = c + d + c^{-1} + d^{-1}$  and let  $E : L(F_2) *_{L(F_2)} L(F_2) \rightarrow L(F_1)$  and  $\tau : L(F_2) *_{L(F_1)} L(F_2) \rightarrow \mathbb{C}$  be the conditional expectation (finding  $h$ -terms) and the canonical trace, respectively. Then

$$\tau((x+y)^8) = \tau(E((x+y)^8)) = 44256.$$

□

**Remark 5.2.** The above result also gotten from the following way ; First, recall that  $L(F_2) *_{L(F_1)} L(F_2) \simeq L(F_2 *_{F_1} F_2)$ . Also, we can regard the group  $F_2 *_{F_1} F_2$  as a (topological) fundamental group of torus with genus 2,

$$G = \langle a, b, c, d : aba^{-1}b^{-1}d^{-1}c^{-1}dc = e \rangle .$$

(Remember that  $aba^{-1}b^{-1} = cdc^{-1}d^{-1}$  is our  $h$  !) We need to recognize that  $aba^{-1}b^{-1}d^{-1}c^{-1}dc$  is a word with length 8, without considering the relation in the group  $G$ . Again, denote  $a + b + a^{-1} + b^{-1}$  and  $c + d + c^{-1} + d^{-1}$  by  $x$  and  $y$ , respectively. Now, define the following trace

$$\tau_4 : L(F_4) \rightarrow \mathbb{C}$$

by

$$\tau_4 \left( \sum_{g \in F_4} \alpha_g g \right) = \alpha_{e_{F_4}},$$

where  $F_4 = \langle a, b, c, d \rangle$ . (Of course, in our case, the word  $aba^{-1}b^{-1}d^{-1}c^{-1}dc$  is  $e$ , but this can be come from making the words with length 8 !) Now, let's compute  $\tau((x+y)^8)$ . This can be computed by using the method introduced in [35], as follows ; this method is also used in Section 3.3.

$$\tau_4((x+y)^8) = 44284.$$

By the relation

$$aba^{-1}b^{-1}d^{-1}c^{-1}dc = e$$

and

$$c^{-1}d^{-1}cdbab^{-1}a^{-1} = e^{-1} = e,$$

we have to add 16 to  $\tau((x+y)^8)$ . i.e

$$\begin{aligned} \tau((x+y)^8) &= \tau(E((x+y)^8)) \\ &= \tau_4((x+y)^8) + 16 \\ &= 44264. \end{aligned}$$

The above method introduced in the previous remark looks much more easy to compute the moments of  $x+y$ . However, when we deal with the higher degree computation, it is very hard to find the suitable relation for the identity  $e$ .

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