

A Note About Universality Theorem as an Enumerative Riemann-Roch Theorem

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This short note is a supplement of the longer paper [Liu6], in which the author gives an algebraic proof of the following universality theorem.

Theorem 1 *Let $\delta \in \mathbf{N}$ denote the number of nodal singularities. Let L be a $5\delta - 1$ very-ample¹ line bundle on an algebraic surface M , then the number of δ -nodes nodal singular curves in a generic δ dimensional linear sub-system of $|L|$ can be expressed as a universal polynomial (independent to M) of $c_1(L)^2$, $c_1(L) \cdot c_1(M)$, $c_1(M)^2$, $c_2(M)$ of degree δ .*

The finiteness of the “number of δ -nodes nodal singular curves in a generic δ dimensional linear sub-system of $|L|$ ” was proved by Göttsche in [Got] proposition 5.2. Our theorem shows that these numbers are topological invariants of (M, L) .

A weaker form of the above statement has appeared in [Got] based on results in [V] and [KP] for small δ . In his conjecture, Göttsche had assumed the existence of a lower bound m_0 , such that for any very ample L_0 , the statement in the above theorem holds for $L = L_0^{\otimes m}$, with $m \geq m_0$.

The purpose of the note is to address upon the geometric and topological meanings of the universality theorem and its relationship with the well known surface Riemann-Roch formula of algebraic surfaces, to compare the difference of the symplectic and the algebraic approaches in resolving this problem, and list the open problems and conjectures related to the solution of the problem, etc. in a less technical term.

The universality theorem has played an important role in giving an in-depth understanding of Yau-Zaslow conjecture [YZ] (see page 6 also). The universality theorem and the various machineries built up in [Liu1], [Liu3], [Liu4], [Liu5], [Liu6] and [Liu7] are used in [Liu2] to solve the Harvey-Moore conjecture [HM] on the counting rational curves of Calabi-Yau K3 fibrations. The generalization of our universality theorem to higher dimensions, counting divisors in a complete linear system, also bring us new machineries to understand the concept of “number of nodal curves” in higher dimensions.

In section 1 we outline the formulation of the general enumerative problem of a complete linear system which relates the universality theorem with the classical Riemann-Roch theorem. In fact the above universality theorem can be interpreted naturally as the prolongation of the surface Riemann-Roch formula.

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¹For the definition of k -very ample line bundles, please consult [Got]. The $5\delta - 1$ power of very ample line bundles are $5\delta - 1$ very ample.

In section 2, we outline the key ideas of the proof in [Liu6] and address the key conceptual and technical issues the paper has resolved. As the algebraic proof in [Liu6] is slightly lengthy, we hope that the sketch of the basic ideas may help the reader to digest the long paper.

In section 3, we list a few open problems related to the solution of universality theorem.

1 The Geometric Meaning of The Universality Theorem

Recall the well known classical surface Riemann-Roch formula (e.g. [GH] or [Z]) that given an algebraic surface M , the Euler characteristic of an algebraic line bundle L on M satisfies,

$$\begin{aligned}\chi(L) &\equiv h^0(M, L) - h^1(M, L) + h^2(M, L) = \chi(\mathcal{O}_M) + \frac{c_1(L)^2 - c_1(K_M) \cdot c_1(L)}{2} \\ &= \frac{c_1(M)^2 + c_2(M)}{12} + \frac{c_1(L)^2 - c_1(K_M) \cdot c_1(L)}{2}.\end{aligned}$$

When $L = \mathcal{O}_M$, it is reduced to the well known Noether formula $\chi(\mathcal{O}_M) = \frac{c_1(M)^2 + c_2(M)}{12}$.

As a special case of the general Hirzebruch-Riemann-Roch formula [Hir], the above formula can be viewed as the grade two term of the expansion of $Todd(M)ch(L)$, where $Todd(M)$ and $ch(L)$ are the well known Todd class and Chern Character (please consult [F] or [Hir] for their definitions).

On the other hand, a more down-to-earth approach to the surface Riemann-Roch formula involves showing that the holomorphic Euler number $\chi(L)$ can be expressed as a universal degree one polynomial of $c_1(M)^2$, $c_2(M)$, $c_1(M) \cdot c_1(L)$ and $c_1(L)^2$. I.e.

$$\chi(L) = A_1 c_1(M)^2 + A_2 c_2(M) + A_3 c_1(M) \cdot c_1(L) + A_4 c_1(L)^2.$$

The above statement can be viewed as an existence result of the universal polynomial in $c_1(M)^2$, $c_2(M)$, $c_1(L)^2$, $c_1(L) \cdot c_1(M)$ to express the ‘‘topological’’ quantity $\chi(L)$. Once such a polynomial is shown to exist, then we identify the four coefficients A_1, A_2, A_3, A_4 through concrete examples. While there are plenty of choices of algebraic surfaces in identifying these coefficients A_1, A_2, A_3 , and A_4 , a natural choice to identify A_1 and A_2 (i.e. the coefficients in the Noether formula) is to consider $M = K3$ and $M = \mathbf{P}^2$. Then by a simple arithmetic and the knowledge about their irregularities q ($= \frac{b_1}{2}$, geometric genus p_g ($= \frac{b_2^+ - 1}{2}$), we find $\chi(\mathcal{O}_{K3}) = 1 - 0 + 1 = 2 = A_1 \times 0 + A_2 \times 24$, so $A_2 = \frac{1}{12}$. Similarly $\chi(\mathcal{O}_{\mathbf{P}^2}) = 1 - 0 + 0 = A_1 \times c_1(\mathbf{P}^2)^2 + A_2 c_2(\mathbf{P}^2) = 9A_1 + 3A_2$, so $A_1 = \frac{1}{12}$, too. Then one may identify A_3 and A_4 by adopting the polarized manifolds $(\mathbf{P}^2, \mathbf{H}^d)$ and (T^4, L) . By the classical calculation $h^0(\mathbf{P}^2, \mathbf{H}^d) = \frac{d(d+3)}{2}$, $h^1(\mathbf{P}^2, \mathbf{H}^d) = h^2(\mathbf{P}^2, \mathbf{H}^d) = 0$ and $h^0(T^4, L) = \frac{c_1(L)^2}{2}$, $h^1(T^4, L) = h^2(T^4, L) = 0$ for any ample L , the coefficients A_3 and A_4 can be identified with $\frac{1}{2}$.

In this way we have recovered the Riemann-Roch formula,

$$\chi(L) = \frac{1}{12}(c_1(M)^2 + c_2(M)) + \frac{1}{2}(c_1(L)^2 - c_1(L) \cdot c_1(K_M)).$$

Remark 1 *The choices of $\mathbf{P}^2, (\mathbf{P}^2, \mathbf{H}^d), K3$ and (T^4, L) can be justified by the following facts,*

(i). *The dimension formulae of $h^i(\mathbf{P}^2, \mathbf{H}^d)$ are well known classically (Consult for example [Ha] chapter III.5).*

(ii). *The $K3$ and T^4 have trivial canonical bundles ² and therefore have vanishing first Chern classes.*

²therefore $h^i(L) = h^{2-i}(L^*)$ under Serre duality.

Remark 2 The identification of the second term $\frac{c_1(L)^2 - c_1(L) \cdot c_1(K_M)}{2}$ with the expected complex dimension, Gromov-Taubes formula $\frac{C^2 - C \cdot c_1(K_M)}{2}$, (with the substitution $c_1(L) = C$) of moduli space of curves is the starting point of our projects to formulate the algebraic Seiberg-Witten theory exploring the correspondence of surface Riemann-Roch theorem and Taubes' version of Gromov invariant. (see [Liu1], [Liu3] and [Liu6]).

1.1 Linear Systems and Singular Curves

Given a line bundle $L \mapsto M$ over the algebraic surface M , the complete linear system $|L| = \mathbf{P}(H^0(M, L))$ parametrizes the effective curves determined by the non-zero global sections $\in H^0(M, L)$. So we may identify $|L|$ with a projective space parametrizing the linear equivalence classes of effective divisors. When L is not "positive" enough, the dimension $h^0(L)$ depends on the algebraic surface M and L explicitly and has to be investigated individually. On the other hand for sufficiently positive L the Kodaira vanishing theorem implies $h^1(L) = h^2(L) = 0$. Then $h^0(L) = h^0(L) - h^1(L) + h^2(L) = \chi(L)$ is a topological quantity (i.e. independent to the complex/holomorphic structures of (M, L)) and is determined by surface Riemann-Roch formula.

Let L satisfies the condition that $L^* \otimes K_M$ is non-nef. Under such a simplifying condition ⁴ the $h^2(L) = h^0(L^* \otimes K_M) = 0$ by Serre duality and the fact $L^* \otimes K_M$ has a negative degree. The primary invariant one may associate with the projective space $|L|$ is its expected dimension $h^0(L) - h^1(L) - 1 = \chi(L) - 1$, computed by the Riemann-Roch formula.

On the other hand, the well known classical Bertini theorem (see e.g. page 179 of [Ha]) implies that for any very ample (or based point free) L , the generic divisors ⁵ in $|L|$ are smooth and irreducible. Moreover the curves tend to develop singularities when they move to the boundary points of the top stratum of smooth curves. This suggests us to stratify the space $|L|$ into the various strata according to the "topological types" ⁶ of the list of singularities of the divisors (curves) and the unique top dimensional open stratum parametrizes smooth curves in $|L|$.

The intersection pairing $\int_{|L|} \mathbf{H}^{\chi(L)-1} = 1$ can be interpreted as,

- (i). The intersection number associated with the following enumerative problem: How many irreducible smooth curves in $|L|$ are there which pass through $\chi(L) - 1$ generic points in M ?
or equivalently,
- (ii). The number of smooth irreducible curves in a generic 0 dimensional linear sub-system of $|L|$.

Even though the sub-scheme of $|L|$ parametrizing smooth curves through $\chi(L) - 1$ generic points in M may not be reduced and smooth of zero dimension, the intersection number defined by intersection theory [F] is still defined and is equal to 1.

If one is familiar with the definition of algebraic Seiberg-Witten invariant of L , it will be clear that the above number is nothing but the algebraic Seiberg-Witten invariant ⁷ of L , $\mathcal{ASW}(t_L, c_1(L))$.

The Relationship with A Degenerated Case of Donaldson-Thomas Invariant

Consider a closed sub-scheme Z of a smooth algebraic scheme M , there is a associated short exact sequence of the ideal sheaf \mathcal{I}_Z ,

³for example, when $L \otimes K_M^{-1}$ is ample.

⁴This condition appears naturally in the definition of algebraic family Seiberg-Witten invariant.

⁵They are curves when $\dim_{\mathbf{C}} M = 2$. In the paper we use the term "curves" and "divisors" interchangeably.

⁶The topological type here refers to the topological type of the link space in $\mathbf{S}^3 \subset \mathbf{C}^2$ of the isolated singularity.

⁷The $t_L \in T(M)$ insertion in the SW invariant indicates that we have restrict the holomorphic structure of $\mathcal{E}_{c_1(L)}$ to L .

$$0 \mapsto \mathcal{I}_Z \mapsto \mathcal{O}_M \mapsto \mathcal{O}_Z \mapsto 0.$$

The Hilbert scheme is the universal object parametrizing the closed sub-schemes (or equivalently ideal sheaves) with a fixed Hilbert polynomial.

In [DT], [Th], S. Donaldson and R. Thomas considered the moduli space of stable sheaves on Fano or Calabi-Yau 3-fold M and defined a version of \mathbf{Z} valued invariant based on the virtual fundamental class technique of [LiT1]. When one specifies the sheaf rank to 1, the general problem collapses to the definition of the virtual fundamental cycle of Hilbert schemes.

In the same paper [Th], R. Thomas also considered the moduli space of ideal sheaves \mathcal{I}_Z with a fixed determinant ⁸ and defined the Donaldson-Thomas invariant similarly.

These Donaldson-Thomas invariant was compared in a recent paper [MNOP] with the Gromov-Witten invariants for the “local Calabi-Yau” in relating the Donaldson-Thomas invariants with the mysterious “Gopakurama-Vafa invariants”.

When $\dim_{\mathbf{C}} M = 2$ and we consider the codimension one $Z \subset M$, the ideal sheaf \mathcal{I}_Z becomes invertible. In such a case the determinant of \mathcal{I}_Z is $\mathcal{I}_Z \cong \mathcal{O}_M(-Z)$ itself and fixing the determinant corresponds to fixing the linear equivalence class of Z .

Therefore the complete linear system $|L|$ is nothing but the $\dim_{\mathbf{C}} M = 2$ analogue of Donaldson-Thomas moduli space of ideal sheaves with a fixed determinant. \square

1.2 Enumeration of Singular Curves

On the other hand, there is no reason to restrict ourself to enumerating smooth curves in $|L|$ only. Instead one may consider all the (non-compact) strata of singular curves parametrized by the list of topological types of singularities.

It is known that the creation of isolated singularities on an algebraic curve drops the “expected dimension” of the moduli space of curves and the geometric genera of the curves. E.g. in the case of nodal singularities, introducing a single node on the curve drops the “expected dimension” and the genus of the curve by 1.

In this section we let the bold character \mathbf{m} denote a finite list of isolated curve singularities in algebraic surfaces ⁹.

Question 1: Let $d_{\mathbf{m}}(L)$ denote the “expected dimension” of the stratum (which can contain more than one component) of singular curves with the prescribed “topological types” of singularities. How many singular curves are there which pass through the generic $d_{\mathbf{m}}(L)$ -number of points? Or how many singular curves are there in the closure of the stratum which lie in the generic $\chi(L) - 1 - d_{\mathbf{m}}(L)$ dimensional linear sub-system of $|L|$?

At first the question may look ill-posed as

- (a). The stratum of singular curves (fixing the topological type) is usually non-compact.

It can be remedied by adding the compactifying strata to the given stratum ¹⁰. One may compactify it inside $|L|$.

- (b). The sub-scheme of the union of compactified strata parametrizing the \mathbf{m} singular curves in the generic $\dim_{\mathbf{C}} |L| - d_{\mathbf{m}}(L) = \chi(L) - 1 - d_{\mathbf{m}}(L)$ dimensional linear sub-system is “expected” to be regular of zero dimension if all the generic conditions are met. But it may not be regular of zero dimension geometrically.

⁸The determinant of a coherent sheaf is defined by taking the alternating product of determinants of a locally free resolution of \mathcal{I}_Z .

⁹Because our focus is to give a conceptual understanding, we will go into the details of how to express \mathbf{m} in terms of the topology and the singular multiplicities of the list of singularities.

¹⁰this means adding strata corresponding to singularities into which the original singularities can degenerate

Nevertheless the well known remedy is to rephrase the above question in terms of intersection theory technique and ask:

Question 2: How can we attach an intersection number (usually called the virtual number) to each compactified stratum of singular curves in $|L|$ with prescribed “topological type of singularities”¹¹ such that it is reduced to the actual curve counting when the sub-scheme of curves in the generic $\chi(L) - 1 - d_{\mathbf{m}}(L)$ dimensional linear sub-system of $|L|$ is regular of the expected dimension 0?

In the following, we will abbreviate and denote these numbers $n_{\mathbf{m}}$ as “the number of singular curves” in $|L|$. It should be understood in the sense of virtual numbers unless we are capable of proving a version of regularity result on the sub-scheme of $|L|$ parametrizing the singular curves in generic $\dim_{\mathbf{C}}|L| - d_{\mathbf{m}}(L)$ dimensional sub-linear system.

It is highly non-trivial to check whether these “virtual numbers of singular curves” are invariants. It makes perfect sense to ask:

Question 3: Are the “number of singular curves” well defined? Suppose that the “number of singular curves” has been defined, how does the intersection number depend on the algebraic surfaces M and the line bundle L ? Are they deformation invariants?

If there is a family of algebraic surfaces with sufficiently very ample relative polarizations which restrict to L on the special fiber, are the corresponding “numbers of singular curves” defined for each member within such a family independent to the deformation?

Are these numbers “topological”, which depend only on the topological data on (M, L) ? As the dimension $\dim_{\mathbf{C}}|L| = \chi(L) - 1$ is topological (by surface Riemann-Roch) when L is sufficiently positive, would it be too much to expect those “virtual numbers of singular curves” do, too?

Before we investigate the relationship of our universality theorem with these structural questions on the “virtual number of singular curves”, let us review how the special cases of these questions have shown up in the researches of the various group of people and how they have answered them.

Among the different type of isolated curve singularities, the nodal singularities, i.e. the normal crossing curve singularity, is the most and well-studied one.

Take $M = \mathbf{CP}^2$ and $L = \mathbf{H}^d$, the varieties of irreducible nodal curves in $|L| = |\mathbf{H}^d|$ over \mathbf{CP}^2 are known to be Severi varieties. In [H] it was proved that the Severi varieties are irreducible. On the other hand, the number of δ -node nodal curves in the generic δ dimensional linear subsystem of $|\mathbf{H}^d|$ is also known to be the Severi degree of the Severi variety and was calculated by [CP] (see also [Ran]) in terms of recursive formulae on the degree $d = c_1(\mathbf{H}^d)$. The same authors had shown that the intersection numbers could be understood in the classical sense, i.e. the “number of nodal curves” really counts the discrete number of nodal singular curves in $|\mathbf{H}^d|$.

On the other hand, when ¹² $M = K3$ and $c_1(L) \in H^2(M, \mathbf{Z})$, the “numbers of nodal curves” usually cannot be understood in the classical sense. This is the perfect examples to observe that the stratum of $|L|$ parametrizing the δ - node nodal curves can be rather ill-behaved for specific choice of algebraic K3 and L .

In particular, when $\delta = \frac{c_1(L)^2}{2} - 1$, these “number of rational nodal curves” attached to all such $|L|$ are predicted by the Yau-Zaslow formula [YZ] to be generated by the modular form $\frac{1}{\eta(q)^{24}}$ (the generating function of the Euler numbers of the Hilbert Schemes of K3, by a beautiful result of Göttsche).

¹¹We fix the topological types instead of the analytic types of the germ of singularities here. If we work over \mathbf{C} , the topological type of isolated curve singularities is determined by the link space of the singularity in $\mathbf{S}^3 \subset \mathbf{C}^2$, which is a multi-component link in \mathbf{S}^3 .

¹²I.e. $c_1(M) = 0$ and $\pi_1(M) = \{1\}$.

Recall the conjecture of Yau-Zaslow makes the following three predictions ¹³.

- (1). The “number of embedded nodal rational curves” in $|L|$ is well defined.
- (2). The “number of embedded nodal rational curves” in $|L|$ depends on L only ¹⁴ through $\delta = \frac{c_1(L)^2}{2} - 1$, or equivalently, the self-intersection number of $c_1(L)$, and can be denoted by n_δ .
- (3). The generating function $\sum_{\delta \geq 0} n_\delta q^\delta$ coincides with $\left(\prod_{i=0}^{\infty} \frac{1}{1-q^i}\right)^{\chi(M)}$.

Due to the difficulty in interpreting the “number of nodal rational curves” classically, some people (e.g. [BL]) had attempted to interpret these numbers as the Gromov-Witten invariants of the \mathbf{S}^2 families in some special cases. Indeed they were able to justify the (iii) statement in the conjecture (replacing the term “number of nodal rational curves” by genus zero Gromov-Witten invariants) of \mathbf{S}^2 hyperkahler families when L is primitive, i.e. $c_1(L)$ being a primitive element in the $K3$ lattice $H^2(K3, \mathbf{Z})$.

Despite that Gromov-Witten invariants on algebraic surfaces are closely related to the concept of “number of nodal curves” we proposed above and it may be tempting to think Gromov-Witten invariants as the exact intersection theoretical interpretation of **Question 2**, we point out the importance to separate these two different concepts. Especially it becomes apparent in some special case on $K3$ [Gat] that the intersection numbers predicted by Yau-Zaslow formula differs from the genus zero \mathbf{S}^2 family Gromov-Witten invariant of the same (non-primitive) class. On the other hand, Yau-Zaslow’s prediction matches up perfectly with calculation by Vainsencher [V] and was the strong evidence in the early days that Yau-Zaslow formula gave the correct prediction (See the final section “conclusions and Prospects” of [YZ]).

A careful investigation upon their subtle difference shows us that genus zero Gromov-Witten invariant enumerates not only maps onto immersed curves but also the multiple-coverings of maps. Yet the “number of nodal curves” in Yau-Zaslow conjecture counts immersed nodal curves in the linear system. Only when the class is primitive ¹⁵, these two concepts become coincide accidentally—as there is no chance of having multiple-covering maps into any primitive class.

On the other hand, besides some special cases of cohomology classes and almost complex structures closed enough to integrable complex structures, the author is not aware of the general definition of “number of nodal curves” in symplectic geometry.

Remark 3 *The above discussion suggests that we should consider the “number of nodal curves” and Gromov-Witten invariants as separated concepts. They coincide only in special cases.*

In fact the definition of Gromov-Witten invariants depend on the moduli stack of marked curves, and is related to the “world-sheet” point of view of string theory. Our theory of universality theorem makes use of the universal spaces of algebraic surfaces, which is closely related to the space-time approach of string theory. Nevertheless, we expect that there should be a combinatorial formula relating these two types of concepts. Our algebraic construction of the algebraic family obstruction bundle ¹⁶ (along with the technique to remove the type II exceptional class contributions when L is not sufficiently positive) provides an algebraic way to define the “number of nodal curves” as a virtual number, even when the corresponding moduli space of nodal curves is not well behaved.

For algebraic surfaces with geometric genera $p_g > 0$, Gromov-Witten invariants are frequently vanishing (due to the simple type condition. See [Liu3] section 4.3), while “the number of nodal

¹³The (3). below is usually known to be the Yau-Zaslow conjecture sometimes. To understand why the (1). and (2). are an integral part of the conjecture, please consult the original argument in the paper of Yau and Zaslow [YZ].

¹⁴This is a very strong statement which claims that the “number of embedded nodal rational curves” does not depend on the choices of the classes $c_1(L) \in H^2(M, \mathbf{Z})$ if they have the same self-intersection number.

¹⁵and the curve cone/Picard lattice is one dimensional.

¹⁶embedded in the proof of the main theorem.

curves” in the linear system $|L|$ can still be very rich! In the case of $K3$ (with $p_g = 1$) it is also the reason why people had adopted the twistor families of $K3$ in counting curves.

Some examples in the table I of [Va] indicates that the “number of nodal curves” may not always be topological–independent to degeneration of complex structures.

Topological or Non-topological?

Does the above discussion mean that we have to give up the idea that the “numbers of nodal curves” are topological? Not really!

In fact the real essence of the universality theorem is to tell us that for the number of nodal singularities $= \delta$:

(1). When L is $5\delta - 1$ -very ample, the “number of nodal curves” on any algebraic surface can be understood in the classical sense, once we take into account of multiplicities. The argument is essentially due to Göttsche [Got].

(2). These numbers are actually topological! Indeed they can be expressed as degree δ universal polynomials of $c_1^2(L), c_1(L) \cdot c_1(M), c_1^2(M), c_2(M)$ in the same way that the expected dimension $\dim_{\mathbb{C}}|L|$ has been given by a degree one polynomial of $c_1^2(L), c_1(L) \cdot c_1(M), c_1^2(M), c_2(M)$ through the Riemann-Roch formula (see page 2).

(3). The theorem implies that the “number of nodal curves” on totally different algebraic surfaces are intimately related, even though the algebraic geometry on these distinct surfaces can be quite different. The theorem allows us to analyze the dependence of these enumerative geometric information upon the underlying surface M and L in terms of homotopic types of the algebraic surfaces M and the cohomology class $c_1(L)$ of L .

On the other hand, we address the following apparent puzzles briefly.

Question: If the universality theorem asserts that the counting of nodal curves is topological, why are there examples of deformation equivalent surfaces (check Table 1 on page 11 of [Va]) which have different “numbers of nodal curves” in the same classes?

In a simple term to answer, it is due to the $5\delta - 1$ -very ample condition imposed on L . We may rephrase the above question by the following one,

Question: Does the universality theorem mean that the nodal curve counting (or more generally singular curve counting) is completely rigid as the “number of singular curves” are always predicted by one single universal formula?

Not really! Each algebraic surface still preserves its own character. The universality theorem only implies that in a complete linear system $|L|$ with L far away from the origin of the ample cone, the “number of nodal curves” behaves topologically. Our universality theorem gives the effective bound on L for such statements to hold.

On the other hand, without the suitable very ample condition, the topological prediction extrapolated from the universality theorem is generally not the exact answer ¹⁷!

Then we may ask:

Question: What happens if L is not $5\delta - 1$ very ample?

For general algebraic surfaces, apparently the predictions on the “numbers of nodal curves” are not accurate without $5\delta - 1$ very ample condition.

For the Calabi-Yau algebraic surfaces, i.e. $K3$ surfaces or Abelian surfaces ($\cong T^4$), the prediction from the universal polynomials of our universality theorem still holds even without the positivity

¹⁷This comment does not apply when $M = K3$ or T^4 in which case the topological answer by the universality theorem matches with the geometric answer perfectly despite that L may not be $5\delta - 1$ very ample. This is because a simple vanishing argument on the contribution from type II exceptional curves. See page 10.

condition on L . The reason that $K3$ and T^4 are special in this aspect has to be understood in a rather theoretical level. We will discuss the vanishing result in subsection 1.3.1 briefly, following the spirit of Riemann-Roch theorem.

On the other hand, the fact that for $\delta < 7$ the Zaslow-Yau prediction and Vainsencher's enumeration of the universal polynomials **Do** match has been a convincing evidence in the early days when Yau-Zaslow had made their conjecture [YZ].

1.3 The Type II Exceptional Curves and The Discrepancy to Universality Theorem

To understand the non-topological nature of the “numbers of nodal curves” generally (without any assumption on L) and how the geometric answers derivate from the universality theorem, we have to reflect the way we deal with the similar phenomenon in the original Riemann-Roch theorem. While it is true that for $L \otimes K_M^{-1}$ ample, $h^1(L) = h^2(L) = 0$ the $h^0(L) = \chi(L)$ and therefore $\dim_{\mathbf{C}}|L|$ is topological, generally the expected dimension $h^0(L) - h^1(L) - 1$ is not topological and can not be predicted by a topological formula.

If we go back to the history of the development of Riemann-Roch theorem (see e.g. chapter VII-2A of [Di]/or chapter IV and its appendix of [Za]), the higher sheaf cohomologies were introduced (in the classical language, they were known to be $h^1(M, L)$ = the super-abundance and $h^2(M, L)$ = index of specialty) exactly to balance the discrepancy between Riemann-Roch formula (of topological nature involving characteristic classes on M and L) and $h^0(L)$.

If one browses through the paper [Liu3] for the construction of the algebraic (family) Seiberg-Witten invariants, it is clear that the construction of the algebraic Seiberg-Witten invariant has been separated into three cases.

(i). The case of regular linear system with $h^1(L) = h^2(L) = 0$: In this case, the space $|L|$ is smooth of the right dimension $h^0(L) - 1 = \chi(L) - 1$, and the algebraic Seiberg-Witten invariant can be defined in the classical sense.

(ii). $h^2(L) = 0$ but $h^1(L)$ may be non-zero: In this case $|L|$ is not of its expected dimension. Yet one may introduce the algebraic Kuranishi model¹⁸ of bundle maps $\Phi_{\mathbf{V}\mathbf{W}} : \mathbf{V} \mapsto \mathbf{W}$ such that $\text{rank}_{\mathbf{C}}\mathbf{V} - \text{rank}_{\mathbf{C}}\mathbf{W} = h^0(L) - h^1(L) = \chi(L)$.

Accordingly the algebraic Seiberg-Witten invariant is defined to be $\int_{\mathbf{P}(\mathbf{V})} c_{\text{top}}(\mathbf{H} \otimes \mathbf{W})$.

(iii). $h^2(L) \neq 0$: In this case $h^2(L)$ is the correction term to the dimension of the class $c_1(L)$ from the Riemann-Roch formula.

This suggests us to group $h^0(L) - h^1(L)$ inside $h^0(L) - h^1(L) + h^2(L)$ and the expected dimension of the linear system $|L|$ in this case is $h^0(L) - h^1(L) - 1 = \chi(L) - 1 - h^2(L)$. Namely, we have to calculate $h^2(L)$ and subtract it from $\chi(L) - 1$ to get the actual “expected dimension” $h^0(L) - h^1(L) - 1$ and $h^2(L)$ plays the role of the “correction term” in the following formula,

$$h^0(L) - h^1(L) - 1 = \{\chi(L) - 1\} - h^2(L).$$

Then we may ask

Question: How does the correction term $h^2(L)$ of the surface Riemann-Roch formula shed light on the discrepancy of “number of nodal curves” to the Universality theorem?

Is there any analogous concept in the enumeration theory of nodal curves parallel to the dimension of second sheaf cohomology $h^2(L)$ in the calculation of the expected dimension $h^0(L) - h^1(L) - 1$?

¹⁸Read [Liu3] for more details.

In fact, such an analogue exists. It is the excess contribution to the family invariants from decomposition of curves involving type *II* exceptional curves. It plays an exact analogue of $h^2(L)$ in the surface Riemann-Roch formula.

Definition 1 *A divisor class e is an exceptional class over an algebraic fibration $\mathcal{X} \mapsto B$ if it satisfies the following condition.*

- (i). *The self intersection number $e^2 < 0$.*
- (ii). *The degree of e with respect to ample polarizations is positive.*

The first condition ensures that any irreducible representatives of e is unique in the corresponding fiber. The second condition is necessary for e to be represented by algebraic curves. The class e is exceptional in the sense that for an irreducible \mathbf{e} representing e , $h^0(\mathbf{e}, \mathcal{O}(\mathbf{e})) = 0$ and the curve \mathbf{e} is infinitesimally rigid.

In the family theory approach to the universality theorem, we encounter two types of exceptional curves on the universal families $M_{\delta+1} \mapsto M_\delta$ and they are called type *I* and type *II* exceptional curves, respectively.

The fibration $M_{\delta+1} \mapsto M_\delta$ can be constructed from the product family $M \times M_\delta \mapsto M_\delta$ by blowing up n -consecutive cross sections of the intermediate blown up fiber bundles¹⁹. So there is a canonical projection map $M_{\delta+1} \mapsto M \times M_\delta \mapsto M$. Following the convention in [Liu6], the exceptional divisors of the intermediate blown up manifolds are denoted by $E_1, E_2, \dots, E_\delta$, respectively.

Definition 2 *A type I exceptional classes is an exceptional class of the form $e = E_i - \sum_{j_i > i} E_{j_i}$.*

It lies in the kernel of the proper push-forward $\mathcal{A}_{2\delta+1}(M_{\delta+1}) \mapsto \mathcal{A}_1(M)$.

An irreducible algebraic curve representing the type *I* exceptional class is said to be a type *I* exceptional curve in the family $M_{\delta+1} \mapsto M_\delta$.

The type *I* exceptional curves have played important roles in the proof of universality theorem.

Definition 3 *A type II exceptional class in the universal family $M_{\delta+1} \mapsto M_\delta$ is an exceptional class which is not in the kernel of the proper push-forward $\mathcal{A}_{2\delta+1}(M_{\delta+1}) \mapsto \mathcal{A}_1(M)$.*

Over the universal families $M_{\delta+1} \mapsto M_\delta$ the appearance of the type *II* exceptional classes and their excess contributions to the family invariants can be understood from three different prospectives,

(a). From the linear system interpretation of the nodal curve counting, there are strata in $|L|$ which consist of curves with non-isolated singularities. These are the curves with non-reduced irreducible components. Just like the counting of nodal curves using the universal family may encounter other type of singular curves with isolated singularities, sometimes the singular curves with non-reduced irreducible components may contribute to the algebraic family Seiberg-Witten invariant $\mathcal{AFSW}_{M_{\delta+1} \times \{t_L\} \mapsto M_\delta \times \{t_L\}}(1, c_1(L) - 2 \sum_{1 \leq i \leq n} E_i)$ as well. The type *I* exceptional curves begin to contribute to the family invariants when $n \geq 4$. On the other hand, when L is $5\delta - 1$ very ample, Göttsche's argument implies implicitly that the type *II* exceptional curves do not contribute to the above family invariant $\mathcal{AFSW}_{M_{\delta+1} \times \{t_L\} \mapsto M_\delta \times \{t_L\}}(1, c_1(L) - 2 \sum_{1 \leq i \leq n} E_i)$. But it is a totally different story when L fails to be $5\delta - 1$ very ample on a "general" algebraic surface.

(b). From the point of view of family Gromov-Taubes theory, the appearance of irreducible type *II* exceptional curves representing e_{II} pairing negatively with $C - \mathbf{M}(E)E$ will force effective representatives of $C - \mathbf{M}(E)E$ lying over the existence locus of irreducible effective e_{II} to break off a certain multiple of curves representing e_{II} and the class $C - \mathbf{M}(E)E$ can be written as $(C - \mathbf{M}(E)E - e_{II}) + (e_{II})$ formally. Potentially they can contribute to the family invariant $\mathcal{AFSW}_{M_{\delta+1} \times \{t_L\} \mapsto M_\delta \times \{t_L\}}(1, C - \mathbf{M}(E)E)$ as well.

¹⁹Read the beginning of section 2 [Liu6] for more details.

(c). The canonical algebraic obstruction²⁰ bundle $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$ over $X = \mathbf{P}(\mathbf{V}_{\text{canon}})$ was introduced to detect the curves with singularities in a complete linear system $|L|$. While the base space M_δ parametrizes all the possible configurations of singular points and the canonical algebraic obstruction bundle detects the curves with local multiplicities greater than one, any non-reduced curve in the linear system $|L|$ will be detected as singular curves with infinitely many singularities. Even if we had subtracted the excess contributions from the type *I* exceptional curves (which are universal), the modified family invariant (calculable to be a universal polynomial of C^2 , $C \cdot c_1(K_M)$, $c_1(K_M)^2$ and $c_2(M)$) would still have contained the excess contributions from the various multiple covering (corresponding to non-reduced curves in algebraic geometry) of exceptional curves. They are exactly the type *II* exceptional curves discussed in definition 3.

On the other hand, asserting that multiple coverings of type *II* curves “may” contribute potentially to the family invariant does not mean that their contributions are always nonzero in all possible situations. Our universality theorem gives us an effective bound universally on the nodal curve counting of L such that the type *II* curves do not contribute to the family invariant of $L \otimes \mathcal{O}(-\mathbf{M}(E)E)$. In these cases, the type *II* exceptional curve may occur in the possible decompositions of the curves dual to $C - \mathbf{M}(E)E$, but they do not contribute to the family invariant. This is because these types of decompositions involving type *II* exceptional classes may have a lower expected family dimension than the original class $C - \mathbf{M}(E)E$.

1.3.1 A Short Discussion of the Vanishing Argument of type II Contribution on $K3$

In the original surface Riemann-Roch problem, by Serre duality the index of specialty $h^2(L) = h^0(L^* \otimes K_M)$. Then $h^2(L)$ vanishes for sufficiently positive L . This corresponds to our statement that when L is $5\delta - 1$ very ample, type *II* curves do not contribute to the family invariant. When K_M is trivial (this happens when $M=T^4$ or $K3$), it implies the vanishing of $h^2(L)$ for any effective L .

Likewise there is a similar vanishing result for the type *II* contributions to the family invariants when $M = T^4$ or $K3$. In the following, we outline the main cause for the vanishing result in a less technical way. More details will appear in the paper [Liu7].

It is well known in Seiberg-Witten theory that $SW(\mathcal{L}) = 0$ for nontrivial $spin_c$ structures \mathcal{L} over all symplectic four-manifolds with trivial canonical bundles. From a differential geometric point of view, this vanishing statement for T^4 and $K3$ are due to the existence of Ricci flat metrics on T^4 or $K3$ and the vanishing result of Witten [W] back to the early days of Seiberg-Witten theory. On the other hand, in the following we point out directly from the point of view of algebraic geometry the cause for such a vanishing result.

Let M be an algebraic $K3$. Let D be an effective divisor on M in the linear system $|L|$. Then there is the following standard short exact sequence on M ,

$$0 \mapsto \mathcal{O}_M \mapsto \mathcal{O}_M(D) \mapsto \mathcal{O}_D(D) \mapsto 0.$$

If we replace D by the universal divisor $\mathbf{D} \subset M \times |L|$ and push-forward the globalized exact sequence

$$0 \mapsto \mathcal{O}_{M \times |L|} \mapsto \pi_{\mathbf{D}}^* \mathcal{H} \otimes \mathcal{O}_{M \times |L|}(\mathbf{D}) \mapsto \pi_{\mathbf{D}}^* \mathcal{H} \otimes \mathcal{O}_{\mathbf{D}}(\mathbf{D}) \mapsto 0$$

along $\pi_{|L|} : M \times |L| \mapsto |L|$, then we will get the following right derived sequence after some simple base change argument,

$$0 \mapsto \mathbf{C}_{|L|} \mapsto \mathbf{H} \otimes H^0(M, L) \otimes \mathbf{C}_{|L|} \mapsto \mathbf{H} \otimes \mathbf{R}^0 \pi_{|L|*} \mathcal{O}_{\mathbf{D}}(\mathbf{D}) \mapsto \otimes H^1(M, \mathcal{O}_M) \otimes \mathbf{C}_{|L|} \mapsto \mathbf{H} \otimes H^0(M, L) \otimes \mathbf{C}_{|L|}$$

²⁰Consult [Liu5], [Liu6] for the detail construction.

$$\mapsto \mathbf{H} \otimes \mathbf{R}^1 \pi_{|L|*}(\mathcal{O}_{\mathbf{D}}(\mathbf{D})) \mapsto \mathbf{R}^2 \pi_{|L|*}(\mathcal{O}_M) \mapsto \mathbf{H} \otimes \mathbf{R}^2 \pi_*(\mathcal{O}(\mathbf{D})) = 0.$$

This exact sequence has been analyzed in [Liu3] in great detail. The key observation here is that by relative Serre duality the second derived image $\mathbf{R}^2 \pi_*(\mathcal{O}_M) \cong (\mathbf{R}^0 \pi_*(\mathcal{O}_M))^*$ is isomorphic to the trivial line bundle over $|L|, \mathbf{C}_{|L|}$, when M has a trivial canonical bundle K_M .

We notice that $\mathbf{R}^1 \pi_{|L|*}(\mathcal{O}_{\mathbf{D}}(\mathbf{D})) \mapsto \mathbf{R}^2 \pi_{|L|*}(\mathcal{O}_M)$ is a surjection. Consider a Kuranishi-model of \mathbf{D} with the obstruction bundle \mathbf{W}_{obs} . The moduli space of curves is the zero locus of a defining section of \mathbf{W}_{obs} . Then by the defining property of the Kuranishi model, the fiber of \mathbf{W}_{obs} above a curve $D \in |L|$ maps surjectively onto $H^1(D, \mathcal{O}_D(D))$.

This implies that the obstruction bundle \mathbf{W}_{obs} of the class D will map surjectively onto the trivial line bundle $\mathbf{C}_{|L|}$, $\mathbf{W}_{obs} \mapsto \mathbf{C}_{|L|} \mapsto 0$. By [F] example 12.1.8 on page 216-217, this implies that the top Chern class of \mathbf{W}_{obs} vanishes. The fundamental cycle class of the moduli space defined by the localized top Chern class of \mathbf{W}_{obs} along the moduli space of curves $\cong |L|$ is zero. In particular, the Seiberg-Witten invariant of D is zero.

This pathetic symptom is exactly why the algebraic (family) Seiberg-Witten invariant has been introduced [Liu3], which removes essentially the troublesome \mathbf{C} (more generally the \mathbf{C}^{p_g} for $p_g > 0$) factor in the obstruction bundle and set the obstruction bundle for algebraic Seiberg-Witten invariants to be $Ker(\mathbf{W}_{obs} \mapsto \mathbf{C}_{|L|})$.

A more intuitive approach is to consider the hyperkahler twistor family of $K3$.

It is well known that $K3$ admits hyperkahler Riemannian metrics. I.e. there exists integrable complex structure $I, J, K \in End(\mathbf{TM})$ satisfying $I^2 = J^2 = K^2 = -Id_{TM}$ and $IJ + JI = JK + KJ = KI + IK = 0$. Then for $\mathbf{x} = (x, y, z) \in \mathbf{R}^3, x^2 + y^2 + z^2 = 1$, we have an \mathbf{S}^2 family of complex structures $I_{\mathbf{x}} = xI + yJ + zK \in End(TM)$. A Ricci-flat metric g is related to the \mathbf{S}^2 family of Kahler forms $\omega_{\mathbf{x}}$ by $g(I_{\mathbf{x}} \cdot, \cdot) = \omega_{\mathbf{x}}(\cdot, \cdot)$.

Consider a special \mathbf{S}^2 hyperkahler family of complex structures on the $K3$ such that $C \in H^2(M, \mathbf{Z})$ becomes a $(1, 1)$ class at $b \in \mathbf{S}^2$. The class C fails to be of $(1, 1)$ type over $\mathbf{S}^2 - \{b\}$. So the ‘‘family moduli space’’ of curves dual to C , \mathcal{M}_C is confined above $b \in \mathbf{S}^2$.

The tangent space of the thickened base $\mathbf{T}_b \mathbf{S}^2$ maps injectively into the obstruction bundle such that the following diagram is commutative,

$$\begin{array}{ccc} \mathbf{T}_b \mathbf{S}^2 & \longrightarrow & \mathbf{W}_{obs} \\ & \searrow & \downarrow \\ & & \mathbf{C}_{|L|} \end{array}$$

For the \mathbf{S}^2 -thickened hyperkahler family, the obstruction bundle $\mathbf{W}_{obs}/Im \mathbf{T}_b \mathbf{S}^2$ of the \mathbf{S}^2 family does not have a trivial \mathbf{C} factor. There is no wonder that non-trivial family invariants can be defined for such families.

Let e_{II} be a type II exceptional class of the universal family $M_{\delta+1} \mapsto M_{\delta}$. Then the family moduli space $\mathcal{M}_{e_{II}}$ over M_{δ} has the structure of a projectified cone over $M_{\delta} \times T(M)$. Over the family moduli space $\mathcal{M}_{e_{II}}$, the universal curve (divisor) is denoted by $\mathcal{D}_{e_{II}}$.

Suppose that $C = c_1(L)$ with $C \cdot e_{II} < 0$ is the class to enumerate the family invariant $\mathcal{AFSW}_{M_{\delta+1} \mapsto M_{\delta}}(1, C - 2 \sum_i E_i)$. Let $\mathcal{M}_C \mapsto M_{\delta}$ be the family moduli space of curve dual to C . As a model example we investigate the curves dual to C which are decomposed into components dual to $C - e_{II}$ and e_{II} . Such curves form a sub-moduli space $\mathcal{M} = \mathcal{M}_{C-e_{II}} \times_{M_{\delta}} \mathcal{M}_{e_{II}} \subset \mathcal{M}_C$. Over the locus $\mathcal{M}_{C-e_{II}} \times_{M_{\delta}} \mathcal{M}_{e_{II}}$, there are a pair of universal curves, $\mathcal{D}_{C-e_{II}}$ and $\mathcal{D}_{e_{II}}$ for $C - e_{II}$ and e_{II} , respectively. Their sum $\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}$ is nothing but the universal curve \mathcal{D}_C of C , restricted to the sub-locus $\mathcal{M} \subset \mathcal{M}_C$.

The key observation is that $\mathbf{R}^1 \pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}}(\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}))$ is isomorphic to the direct sum of a rank two sub-sheaf $\mathcal{O}_{\mathcal{M}} \oplus \mathcal{O}_{\mathcal{M}}$ and a coherent sheaf, because of the curve decomposition $\mathcal{D}_C \mapsto \mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}$.

Firstly both $\mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}}}(\mathcal{D}_{C-e_{II}}))$ and $\mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{e_{II}}}(\mathcal{D}_{e_{II}}))$ map surjectively onto $\mathcal{O}_{\mathcal{M}}$, following the same discussion earlier. Then the mappings ²¹

$$\mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}}}(\mathcal{D}_{C-e_{II}})) \rightarrow \mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}}+\mathcal{D}_{e_{II}}}(\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}})) \leftarrow \mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{e_{II}}}(\mathcal{D}_{e_{II}})),$$

send their quotients $\mathcal{O}_{\mathcal{M}}$ injectively into $\mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}}+\mathcal{D}_{e_{II}}}(\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}))$. It is not hard to check that the direct sum morphism $\mathcal{O}_{\mathcal{M}} \oplus \mathcal{O}_{\mathcal{M}} \mapsto \mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}}+\mathcal{D}_{e_{II}}}(\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}))$ is also an injection and the corresponding short exact sequence splits.

Similar to our earlier discussion, over the whole family moduli space \mathcal{M}_C the family obstruction bundle \mathbf{W}_{obs} has a trivial quotient $\cong \mathbf{C}$. This trivial factor is either removed in defining algebraic family Seiberg-Witten invariant, or killed by the injection of $\mathbf{T}_b\mathbf{S}^2$ if we adopt hyperkahler families. When the family obstruction bundle \mathbf{W}_{obs} is restricted to the sub-moduli space $\mathcal{M}_{C-e_{II}} \times_{M_\delta} \mathcal{M}_{e_{II}}$, it has a \mathbf{C}^2 quotient ²² because of the presence of the $\mathcal{O}_{\mathcal{M}} \oplus \mathcal{O}_{\mathcal{M}}$ factor in $\mathbf{R}^1\pi_*(\mathcal{O}_{\mathcal{D}_{C-e_{II}}+\mathcal{D}_{e_{II}}}(\mathcal{D}_{C-e_{II}} + \mathcal{D}_{e_{II}}))$.

This implies that the localized contribution of this sub-moduli space $\mathcal{M}_{C-e_{II}} \times_{M_n} \mathcal{M}_{e_{II}}$ to the family invariant vanishes, due to the presence of additional trivial quotient in the obstruction bundle!

The same argument can be carried out for decompositions with more than one type II components and we find that the type II curves contribute trivially to the family invariants of $K3$. This is the reason why the type II correction terms never appear in the nodal curve counting on $K3$.

2 The Outline of Some Key Ideas in the Algebraic Proof of Universality Theorem

Even though the algebraic proof of universality theorem in [Liu6] has been reduced to a purely algebraic argument and hence the conceptual dependence on the symplectic "SW = Gr" used in [Liu1] has been relieved, the argument is still of a considerable length. To guide the reader through the paper, the goal of the section is to provide a short sketch of the ideas involved in the proof, the various difficulties it has overcome, etc., in less technical terms.

Let M_δ denote the n -th universal space associated to the algebraic surface M . Then $f_\delta : M_{\delta+1} \mapsto M_\delta$ is smooth of relative dimension two and is the universal space parametrizing the n -consecutive codimension two blowing ups of M . In [Liu3] section 5.1 we have introduced the concept of canonical algebraic family Kuranishi model²³ of $C - \mathbf{M}(E)E$. Assuming that $\mathcal{R}^i\pi_*(\mathcal{E}_C) = 0$ for $i > 0$, the pair of vector bundles $\mathbf{V}_{canon}, \mathbf{W}_{canon}$ were introduced, where $\mathcal{V}_{canon} = \mathcal{R}^0\pi_*(\mathcal{E}_C)$ and $\mathcal{W}_{canon} = \mathcal{R}^0\pi_*(\mathcal{O}_{\mathbf{M}(E)E} \otimes \mathcal{E}_C)$ are the corresponding locally free sheaves of sections. The sheaf morphism $\mathcal{R}^0\pi_*(\mathcal{E}_C) \mapsto \mathcal{R}^0\pi_*(\mathcal{O}_{\mathbf{M}(E)E} \otimes \mathcal{E}_C)$ between the locally free sheaves (see definition 5.3 of [Liu3] for more details) induces a bundle map between the vector bundles $\mathbf{V}_{canon} \mapsto \mathbf{W}_{canon}$. The vector bundle $\mathbf{V}_{canon} \mapsto M_\delta \times T(M)$ parametrizes the non-linear system of curves dual to C in the family $M_{\delta+1} \times T(M) \mapsto M_\delta \times T(M)$. A curve (corresponding to a ray in a fiber of \mathbf{V}_{canon}) dual to C can be splitted into components dual to $C - \mathbf{M}(E)E$ and dual to $\mathbf{M}(E)E$ if and only if the restriction map to $\mathbf{M}(E)E$ of the curve dual to C vanishes along the whole $\mathbf{M}(E)E$, i.e. the ray corresponding to the definition section of the curve is in the kernel $Ker(\mathbf{V}_{canon} \mapsto \mathbf{W}_{canon})$ of the bundle map²⁴ $\mathbf{V}_{canon} \mapsto \mathbf{W}_{canon}$.

²¹Both are induced by $\mathcal{O}_A(A) \mapsto \mathcal{O}_{A+B}(A+B)$ with $A = \mathcal{D}_{C-e_{II}}$ or $A = \mathcal{D}_{e_{II}}$.

²²Instead of only one \mathbf{C} .

²³Here $C = c_1(L)$. We use C in the paper to allow the extension to non-linear systems instead of linear system $|L|$.

²⁴This canonical bundle map is constructed by the restriction morphism to $\mathbf{M}(E)E$.

The bundle map $\mathbf{V}_{\text{canon}} \mapsto \mathbf{W}_{\text{canon}}$ induces a bundle map $\mathbf{H}^* \mapsto \pi_X^* \mathbf{W}_{\text{canon}}$ over $X = \mathbf{P}(\mathbf{V}_{\text{canon}})$, which is nothing but a global section of $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$. Thus the family moduli space of $C - \mathbf{M}(E)E$ can be thought to be a projectified cone embedded in $X = \mathbf{P}(\mathbf{V}_{\text{canon}})$, defined by a canonical section $s_{\text{canon}} \in \Gamma(X, \pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H})$ induced by $\mathbf{V}_{\text{canon}} \mapsto \mathbf{W}_{\text{canon}}$. It is the union of the projectified (non-)linear systems of the fibers. Thus, we may rewrite $\mathcal{M}_{C-\mathbf{M}(E)E} = Z(s_{\text{canon}})$. If we want to count curves in the linear system $|L|$ instead of the non-linear system, then we work with $\mathcal{M}_{C-\mathbf{M}(E)E} \times_{T(M)} \{t_L\} = Z(s_{\text{canon}}) \times_{T(M)} \{t_L\}$. Over here $t_L \in T(M)$ is a point in the component of Picard variety $T(M) \cong \text{Pic}^0(M)$ corresponding to L .

By intersection theory [F] construction, the localized top Chern class defines a cycle class supported in $Z(s_{\text{canon}})$, it is denoted by $\mathbf{Z}(s_{\text{canon}})$, following the notations of [F]. Consider the top open stratum Y_{γ_δ} of the universal space M_δ which parametrizes the ordered distinct n points on M . Then $M_\delta - Y(\gamma_\delta)$ is a closed sub-scheme of M_δ . In the following, we denote $Y = X \times_{M_\delta} (M_\delta - Y_{\gamma_\delta})$.

We can split $Z(s_{\text{canon}})$ into the closed $Z_o = \overline{Z(s_{\text{canon}}) \times_{M_\delta} Y_{\gamma_\delta}}$ and the closed $Z_i = \overline{Z(s_{\text{canon}}) - Z_o}$. $Z(s_{\text{canon}}) - \overline{Z(s_{\text{canon}}) \times_{M_\delta} Y_{\gamma_\delta}} \subset Z(s_{\text{canon}}) \times_{M_\delta} Y$. The latter is the union of all the irreducible components in $Z(s_{\text{canon}}) \times_{M_\delta} Y$.

We also know that $X - Y = X \times_{M_\delta} Y_{\gamma_\delta} \mapsto X$ is the top open stratum of X , where the stratification on X has been induced from the admissible stratification²⁵ of M_δ by the surjection $X \mapsto M_\delta$. On the other hand, the standard exact sequence for $U = Z(s_{\text{canon}}) - Z(s_{\text{canon}}) \times_{M_\delta} Y$,

$$\mathcal{Z}(Y \cap Z(s_{\text{canon}})) \mapsto \mathcal{Z}(Z(s_{\text{canon}})) \xrightarrow{j^*} \mathcal{Z}(Z(s_{\text{canon}}) - Z(s_{\text{canon}}) \cap Y) \mapsto 0$$

and the induced exact sequence on their quotients \mathcal{A} ,

$$\mathcal{A}(Y \cap Z(s_{\text{canon}})) \mapsto \mathcal{A}(Z(s_{\text{canon}})) \xrightarrow{j^*} \mathcal{A}(Z(s_{\text{canon}}) - Z(s_{\text{canon}}) \cap Y) \mapsto 0$$

on page 21 of [F] imply that $\mathbf{Z}(s_{\text{canon}}) \in \mathcal{A}(Z(s_{\text{canon}}))$ can be splitted into a unique component in $\mathcal{A}(Z(s_{\text{canon}}) \cap Y)$ and a unique image in $\mathcal{A}(Z(s_{\text{canon}}) - Z(s_{\text{canon}}) \cap Y)$.

As Z_o and Z_i are both unions of irreducible components of $Z(s_{\text{canon}})$, the decomposition $Z(s_{\text{canon}}) = Z_o \cup Z_i$ induces a decomposition on the normal cones $\mathbf{C}_{Z(s_{\text{canon}})}X = \mathbf{C}_{Z_o}X \cup \mathbf{C}_{Z_i}X$, which induces a corresponding decomposition of $s(\mathbf{C}_{Z(s_{\text{canon}})}X) \in \mathcal{A}(Z(s_{\text{canon}}))$ into $s(\mathbf{C}_{Z_o}X) \in \mathcal{A}(Z_o)$ and $s(\mathbf{C}_{Z_i}X) \in \mathcal{A}(Z_i)$. By capping with $c_{\text{total}}(\mathbf{H} \otimes \pi_X^* \mathbf{W}_{\text{canon}}|_{Z(s_{\text{canon}})})$, we get a canonical decomposition of $\mathbf{Z}(s_{\text{canon}})$ into a unique component in $\mathcal{A}(Z(s_{\text{canon}}) \cap Y)$, called localized contribution of top Chern class $\mathbf{Z}_{Z_o}(s_{\text{canon}})$ and a component extended from $\mathcal{A}(Z(s_{\text{canon}}) - Z(s_{\text{canon}}) \cap Y)$.

Definition 4 Let $\eta \in \mathcal{A}(Z(s_{\text{canon}}) - Z(s_{\text{canon}}) \cap Y) = \mathcal{A}(Z_o - Z_o \cap Y)$. Lift η to an explicit closed cycle in $\mathcal{Z}(Z_o - Z_o \cap Y)$. Then such a lifting can be extended to $\mathcal{Z}(Z(s_{\text{canon}}))$. Define the resulting extension of η to $\mathcal{A}(Z(s_{\text{canon}}))$ by $\bar{\eta}$.

The extension $\bar{\eta}$ a priori depends on the lifting of η from the cycle class group to $\mathcal{Z}(Z_o - Z_o \cap Y)$. On the other hand, for $\eta = j^* \mathbf{Z}(s_{\text{canon}})$ we can take $\bar{\eta}$ to be $\mathbf{Z}_{Z_o}(s_{\text{canon}}) = \{c_{\text{total}}(\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}|_{Z_o}) \cap s_{\text{total}}(Z_o, X)\}_{\dim_{\mathbf{C}} X - \text{rank}_{\mathbf{C}} \mathbf{W}_{\text{canon}}}$ and it is unique.

Suppose that we had had the equality $j^* \mathbf{Z}(s_{\text{canon}}) = \mathbf{Z}(s_{\text{canon}})$, then $\mathbf{Z}(s_{\text{canon}})$ would have been viewed as an extension of the cycle class $j^* \mathbf{Z}(s_{\text{canon}})$ into $Z(s_{\text{canon}})$ (by taking the closure!) and it would have represented the fundamental cycle class of the “moduli space of curves from the fiber-wise sections of $\mathcal{E}_C \otimes \mathcal{O}(-\mathbf{M}(E)E)$. Each curve in $Z(s_{\text{canon}}) \times_{M_\delta} Y_{\gamma_\delta}$ projects into M by the restricted blowing down map $M_{\delta+1} \times_{M_\delta} Y_{\gamma_\delta} \mapsto M \times Y_{\gamma_\delta}$ and produces curves with at least n singularities. One may consider the intersection number by capping the fundamental class $\mathbf{Z}(s_{\text{canon}})$ with $c_1(\mathbf{H})^{p_g - q + \frac{(C - \mathbf{M}(E)E)^2 - (C - \mathbf{M}(E)E) \cdot c_1(K_{M_{\delta+1}/M_\delta})}{2}}$ and the answer can be shown to be a universal degree n

²⁵See section 2 of [Liu6] for more details about how to stratify M_δ by Y_Γ , $\Gamma \in \text{adm}(n)$.

polynomial of $c_1(L)^2$, $c_1(L) \cdot c_1(M)$, $c_1^2(M)$ and $c_2(M)$, by applying the family blowup formula [Liu3] inductively.

The possible failure of the equality $\mathbf{Z}(s_{\text{canon}}) = \overline{j^*\mathbf{Z}(s_{\text{canon}})}$ indicates that there are excess cycle classes $\mathbf{Z}(s_{\text{canon}}) - \overline{j^*\mathbf{Z}(s_{\text{canon}})}$ localized in $\mathcal{A}(Y \cap Z(s_{\text{canon}}))$ which also contributes to $\overline{\mathbf{Z}(s_{\text{canon}})}$ besides the generic component $j^*\mathbf{Z}(s_{\text{canon}})$. Thus it becomes a subtle issue to separate $\overline{j^*\mathbf{Z}(s_{\text{canon}})}$ from the localized top Chern class $\mathbf{Z}(s_{\text{canon}})$, which can be pushed-forward into X as the global object—the top Chern class $c_{\text{top}}(\pi_X^*\mathbf{W}_{\text{canon}} \otimes \mathbf{H})$.

Different groups of people had chosen quite different paths to either address or to bypass this issue. In [Got] Göttsche had proposed a different approach using Hilbert schemes of surfaces. Instead of using universal spaces and counting the resolved curves, he works with the singular curves themselves instead of the resolved smooth curves. When L is $5\delta - 1$ very ample, he could construct a finite scheme (see [Got]) in an ambient scheme birational to some Hilbert scheme of M , representing the fundamental cycle class of moduli space of nodal curves in generic n dimensional linear sub-system. In his approach, he completely avoided the above problem. On the other hand, the topological nature of the resulting “number of nodal curves” has been less transparent.

On the other hand, Vainsencher [V], Kleiman-Piene had adopted the universal space approach without using the admissible stratification explicitly and fought with the problem directly. By assuming that L is sufficiently very ample and $\delta \leq 6, \leq 8$, respectively, they were able to prove regularity results on the $Z(s_{\text{canon}})$ and identify the non-trivial excess contribution of intersection numbers from $\mathcal{A}(Z(s_{\text{canon}}) \cap Y)$ in the forms of sums of excess contributions of singular curves from other types of non-nodal singularities²⁶.

In [Liu1], [Liu6] the author has taken a new approach to tackle the problem. We realize that the problem is in fact the natural prolongation of surface Riemann-Roch formula. Then we migrate tools from differential topology and symplectic geometry to algebraic geometry in resolving the problem. The family blowup formula and the family switching formula provide the necessary bridges between our non-linear enumeration problem and the Grothendieck Riemann-Roch theorem.

2.1 Some Problems Addressed in Our Paper

In the following we list all the key problems we address in the paper [Liu6],

Problem 1: As has been commented earlier, we do not expect that for all L and the general M , the numbers of nodal curves in linear sub-system of $|L|$ are topological numbers. What are the effective range of L in which the numbers of nodal curves become topological?

If one always has to raise L to an extremely high power of a very ample line bundle, without an effective control over its very-ampleness, it limits the application of the universal formulae to enumerative predictions.

Problem 2: It is rather unclear from a geometric prospective that the excess localized contributions of the cycle classes in $\mathcal{A}(Z(s_{\text{canon}}) \cap Y)$ are “intrinsic”. I.e. it is not clear at all that the cycle classes localized inside $\mathcal{A}(Y)$ are “invariant” to the deformations of the complex structures of M and/or the holomorphic structures of L . Not to mention that it may or may not have any intrinsic properties across different algebraic surfaces.

Consider an ideal situation that s_{canon} can be slightly deformed (in the algebraic category) into a new cross section s' with the zero locus $Z(s')$. It is usually not true that $\mathbf{Z}(s') - \overline{j^*\mathbf{Z}(s')}$ is equal to $\mathbf{Z}(s_{\text{canon}}) - \overline{j^*\mathbf{Z}(s_{\text{canon}})}$.

In what sense does the excess localized contributions to top Chern class $\mathbf{Z}(s_{\text{canon}}) - \overline{j^*\mathbf{Z}(s_{\text{canon}})}$ have any intrinsic geometric meaning?

²⁶See e.g. [V] for more details and examples.

On the other hand,

Problem 3: The space $Y = X \times_{M_\delta} (M_\delta - Y_\gamma)$ itself is not smooth. Potentially this adds to the technical issues in identifying the excess cycle class localized in $Z(s_{\text{canon}}) \cap Y$. In general the locus $Z_i \subset Y$ is highly singular. How do we study enumerative geometry upon such singular objects?

Our strategy to simplify **Problem 3** is to use the admissible stratification on the universal space $M_\delta = \coprod_{\Gamma \in \text{adm}(n)} Y_\Gamma$ and rewrite Y as $X \times_{M_\delta} (\cup_{\Gamma \in \text{adm}(n); \Gamma \neq \gamma_\delta} Y(\Gamma))$, which is a finite union of smooth subspaces $X \times_{M_\delta} Y(\Gamma)$ in X .

In this way we have replaced a single non-smooth Y by a hierarchy of smooth $X \times_{M_\delta} Y(\Gamma)$ and different $X \times_{M_\delta} Y(\Gamma)$ may intersect each other.

One key observation in the algebraic proof of universality theorem is the following simple fact: The algebraic analogue of the Kuranishi model type perturbation argument in the differentiable or symplectic category is exactly the localized top Chern class of the zero locus of a section of the algebraic obstruction vector bundle. We generalize this concept slightly and call the following expression $\{c_{\text{total}}(E) \cap s_{\text{total}}(Z, X) \cap [X]\}_{\dim_{\mathbf{C}} X - \text{rank}_{\mathbf{C}} E}$ the localized contribution of top Chern class for $Z \subset Z(s)$ and some $s \in \Gamma(X, E)$.

On the other hand, one may blow up X along Z with an exceptional divisor D . Then the above expression of localized contribution of top Chern class can be viewed as the push-forward²⁷ of $\sum_{1 \leq i \leq \text{rank}_{\mathbf{C}} E} c_{\text{rank}_{\mathbf{C}} E - i}(E) \cap (-1)^{i-1} D^{i-1}[D]$, which appears naturally in the residual intersection formula of top Chern class. See example 14.1.4 on page 245 of [F] for more details.

Our algebraic geometric construction to enumerate the localized contribution of top Chern classes along $Z(s_{\text{canon}}) \cap Y$ is to blow up X repeatedly along the sub-loci. But we do not blow up X along $Z(s_{\text{canon}}) \cap Y$ at once because it leads to an un-identifiable localized contribution of top Chern classes and is not helpful to us. Instead our scheme to tackle this problem requires us to rewrite $Z(s_{\text{canon}}) \cap Y$ as a seemingly more complicated union $\cup_{\Gamma \in \Delta(n) - \{\gamma_\delta\}} Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$. Notice that quite a lot of admissible strata $Y(\Gamma)$ with $\Gamma \in \text{adm}(n)$ ²⁸

have been thrown away from the union as they are in the closure of the other admissible strata. One can show that we only need to consider those $\Gamma \in \Delta(n)$, satisfying special maximality conditions²⁹. The set $\Delta(n)$ collects the admissible graphs such that their associated admissible strata are kept in the union. Then we blow up X inductively along the various $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$, $\Gamma \in \Delta(n)$, and apply the residual intersection formula of top Chern class to $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$ and to the canonical section s_{canon} defining $\mathcal{M}_{C - \mathbf{M}(E)E}$.

But this approach also rises three additional issues that we have to address, in order to identify all these localized contributions of top Chern classes.

Problem 4: Is the contribution of localized top Chern classes along $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$ by an inductive application of residual intersection theory of top Chern classes sensitive to the ordering that we choose for the blowing ups of the various $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma')$ applied prior to the given $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$?

and,

Problem 5: The canonical algebraic obstruction bundle $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$ defining algebraic family Seiberg-Witten invariants of $C - \mathbf{M}(E)E$ gets modified into $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H} \otimes \mathcal{O}(-D)$ even after a single usage of residual intersection formula of top Chern class. The symbol here D stands for the exceptional divisor of the blowing up.

Suppose we blow up along each $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$ individually, potentially we have to perform millions of blowing ups on X when n goes large.

²⁷by using the definition of total Segre class of the normal cone $\mathbf{C}_Z X$.

²⁸The set $\text{adm}(n)$ is the set of n -vertex admissible graphs. See [Liu6] section 2 for its definition.

²⁹Consult definition 6 on page 47 of the paper [Liu6].

Are we still able to work with the seemingly complicated residual obstruction vector bundle ³⁰ which get modified and identify the various localized contributions of top Chern class localized in different $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$, $\Gamma \in \Delta(n) - \{\gamma_\delta\}$?

Problem 6: As we have mentioned that different $Y(\Gamma)$, $\Gamma \in \Delta(n) - \{\gamma_\delta\}$, may intersect non-trivially. As a consequence different sub-loci $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$ can touch. There can be a potential danger of “over counting”. How do we avoid over-counting in our scheme?

2.2 Responses to the Above Problems

All these six problems have addressed in the paper [Liu6], which is responsible of its length and complicated notations. Let us sketch how we have addressed these issues in the paper [Liu6], phrased in a less technical term. Hopefully it can provide the reader a guide to read the long paper.

Response to **Problem 1:** While identifying the cycle class $\overline{j^* \mathbf{Z}(s_{\text{canon}})}$ geometrically, we do not attempt to raise L to a very high power to control the regularity of the scheme $Z(s_{\text{canon}}) \times_{M_\delta} Y$. Even though the regularity of $Z(s_{\text{canon}}) \times_{M_\delta} Y$ (or $Z(s_{\text{canon}}) \times_{M_\delta} Y \cap V$) ³¹ may be helpful in identifying the excess contribution $(\mathbf{Z}(s_{\text{canon}}) - \overline{j^* \mathbf{Z}(s_{\text{canon}})}) \cap c_1(\mathbf{H})^{\text{rank}_{\mathbf{C}} \mathbf{V}_{\text{canon}} - \text{rank}_{\mathbf{C}} \mathbf{W}_{\text{canon}} - 1 + 2n} \cap [t_L]$, we construct new machineries to bypass the problem—by using the combinatorial structure on the universal spaces suggested by Gromog-Taubes theory in symplectic geometry.

We only use a slightly strengthened form of Göttsche’s argument to control the locus $Z(s_{\text{canon}}) \times_{M_\delta} Y_{\gamma_\delta}$. Under the $5\delta - 1$ very ampleness assumption on L , $V \cap Z(s_{\text{canon}}) \times_{M_\delta} Y_{\gamma_\delta}$ is a finite sub-scheme in X for generic δ dimensional V . The main theme of the paper [Liu6] is to identify the excess contributions to the family invariant $\mathcal{AFSW}_{M_{\delta+1} \times \{t_L\} \rightarrow M_\delta \times \{t_L\}}(1, c_1(L) - 2 \sum_{1 \leq i \leq n} E_i)$ using only the smoothness of the family moduli spaces of type I exceptional classes and their intersections: $Y(\Gamma)$. We do not try to control the regularity of $Z(s_{\text{canon}}) \times_{M_\delta} Y$.

Response to **Problem 2:** The original problem of enumerating the localized contribution of top Chern classes is formulated in terms of intersection theory [F]. Yet to answer **problem 2**, the ideas inspired from family Gromov-Taubes theory have played crucial roles. The algebraic curves representing the class $C - \mathbf{M}(E)E$ within the family $M_{\delta+1} \times T(M) \mapsto M_\delta \times T(M)$ may fail to be irreducible and may break into more than one irreducible component. While the curves breaking into different components is a purely geometric phenomenon, we have found a topological constraint which forces the degeneration to occur.

Lemma 1 *Let \mathbf{e} be an irreducible exceptional curve representing the exceptiona class e over $b \in M_\delta$ in the family $M_{\delta+1} \mapsto M_\delta$. Suppose that $(C - \mathbf{M}(E)E) \cdot e < 0$, then any effective representative of $C - \mathbf{M}(E)E$ over b has to contain \mathbf{e} as one of its irreducible components.*

Proof: If the curve Σ representing $C - \mathbf{M}(E)E$ lies in the same fiber $M_{\delta+1} \times_{M_\delta} \{b\}$ as \mathbf{e} but does not contain \mathbf{e} as one of its irreducible components. Then all irreducible components of Σ intersect non-negatively with \mathbf{e} . As a consequence their sum $= (C - \mathbf{M}(E)E) \cdot e \geq 0$, violating the assumption! \square

Over the various locally closed and smooth admissible strata Y_Γ , $\Gamma \in \Delta(n) - \{\gamma_\delta\}$, those type I exceptional classes e_i with $e_i^2 < -1$, effective and irreducible over Y_Γ , ³² pair negatively ³³ with

³⁰They are denoted as $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H} \otimes_{\Gamma' \in I_\Gamma} \mathcal{O}(-D_{\Gamma'})$ in the paper [Liu6].

³¹here V is the generic δ -dimensional linear sub-system

³²Suppose j_i denote the direct descendent indexes of i in Γ , then $e_i = E_i - \sum_{j_i} E_{j_i}$.

³³They are denoted as e_{k_i} , $1 \leq i \leq p$ in [Liu6].

$C - \mathbf{M}(E)E$. When e_i is represented by irreducible³⁴ \mathbf{P}^1 , by lemma 1 the condition $(C - \mathbf{M}(E)E) \cdot e_i < 0$ forces any algebraic curve representing $C - \mathbf{M}(E)E$ above the same fiber to break off at least an irreducible \mathbf{P}^1 component representing e_i . This suggests that the canonical algebraic family obstruction bundles³⁵ $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}$ of the new class $C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i$ over $Y(\Gamma)$ will be involved. In [Liu5] and [Liu6], we discuss extensively the relationship of $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}$, its canonical section s_{canon}° and $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$, with its canonical section s_{canon} . The unique factorization of curves from $C - \mathbf{M}(E)E$ to $C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i$ can be formulated as the isomorphism $Z(s_{\text{canon}}^\circ) \times_{M_\delta} Y_\Gamma \cong Z(s_{\text{canon}}) \times_{M_\delta} Y_\Gamma$ of moduli spaces of curves. The algebraic counter-part of family switching formula [Liu5] suggests a natural bundle morphism

$$\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)} \longrightarrow \pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)}$$

whose injectivity over $X \times_{M_\delta} Y_\Gamma$ is responsible for the above isomorphism of moduli space of curves.

The above algebraic structures are suggested by the perturbation argument on Kuranishi models in the differentiable category [Liu1].

We analyze the cycle class localized in $Z(s_{\text{canon}}) \cap Y$ and show that it can be identified with a huge sum of terms (one for each $\Gamma \in \Delta(n) - \{\gamma_\delta\}$, through the residual intersection formula of top Chern class) of certain cycle classes associated to the different obstruction bundles $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}$, one for each statum $Y(\Gamma)$. This procedure produces intersection numbers known as the modified algebraic family Seiberg-Witten invariants of $C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i$, denoted as $\mathcal{AFSW}_{M_{\delta+1} \times_{M_\delta} Y(\Gamma) \rightarrow Y(\Gamma)}^*(c_{\text{total}}(\tau_\Gamma), C - \mathbf{M}(E)E - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i)$.

Once we can identify the intersection number ‘‘localized’’ to each $X \times_{M_\delta} Y_\Gamma$ with the modified invariant, the fact that all these modified algebraic family Seiberg-Witten invariants are ‘‘topological objects’’, i.e. can be expressed as characteristic classes implies that all the intersection numbers attached to $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma) \cap V$ are topological objects.

Response to **Problem 3**: Even though Y itself is not smooth, Y can be identified with the unions of the various $X \times_{M_\delta} Y(\Gamma)$, which are all smooth. The type I exceptional classes effective over $Y(\Gamma)$ have played an essential role in answering Problem 2. On the other hand, each $Y(\Gamma)$ can be interpreted as the locus of co-existence of the type I exceptional class e_i defined by Γ (see section 2, proposition 4 of [Liu6]),

$$Y(\Gamma) = \cap_{1 \leq i \leq n} Y(\Gamma_{e_i}).$$

The above intersection is a regular intersection of smooth spaces $Y(\Gamma_{e_i})$. The Γ_{e_i} is an admissible sub-graph of Γ with one-edges from i -th vertex to all its direct descendants in Γ . They are called fan-like admissible graphs in [Liu6]. The fan-like admissible graphs Γ_{e_i} and Γ give us combinatorial tools to express the enumerative geometric data.

For each $Y(\Gamma) \subset M_\delta$ the normal bundle $\mathbf{N}_{Y(\Gamma)} M_\delta$ is isomorphic to the restriction of the direct sum of normal bundles of $Y(\Gamma_{e_i}) \subset M_\delta$, which are isomorphic to the restriction of canonical algebraic obstruction bundles of e_i , $1 \leq i \leq n$ to $Y(\Gamma)$. This information is crucial for us to identify the individual terms of excess contributions from the residual intersection formula.

The key is the following four-term sheaf exact sequence

$$\begin{aligned} 0 \mapsto \mathcal{R}^0 \pi_* (\mathcal{O}_{\sum_{1 \leq i \leq \Xi_{k_i}}} \otimes \mathcal{E}_{C - \mathbf{M}(E)E}) \mapsto \mathcal{R}^0 \pi_* (\mathcal{O}_{\mathbf{M}(E)E + \sum_{1 \leq i \leq p} \Xi_{k_i}} \otimes \mathcal{E}_C)|_{Y(\Gamma) \times T(M)} \mapsto \\ \mathcal{R}^0 \pi_* (\mathcal{O}_{\mathbf{M}(E)E} \otimes \mathcal{E}_C)|_{Y(\Gamma) \times T(M)} \mapsto \mathcal{R}^1 \pi_* (\mathcal{O}_{\sum_{1 \leq i \leq \Xi_{k_i}}} \otimes \mathcal{E}_{C - \mathbf{M}(E)E}) \mapsto 0, \end{aligned}$$

where Ξ_{k_i} are the universal type I exceptional curve fibrations representing e_{k_i} .

³⁴It is the case above points in Y_Γ .

³⁵Refer to proposition 9 of [Liu5] for more details.

The second and the third terms in the above sequence are locally free and their associated vector bundles are $\mathbf{W}_{\text{canon}}^\circ$ and $\mathbf{W}_{\text{canon}}$, respectively.

This four-term exact sequence allows us to “transfer” the localized contribution of top Chern class of $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$ along $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$ to the top Chern class of $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}$ and chern classes of some additional obstruction virtual bundle τ_Γ (see definition 10 of [Liu6]).

Response to **Problem 4**: In proposition 16 of [Liu6], we show that suitable permuting the blowing up orders or some special collapsing/grouping of family moduli spaces does not affect the answer of the evaluation. Let P be an index subset of $\Delta(n) - \{\gamma_\delta\}$. The key observation is that no matter which $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma')$ we have blown up earlier, when we push-forward the sum of the various localized contributions of top Chern classes (indexed by elements in P) to X , the total sum is always equal to $\{c_{\text{total}}(\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}) \cap s_{\text{total}}(Z(s_{\text{canon}}) \times_{M_\delta} (\cup_{\Gamma \in P} Y(\Gamma)), X) \cap [X]\}_{\dim_{\mathbf{C}} X - \text{rank}_{\mathbf{C}} \mathbf{W}_{\text{canon}}}$. The rationale behind this statement is based on the well known property that total Segre class is invariant under proper birational push-forward (see proposition 4.2 on page 74 of [F]).

This observation allows us to permute the blowup orderings ahead of any particular blowing up while identifying a particular localized contribution of the top Chern class.

Remark 4 *The permutation of the blowing up orderings depend on each individual $Y(\Gamma)$ we focus upon and is not universal.*

Response to **Problem 5**: **Problem 5** is conceptually the most challenging one to answer in our approach. Our key observation, proposition 9 of [Liu6], answers the question in a slightly surprising way. Quite opposite to the naive intuition, the repeated blowing ups does not spoil our enumeration program. In fact it is necessary in identifying the localized contribution of top Chern classes!

In the earlier responses to **problem 2 & 3**, we has made use of $\mathbf{W}_{\text{canon}}^\circ$ to replace $\mathbf{W}_{\text{canon}}$. On the other hand, the bundle map $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)} \mapsto \pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)}$ is generically injective over $X \times_{M_\delta} Y_\Gamma$ but may fail to be injective on a closed subset of $X \times_{M_\delta} (Y(\Gamma) - Y_\Gamma)$. The failure of the bundle injection is related to the appearance of some other $Y(\Gamma')$, $\Gamma \succ \Gamma'$; $Y(\Gamma) \cap Y(\Gamma') \neq \emptyset$ ³⁶. The failure of the bundle map to be injective throughout $X \times_{M_\delta} Y(\Gamma)$ forbids us to replace $c_{\text{top}}(\pi_X^* \mathbf{W}_{\text{canon}}|_{Y(\Gamma)} \otimes \mathbf{H})$ directly by the cap product of $c_{\text{top}}(\pi_X^* \mathbf{W}_{\text{canon}}^\circ|_{Y(\Gamma)} \otimes \mathbf{H})$ with $c_{\text{top}}(\pi_X^* \mathbf{W}_{\text{canon}}/\mathbf{W}_{\text{canon}}^\circ|_{Y(\Gamma) \times T(M)} \otimes \mathbf{H})$.

In this crucial proposition, we observe that the inductively blowup procedure modifies the top Chern class of $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}$ in a magical way that numerically it is possible to identify the top Chern class of the modified vector bundle with the cap product of the top Chern classes of \mathbf{V}_{quot} and³⁷ of $\pi_X^* \mathbf{W}_{\text{canon}}^\circ|_{Y(\Gamma)} \otimes \mathbf{H}$, the canonical algebraic obstruction bundle of the new class $C - \mathbf{M}(E) - \sum_{e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i$.

The key observation is that the blowing ups along these $Z(s_{\text{canon}}) \cap Y(\Gamma')$, $\Gamma' \in \bar{I}_\Gamma - \bar{I}_\Gamma^{\gg}$, we³⁸ have performed are exactly along the sub-loci of $(Z(s_{\text{canon}}) - Z(s_{\text{canon}}^\circ)) \times_{M_\delta} Y(\Gamma)$ which account for the discrepancy $Z(s_{\text{canon}}) - Z(s_{\text{canon}}^\circ)$ in $Y(\Gamma)$. On the other hand, the discrepancy occurs exactly because the bundle map $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)} \xrightarrow{f} \pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)}$ fails to be injective over some sub-locus of $X \times_{M_\delta} (Y(\Gamma) - Y_\Gamma)$. Moreover we actually identify $\overline{Z(s_{\text{canon}}) - Z(s_{\text{canon}}^\circ)} \times_{M_\delta} Y(\Gamma)$ with the intersection of s_{canon}° and the kernel cone of the above bundle map f .

Then in proposition 9 on page 40 of [Liu6], we justify the usage of the blowup construction to be the right procedure to relate the top Chern classes of $\pi_X^* \mathbf{W}_{\text{canon}}^\circ \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)}$ and of $\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H}|_{X \times_{M_\delta} Y(\Gamma)}$

³⁶Consult corollary 3 on page 40 of [Liu5] for more details.

³⁷The quotient bundle \mathbf{V}_{quot} of $\mathbf{W}_{\text{canon}}|_{Y(\Gamma) \times T(M)}$ is defined in the start of section 3 of [Liu6].

³⁸The index set $\bar{I}_\Gamma - \bar{I}_\Gamma^{\gg} \subset \Delta(n)$ consists of the admissible graphs Γ' such that (i). the strict transforms of $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma')$ are blown up ahead of the strict transform of $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma)$ in our initial scheme, (ii). $\Gamma \succ \Gamma'$. See definition 6 on page 47 of [Liu6] for more details.

Response to **Problem 6**: The way that we avoid over-counting is to demonstrate that our blowup construction is compatible with the inclusion-exclusion principle. To illustrate our basic idea, we consider a toy model regarding the number of elements in a unions of finite sets.

Recall a simple fact from the theory of finite sets. Let $|A|$ denote the cardinality of the finite set A . Then $|A \cup B| + |A \cap B| = |A| + |B|$. In other words, we may use $|A| + |B| - |A \cap B|$ to calculate $|A \cup B|$. This suggests that one may subtract the “over-counting” term from $|A| + |B|$ to get the correct answer. On the other hand, an alternative way to calculate $|A \cup B|$ is to re-express $A \cup B = (A - A \cap B) \cup (A \cap B) \cup (B - A \cap B)$, a disjoint union of subsets of $A \cup B$. Then

$$|A \cup B| = |A - A \cap B| + |A \cap B| + |B - A \cap B| = (|A| - |A \cap B|) + |A \cap B| + (|B| - |A \cap B|).$$

For a finite union of finite sets $A_1, A_2, A_3, \dots, A_k$, these two counting schemes lead to two equivalent yet distinct formulae.

In the first scheme we write $|\cup_{1 \leq i \leq k} A_i| = \sum_{I \subset \{1, 2, \dots, k\}} (-1)^{|I|} |\cap_{i \in I} A_i|$, where the index sets run through all the subsets of $\{1, 2, \dots, k\}$. This is the standard form of inclusion/exclusion formula. In the second case we define the modified cardinality $|\cap_{i \in I} A_i|^*$ to be $|\cap_{i \in I} A_i - \cup_{I \subset J \neq I} \cap_{i \in J} A_i|$, i.e. the cardinality of the elements in $\cap_{i \in I} A_i$ which are not in any refined intersection $\cap_{i \in J} A_i$ for any $J \supset I$, $J \neq I$.

Lemma 2 *For all $I \subset \{1, 2, \dots, k\}$, let $|\cap_{i \in I} A_i|^*$ denote the modified cardinality of $\cap_{i \in I} A_i$. Then we have the following identities relating the modified and the original cardinalities,*

$$|\cap_{i \in I} A_i|^* = |\cap_{i \in I} A_i| - \sum_{J \supset I; J \neq I} |\cap_{i \in J} A_i|^*,$$

for all $I \subset \{1, 2, \dots, k\}$.

Proof: By adjusting the terms in the above identities, it suffices to show that for all $I \subset \{1, 2, \dots, k\}$,

$$|\cap_{i \in I} A_i|^* + \sum_{J \supset I; J \neq I} |\cap_{i \in J} A_i|^* = |\cap_{i \in I} A_i|.$$

One may write $\cap_{i \in I} A_i$ as the disjoint union

$$\cap_{i \in I} A_i = \coprod_{I \subset I'} (\cap_{i \in I'} A_i - \cup_{I' \subset J \neq I'} (\cap_{i \in J} A_i)),$$

as each element $\in \cap_{i \in I} A_i$ appears in the disjoint union on the right hand side exactly once.

By taking cardinalities on both sides and by using the definitions of the modified cardinalities, we get the desired equality. \square

In this toy model we “stratify” the union $\cup_{i=1}^k A_i$ into different set theoretical strata according to the complete collections of A_j that an element $\in \cup_{i=1}^k A_i$ belongs to.

The “modified” cardinalities of a finite set counts the number of elements in a finite intersection of A_j which do not lie in the intersection of more A_j s. This concept is the prototype of the modified family invariants in [Liu1] and [Liu6].

The important characteristics are:

- (a). When there are more and more finite sets involved, the various intersections of distinct sets lead to a hierachy of intersections and a hierachy of “modified” cardinalities.
- (b). To define the modified cardinality numerically, by lemma 2 we may start with the un-modified cardinalities and subtract away all the “correction terms” of modified cardinalities involving intersections of more A_j s.

It involves an inductive definition. If there are m finite sets A_i , $1 \leq i \leq k$, all together. Then the modified cardinality of $\cap_{i=1}^{i=k} A_i$ is set to coincide with the usual (un-modified) cardinality. By a backward induction decreasing the number of intersections of finite sets A_j , at the end one can define the modified cardinalities for all A_i .

(c). This alternative approach does not involve the alternating sum of terms which are typical to the first formulation of the inclusion-exclusion principle.

For simplicity, let us illustrate the basic idea how the modified algebraic family invariants are defined in [Liu6] by working on the simplified situation $\Gamma_1, \Gamma_2 \in \Delta(n) - \{\gamma_\delta\}$.

The inductive blowup construction in section 5 of [Liu6] has been performed in such a way that when X is blown up along $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma_1)$ and along $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma_2)$, a blowing up along the locus $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma') \supset Z(s_{\text{canon}}) \times_{M_\delta} (Y(\Gamma_1) \cap Y(\Gamma_2))$, $\Gamma' \in \Delta(n) - \{\gamma_\delta\}$ has been done in advance. That is why the family invariants we attach to $Y(\Gamma_1)$ or $Y(\Gamma_2)$ get “modified” in this process.

Symbollically let $n_{Y(\Gamma_1) \cup Y(\Gamma_2)}$, $n_{Y(\Gamma_1)}$, $n_{Y(\Gamma_2)}$, and $n_{Y(\Gamma')}$ denote the intersection numbers

$$\{c_{\text{total}}(\pi_X^* \mathbf{W}_{\text{canon}} \otimes \mathbf{H} \otimes \mathcal{O}(-D_Z)) \cap s_{\text{total}}(Z, X)\}_{\dim_{\mathbf{C}} X - \text{rank}_{\mathbf{C}} \mathbf{W}_{\text{canon}}} \cap c_1(\mathbf{H})^{\text{rank}_{\mathbf{C}}(\mathbf{V}_{\text{canon}} - \mathbf{W}_{\text{canon}}) - 1 + \dim_{\mathbf{C}} M_\delta} \cap \{t_L\}$$

built up from the localized contributions of top Chern classes attached to (a). $Z = Y(\Gamma_1) \cup Y(\Gamma_2)$, setting $D_Z = \emptyset$, (b). $Z = Y(\Gamma_1)$, or $Z = Y(\Gamma_2)$ and setting D_Z to be the blowup exceptional divisor of $Z(s_{\text{canon}}) \times_{M_\delta} Y(\Gamma')$, and (c). $Z = Y(\Gamma')$, setting $D_Z = \emptyset$, respectively. In this simplified model example the modified intersection numbers (called modified algebraic family Seiberg-Witten invariants in the long paper [Liu6]) $n_{Y(\Gamma_1)}^*$, $n_{Y(\Gamma_2)}^*$, and $n_{Y(\Gamma')}^*$ have to satisfy

- (i). $n_{Y(\Gamma')}^* = n_{Y(\Gamma')}$.
- (ii). $n_{Y(\Gamma_1)}^* = n_{Y(\Gamma_1)} - n_{Y(\Gamma')}^*$.
- (iii). $n_{Y(\Gamma_2)}^* = n_{Y(\Gamma_2)} - n_{Y(\Gamma')}^*$.

And finally we have

$$n_{Y(\Gamma_1) \cup Y(\Gamma_2)} = n_{Y(\Gamma_1)}^* + n_{Y(\Gamma_2)}^* + n_{Y(\Gamma')}^*.$$

The general cases are dealt with by the same philosophy. In the following we outline the parallelism between the general case and the above toy model.

I. The analogue of the inclusion relationship \subset among different finite intersections of the finite sets A_i in family Seiberg-Witten theory is coded by the partial ordering \gg among the graphs Γ defined in definition 11 of the long paper [Liu6].

The partial ordering \gg gives a sufficient condition which guarantees the subscheme $\mathcal{M}_{C-\mathbf{M}(E)E - \sum_{(C-\mathbf{M}(E)E) \cdot e'_i < 0} e'_i} \times_{M_\delta} Y(\Gamma')$ is included in $\mathcal{M}_{C-\mathbf{M}(E)E - \sum_{(C-\mathbf{M}(E)E) \cdot e_i < 0} e_i} \times_{M_\delta} Y(\Gamma) = Z(s_{\text{canon}}^\circ) \times_{M_\delta} Y(\Gamma)$.

II. The “modified algebraic family invariants” are defined following exactly the same pattern that the “modified cardinalities” are defined above. It involves more technical arguments and the introduction of the classes $\tau_\Gamma \in K_0(Y(\Gamma) \times T(M))$.

Starting with some admissible graph Γ smallest under \gg , the modified invariant attached to $\mathcal{M}_{C-\mathbf{M}(E)E - \sum_{(C-\mathbf{M}(E)E) \cdot e_i < 0} e_i} \times_{M_\delta} Y(\Gamma)$ does not need any correction terms and is set to be equal to some mixed family invariant. See definition 12 of [Liu6] for more details.

By an induction argument upon the reversed ordering of \gg , we define the modified family invariants attached to each $Y(\Gamma)$, $\Gamma \in \Delta(n)$. This has been done in definition 13 and 14 of [Liu6].

III. In our definitions of modified invariants, the alternating sign does not occur. This is because we adopt the additive formulation of the inclusion-exclusion principle. It is not hard to reformulate our approach into the standard inclusion-exclusion formulae involving alternating sums.

3 The Open Problems Related to the Proof of Universality Theorem

In this sub-section, we list some open problems related to or inspired from the proof of the universality theorem.

In [Liu1] we had identified the coefficients of the universal polynomials using Göttsche' proposal, identifying them with parts of Gromov-Witten invariants on \mathbf{P}^2 , $K3$, T^4 and using symplectic technique (including Taubes' identification of $SW = Gr$ for smooth pseudo-holomorphic curves). On the other hand, the construction of residual intersection formula of top Chern classes and the repeatedly scheme-theoretical blowing ups of $X = \mathbf{P}(\mathbf{V}_{\text{canon}})$ in the algebraic category enables us to construct a cycle class (the localized top Chern class of some modified algebraic obstruction bundle³⁹) representing the virtual fundamental class of moduli scheme of curves with δ -node nodal singularities.

Encouraged by the \mathcal{C}^∞ identification of the intersection numbers attached to the cycle classes with the explicit enumerations of Gromov-Witten invariants in the concrete examples through pseudo-holomorphic curves, we may post the following open question:

Open Question: Identify the generating functions of the universal polynomials algebraically.

The universality theorem asserts the existence of these universal polynomials for all $\delta \in \mathbf{N}$ which code the enumerative information on the “number of nodal curves”. By an ingenious argument of Göttsche he has shown [Got] that the generating function of these universal polynomials takes a factorizable form

$$\mathcal{F}(q) = A_1(q)^{c_1(K_M)^2} A_2(q)^{c_2(M)} A_3(q)^{c_1(L)^2} A_4(q)^{c_1(L) \cdot c_1(K_M)}.$$

Substituting q by⁴⁰ $DG_2(q)$, then $\mathcal{F}(DG_2(q))$ can be identified with

$$\frac{(DG_2(q)/q)^{\chi(L)} B_1(q)^{c_1(K_M)^2} B_2(q)^{c_1(L) \cdot c_1(K_M)}}{(\Delta(q)D^2G_2(q)/q^2)^{\frac{\chi(O_M)}{2}}},$$

where B_1 and B_2 are the two power series derived by Göttsche [Got] starting with

$$B_1(q) = 1 - q - 5q^2 + 30q^3 - 345q^4 + 2961q^5 \dots,$$

and

$$B_2(q) = 1 + 5q + 2q^2 + 35q^3 - 140q^4 + 986q^5 + \dots$$

On the other hand, the original \mathcal{C}^∞ approach of the problem suggests a symplectic generalization of the universality theorem.

Open Question: Formulate and Generalize the universality theorem to symplectic four-manifolds M with/without the $b_2^+ = 1$ condition.

Open Sub-Question: For symplectic four-manifolds with $b_2^+ > 1$ (which correspond to algebraic surfaces with $p_g > 0$), find the right analogue for the algebraic family Seiberg-Witten invariants which correspond non-trivial family Seiberg-Witten invariants.

Open Sub-Question: Find the suitable definition of “number of nodal curves” or more generally the “number of singular curves” in the symplectic category. Our study based on algebraic geometric means suggests that one may need to work in the framework of the “virtual numbers” instead of the discrete number count of curves in a given class $\in H^2(M, \mathbf{Z})$.

³⁹We do not write down the modified bundle here. Please consult the paper [Liu6] for the detail expression.

⁴⁰ $D = q \frac{d}{dq}$ and $G_2(q)$ is the quasi-modular form $\frac{-1}{24} + \sum_{k \geq 1} \sigma_1(k)q^k$.

The residual obstruction vector bundle construction in the proof of the universality theorem not only defines the integer valued “number of nodal curves”, it also constructs the virtual fundamental class of δ -node singular curves in a “ $5\delta - 1$ ”-very ample linear system. On the other hand, the technique of residual intersection theory upon type *II* exceptional curves [Liu7] allows us to drop the $5\delta - 1$ -very ample condition and define the virtual fundamental class of the moduli space of nodal curves for general linear systems on general algebraic surfaces.

In our foundation the nodal curve invariant counts embedded nodal curves. On the other hand, the Gromov-Witten theory counts maps, including the various types of multiple covering maps induced from different fractions of the class.

Unlike Gromov-Witten theory which are \mathbf{Q} valued, our theory does not make use of the moduli stacks of curves. Thus our invariants are naturally \mathbf{Z} valued. This fits to the philosophy of Gopakumar-Vafa conjecture [GV], [BP], [HST] in constructing \mathbf{Z} -valued curve counting invariants.

Thus, it is very desirable to find out:

Open Question: Multiple Covering Formula—Find out the weighted contribution of the multiple covering maps and relate our “nodal curve” invariant with the usual Gromov-Witten invariant when $p_g = 0$. When $p_g = 1$ and $M = K3$ or T^4 , we may consider the \mathbf{S}^2 hyperkahler families of $K3$ and the \mathbf{S}^2 family Gromov-Witten invariant and formulate a similar problem.

Another interesting question worthy to pursue is,

Open Question: Go beyond the range of nodal curve singularities and study the structure of “number of singular curves” with non-nodal singularities.

In particular, the algebraic proof of the universality theorem along with the finiteness result of the curves $\in |L|$ for an effective bound on the very-ampleness of L also work for curves with ordinary singularities with multiplicity $m > 2$.

Open Sub-Question: Generalize the Caporaso-Harris [CH] recursive formula of counting of nodal curves on \mathbf{P}^2 to curves with ordinary singularities with multiplicity $m > 2$.

Caporaso-Harris original argument could be interpreted naturally in terms of Gromov-Witten invariants of \mathbf{P}^2 (see e.g. section 7 of [Va]), the fact that their original argument does not rely on the usage of Gromov-Witten invariants⁴¹ may look encouraging to us.

Open Problem: The local proof of the Blowup formula.

If one blows up the algebraic surface M at a single point and get \tilde{M} , then the pull-back of a $5\delta - 1$ -very ample line bundle L on M fails to be very ample on the blown up surface \tilde{M} exactly at the exceptional locus of $\tilde{M} \mapsto M$. On the other hand, a simple calculation on Göttsche’s formula indicates that the formula of nodal curves changes by the following universal formula (again substituting q by $DG_2(q)$).

$$\mathcal{F}_{\tilde{M}}(DG_2) = \mathcal{F}_M(DG_2) \cdot \left(\frac{B_2(q)}{B_1(q)}\right) \cdot \left(\frac{DG_2}{q}\right)^{-1}.$$

The power series $B_1(q)$, $B_2(q)$ are determined from J. Harris and Caporaso’s calculation of Severi degrees [CH] by Göttsche [Got] modulo Yau-Zaslow formula, etc.

The blowup formula implies that a purely “local way” to identify the coefficients of $B_1(q)$, $B_2(q)$ is possible. In fact, such an identification will relate intersection numbers of local nature to the global enumerative geometry datum gathered from Caporaso-Harris calculation [CH] and it is very interesting to understand their relationship.

⁴¹for non-nodal curves, it is unclear at this moment that there are “Gromov-Witten type invariants” corresponding to them.

Open Sub-Problem: Understand the intrinsic geometric meaning of the power series $B_1(q)$ and $B_2(q)$.

By comparing the Yau-Zaslow formula $\prod_i \frac{1}{(1-q^i)} c_2(M)$, $M = K3$ with the Riemann-Roch formula, it is clear that $\prod_i \frac{1}{(1-q^i)}$ is the natural prolongation of the coefficient $\frac{1}{12}$ in front of $c_2(M)$ in the classical Noether formula. It is desirable to understand the relationship of $B_1(q)$ and $B_2(q)$ with the other coefficients of the surface Riemann-Roch formula.

Our formulation of the universality theorem as the natural prolongation of Riemann-Roch theorem suggests that the generating power series of “numbers of singular curves” with different prescribed topological types of curve singularities should be related to each other and prolong the classical Riemann-Roch formula in a natural way. So we propose to study the topological structure of curve singularities in order to find recursive formulae between different singularities.

It is known that the topology of an isolated algebraic curve singularity is completely coded by the topology of the link space $\subset \mathbf{S}^3$ of the singularity. Then the topology of the link is coded by the Alexander polynomial of the link.

Thus, one may ask the following question,

Open Question: Find out the dependence of the universality formulae on the alexander polynomials of the link spaces of the singularities.

Evidence of the nodal curve case shows that a part of the generating function has been modular on Calabi-Yau algebraic surfaces (i.e. $K3$ or T^4).

We expect to get liftings from Alexander polynomials of links to modular objects in this generalization.

We also mention that by generalizing the tools we have used to higher dimensions, the universality theorem can be generalized to higher dimensions. Instead of counting curves with prescribed singularities, we enumerate divisors with prescribed singularities in very ample linear systems. Then the program involves prolonging the whole $Todd_M \cdot ch(L)$ expression into multiplicative power series of powers of the Chern numbers $c_1^{a_1}(M)c_2^{a_2}(M) \cdots c_n^{a_n}(M)c_1^k(L)$, with $\sum ia_i + k = n$.

Finally, we have noticed earlier (see page 4) that the simple invariants that we have attached to linear systems are the degenerated cases (reduced to $dim_{\mathbf{C}} M = 2$) of Donaldson-Thomas invariants of Hilbert schemes of curves. Our machinery has provided a tower of enumerative invariants of singular curves from the leading invariant of linear systems.

Open Question: Generalize the concept of Nodal Curve Invariants to the case of Calabi-Yau three-folds. The algebraic construction of nodal curve invariants is supposed to probe the singular curves inside the Hilbert schemes of curves such that Donaldson-Thomas invariants are the first order leading invariants. It should be closed related to the hypothetical integral valued Gopakumar-Vafa invariants proposed in [GV].

References

- [BL] Jim Bryant and N. Leung, *The Enumerative Geometry of K3 Surfaces and Modular Forms*. **alg-geom 9711031** (1997).
- [BP] J. Bryant, R. Pandharipande, *BPS States of Curves in Calabi-Yau 3-folds*, **5** Geometry and Topology, AG/0009025 (2001).
- [BPV] W. Barth, C. Peters, A. Van De Ven, *Compact Complex Surfaces* Ergebnisse der Math. **4** New York: Springer-Verlag (1984)

- [BS] M. Beltrametti, A. Sommese, *Problems on surfaces and their classification* INDAM, Academic Press.
- [CH] Joe Harris and Caporaso. *Counting plane Curves of any Genus*. **alg-geom/9608025** (1996).
- [Di] J. Dieudonne, *A History of Algebraic and Differential Topology 1900-1960*, Birkhauser (1989).
- [DT] S. Donaldson and R. Thomas, *Gauge Theory in Higher Dimensions*, in the Geometric Universe: Science, Geometry and the work of Roger Penrose, S. Huggett et. al eds., Oxford University Press, (1998).
- [F] W. Fulton. *Intersection Theory*. A series of Modern Surveys in Mathematics, Springer-Verlag, (1984).
- [FM] R. Friedman and J. Morgan *Algebraic Surfaces and Seiberg-Witten Invariants*. Journal of Algebraic Geometry, **6**, no.6, pp445-479, (1997).
- [FM2] R. Friedman and J. Morgan *Obstruction Bundles, Semiregularity, and Seiberg-Witten Invariants*. Communications of Analysis and Geometry, **7**, no.3, pp451-495, (1999).
- [GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. New York: John Wiley and Sons, (1978).
- [GV] R. Gopakurma and C. Vafa, *M Theory and Topological Strings-II*, hep-th/9812127, (1998).
- [Gat] A. Gathmann *The Number of Plane Conics 5-Fold Tangent to a Given Curve*. preprint, **arXiv:math.AG/0202002**, (2002).
- [Got] L. Göttsche. *A Conjectural Generating Function for Numbers of Curves on Surfaces*. Preprint. **alg-geom 9711012** (1997).
- [H] J. Harris, *On the Severi Problem*, Invent. Math. **84** pp445-461, (1986).
- [Ha] R. Hartshorne, *Algebraic Geometry*. Graduate texts in Mathematics, Springer-Verlag **52**
- [Hir] F. Hirzebruch *Topological Methods in Algebraic Geometry*. Grundlehren der math. Wissenschaften, **131**, Third enlarged edition, Springer-Verlag, (1966).
- [HST] S. Hosono, M. Saito, A. Takahashi *Relative Lefschetz Action and BPS State Counting*. Internat. Math. Res. Notices **15**, pp783–816, (2001).
- [KP] S. Kleiman and Piene. *Enumerating Curves on Singular Surfaces, Algebraic Geometry: Hirzebruch 70* **241**, Contemp. Math. Amer. Math. Soc. pp. 209-238, Providence, RI, (1999).
- [HM1] J. Harvey and G. Moore, *Algebras, BPS states, and Strings*. hep-th 9510182, (1995).
- [LiT1] J. Li, G. Tian *Virtual Moduli Spaces and Gromov-Witten Invariants of Algebraic Varieties*. Journal of Amer. Math. Soc. **11**, pp119-174 (1998)
- [LiT2] J. Li, G. Tian *Virtual Moduli Spaces and Gromov-Witten Invariants of General Symplectic Manifolds*. Preprint alg-geom/9608032
- [Liu] A. K. Liu, *Ph.D thesis* Harvard University, (1996).
- [Liu1] A. K. Liu, *Family Blowup Formula, Admissible Graphs and the Enumeration of Singular Curves (I)*. Journal of Differential Geometry **56** pp381-579 (2001).

- [Liu2] A. K. Liu, *Cosmic String and Family Seiberg-Witten Theory* preprint, in preparation (2004).
- [Liu3] A. K. Liu, *The Family Blowup Formula of the Family Seiberg-Witten Invariants* preprint, DG0305294, (2003).
- [Liu4] A. K. Liu, *A Note on Curve Counting Scheme in an Algebraic Family and the Admissible Decomposition Classes* preprint DG/0308196, (2003).
- [Liu5] A. K. Liu, *Family Switching Formula and the $-n$ Exceptional Rational Curves* preprint DG/0308197 (2003).
- [Liu6] A. K. Liu, *the Algebraic Proof of the Universality Theorem* preprint AG/0402045 (2004).
- [Liu7] A. K. Liu, *The Residual Intersection Formula of Type II Exceptional Curves*, in preparation (2004).
- [LL1] T. J. Li and A. K. Liu, *Family Seiberg-Witten Invariant and Wall Crossing Formula*. to appear in Communications of Analysis and Geometry.
- [LL2] T. J. Li and A. K. Liu. *General Wall Crossing Formula*. Mathematical Research Letters. **2** pp. 97-118, (1995).
- [LL3] T. J. Li and A. K. Liu. *The Symplectic Structures of Rational and Ruled Surfaces and the generalized Adjunction inequality*. **2** pp. 453-471, (1995).
- [Law] B.H. Lawson Jr. and M-L. Michelson. *Spin Geometry* Princeton Mathematical Series **38** Princeton Univ. Press, Princeton (1989).
- [MNOP] D.Maulik, N. Nekrasov, A. Okounkov and R. Randharipande. *Gromov-Witten Theory and Donaldson-Thomas Theory*, AG/0312059, (2003).
- [Mor] J. W. Morgan, *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Mathematical Notes (Princeton University Press) **44** (1996).
- [Mum] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties* Grundlehren der math. Wissenschaften **221**, Springer-Verlag, (1976).
- [Ran] *Enumerative Geometry of Singular Plane Curves*, Invent. Math. **97** pp447-465, (1989).
- [T1] C.H. Taubes. *SW \rightarrow Gr, from the Seiberg-Witten Equations to Pseudo-holomorphic Curves*. Journal of American Mathematical Society. **9** no. 3. (1996).
- [T2] C.H. Taubes, *Gr \rightarrow SW, from Pseudo-holomorphic Curves to Seiberg-Witten Solutions*, Journal of Differential Geometry, **51**, pp203-334. (1999).
- [T3] C.H. Taubes, *The Seiberg Witten Invariants and Gromov Invariant*, Mathematical Research Letters. **2** pp221-238, (1995).
- [Th] R. Thomas, *A Holomorphic Casson Invariant for Calabi-Yau 3-folds and Bundles on K3 fibrations* Journal of Differential Geometry **54** pp 367-438,(2000).
- [V] Israel Vainsencher. *Enumeration of n -fold tangent hypersurfaces to a surface*, Journal of Algebraic Geometry. **4** pp503-526. (1995).
- [Va] Ravi Vakil. *Counting Curves of Any Genus of Rational Ruled Surfaces*, **alg-geom 9709003** (1997).

- [W] Edward Witten. *Monopoles and Four Manifolds*. Math. Research Letters. **1**, pp. 769-796. (1994).
- [YZ] S. T. Yau and E. Zaslow. *BPS states, string duality, and Nodal Curves on K3*. Nuclear Physics B. **471** no. 3. pp503-512. (1996).
- [Za] O. Zariski *Algebraic Surfaces*. Springer, Classics in Mathematics, Reprint of 1971 edition. (1995).