# Twisted Hochschild Homology
# of Quantum Hyperplanes

Andrzej Sitarz [*]

*Institute of Physics, Jagiellonian University,*
*Reymonta 4, 30-059 Kraków, Poland*
e-mail: sitarz@if.uj.edu.pl

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**Abstract**

We calculate the Hochschild dimension of quantum hyperplanes using the twisted Hochschild homology.

## 1 Introduction

In noncommutative geometry the Hochschild and cyclic homology take the role of differential de Rham complexes and de Rham homology. It was observed quite early that in many canonical noncommutative examples the Hochschild dimension of a noncommutative space differs from the one of its commutative limit. This seems to be the feature of quantum deformations, in particular the quantum $SU_q(2)$ group and its homogeneous spaces (Podles spheres), which show the "dimension drop" [5]. Similarly, in the case of generic quantum deformations of hyperplanes the effect is even bigger, as irrespectively of the rank their Hochschild dimension is 1 [7].

The twisted (modular) Hochschild and cyclic homology was introduced by Kustermans, Murphy and Tuset [4] based on the notion of modular automorphisms and

its relation with the Haar state on the algebra on the quantum group. The latter satisfies *twisted* cyclicity, which could be extended to closed graded twisted traces on the differential algebras over the quantum group, the latter giving explicit presentation of twisted cyclic cocyles. The notions of the twisted (modular) Hochschild and cyclic homologies became more attractive after explicit demonstration of twisted cyclic cocycles of dimension $3$ for $SU_q(2)$ [4] and of dimension $2$ for the standard Podles sphere, as well as the demonstration of the Connes-Moscovici local formula or this cocycle [6] using the spectral triple derived in [1]. More recent calculations of examples [2, 3] of twisted Hochschild cohomology for quantum spaces confirm that for certain automorphisms the dimension drop does not occur.

Apart from the motivation by the quantum group symmetries and the Haar state, the twisted Hochschild and cyclic complexes can be introduced for any algebra automorphism. In this paper we shall investigate the example of quantum hyperplanes.

## 2 Preliminaries

Throughout the paper we work over the field of complex numbers. Let $\sigma$ be an automorphism of the algebra $\mathcal{A}$ and let $\mathcal{A}_\sigma$ be the vector space $\mathcal{A}$ as a $\mathcal{A} \otimes \mathcal{A}^{op}$ right module with the following right-$\mathcal{A}$-module structure:

$$b \cdot (a_0, a_1) := \sigma(a_1) b a_0, \tag{1}$$

We can consider now three types of homologies (we use three different names in order to distinguish between them later).

**Natural twisted Hochschild homology.** This, we define as the usual Hochschild homology of $\mathcal{A}$ with values in the bimodule $\mathcal{A}_\sigma$, $HH(\mathcal{A}, \mathcal{A}_\sigma)$. It is the homology of the following Hochschild chain complex:

$$\cdots \xrightarrow{b} C_{n+1} \xrightarrow{b} C_n \xrightarrow{b} \cdots \xrightarrow{b} C_1 \xrightarrow{b} C_0, \tag{2}$$

where $C_n = \mathcal{A}_\sigma \otimes \mathcal{A}^{\otimes(n)}$.

**Invariant twisted Hochschild homology**. We can view the automorphism $\sigma$ as the generator of the action of $\mathbb{Z}$ on the algebra $\mathcal{A}$. One may easily verify that the map $b$ of the chain complex (2) is equivariant with respect to this action, that is, it commutes with the action of the group. Therefore, if we take

a subspace containing only invariant elements $C_n^{inv} = (\mathcal{A}_\sigma \otimes \mathcal{A}^{\otimes n})_{inv}$, that is: $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in C_n^{inv}$ if and only if:

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n = \sigma(a_0) \otimes \sigma(a_1) \otimes \cdots \otimes \sigma(a_n).$$

then the following is a subcomplex of (2):

$$\cdots \xrightarrow{b} C_{n+1}^{inv} \xrightarrow{b} C_n^{inv} \xrightarrow{b} \cdots \xrightarrow{b} C_1^{inv} \xrightarrow{b} C_0^{inv}, \tag{3}$$

and we shall denote its homology by $H(\mathcal{A}, \mathcal{A}_\sigma)_{inv}$.

**Twisted Hochschild homology.** Finally, we can consider the quotient of the Hochschild complex by the image of the $1 - \sigma$ map, $C^n/(1 - \sigma)$. We obtain in this way another chain complex:

$$\cdots \xrightarrow{b} C_{n+1}/(1 - \sigma) \xrightarrow{b} C_n/(1 - \sigma) \xrightarrow{b} \cdots \xrightarrow{b} C_1/(1 - \sigma) \xrightarrow{b} C_0/(1 - \sigma), \tag{4}$$

and we shall denote its homology by $HH_\sigma(\mathcal{A})$ and call it (after [4]) the twisted Hochschild homology of $\mathcal{A}$. Since only for such chains twisted cyclicity condition makes sense, this complex is the one related to twisted cyclic homology.

Later, we shall need the following lemma.

**Lemma 2.1.** *Let $(K_*, d)$ and $(C_*, b)$ be chain complexes with the defined action of a group $G$ such that both $b$ and $d$ are equivariant maps. If $f : K_* \to C_*$ is an equivariant quasi-isomorphism of chain complexes and:*

$$b^{-1}(K_n^{inv} \cap B(K_n)) = K_{n+1}^{inv} + Z(K_{n+1}), \tag{5}$$

*then the restriction of $f$ to the invariant subcomplexes is also a quasi-isomorphisms.*

*Proof.* Clearly $f : K_*^{inv} \to C_*^{inv}$ is a morphism of chain complexes, so we need to proof that the induced map in their homology is an isomorphism. Take $k \in K_n^{inv}$ such that $bk = 0$ and assume that the class in $H(C_*^{inv})$ of its image $f(k)$ is trivial. Clearly, then the class of $f(k)$ in $H(C_*)$ is trivial and since $f_*$ is a quasi-isomorphism we know that the class of $k$ in $H(K_*)$ is trivial, that is that $k = bk_0$ for some $k_0 \in K_{n+1}$. Now, using the assumption (5) we know that $k_0$ could be presented as $k_0' + k'$, where $k_0' \in K_{n+1}^{inv}$ and $k' \in Z(K_{n+1})$. Then $bk_0' = k$ within $K_*^{inv}$ and the class of $k$ is indeed trivial in $H(K_*^{inv})$. $\qquad\square$

We shall end the section by introducing the definition of a *scaling automorphism* and proving two important lemmas on twisted Hochschild homology in that case.

**Definition 2.2.** We say that $\sigma$ is a *scaling automorphism* of the algebra $\mathcal{A}$, if there exists a basis of $\mathcal{A}$ as a vector space $\{a_i\}_{i \in I}$ such that for each $i \in I$: $\sigma(a_i) = p_i a_i$, $p_i \in \mathbb{C}, p_i \neq 0$.

Note that this property of $\mathcal{A}$ extends naturally on tensor products of $\mathcal{A}$ with itself, so it remains true for spaces of chains $C_n$, $n = 0, 1, 2, \ldots$.

**Lemma 2.3.** *For any scaling automorphism $\sigma$ (2.2) the invariant twisted Hochschild complex $C_*^{inv}$ is isomorphic to the quotient complex $C_*/(1 - \sigma)$.*

*Proof.* It is sufficient to show that for each $n$, $C_n = (1 - \sigma)C_n \oplus C_n^{inv}$. Since the boundary $b$ commutes with $\sigma$, then the natural isomorphism, $C_n^{inv} \to C_n/(1 - \sigma)$, which we have for each $n = 0, 1, 2, \ldots$, commutes with $b$.
Since, as we observed earlier, there exists a basis $\{c_i\}$ of $C_n$, such that $\sigma(c_i) = \gamma_i c_i$ we easily see that $c_i$ is in the image of $(1 - \sigma)$ unless $\gamma_i = 1$ when it is $\sigma$-invariant. $\qquad\square$

# 3   The multidimensional quantum hyperplanes

Let $V$ be a vector space of dimension $N$ and $T(V)$ its tensor algebra. If $q_{ij}$ is a complex matrix we denote by $S_Q(V)$ the quantum symmetric algebra:

$$S_Q(V) = T(V)/I_Q,$$

where $I_Q$ is the ideal in $T(V)$ generated by all elements

$$X_{ij} = x_i \otimes x_j - q_{ij} x_j \otimes x_i, \quad i < j, x_i, x_j \in V.$$

We always assume $q_{ii} = 1$ for all $i = 1, 2, \ldots, N$ and $q_{ji} = q_{ij}^{-1}$ for $i < j$.
We introduce also the quantum antisymmetric algebra $\Lambda_Q(V)$ defined as $T(V)/A_Q$, where $A_Q$ is the ideal in $T(V)$ generated by elements:

$$A_{ij} = x_i \otimes x_j + (q_{ij})^{-1} x_j \otimes x_i, \quad i < j, x_i, x_j \in V.$$

(Note that in case of an arbitrary field one needs some extra generators to define correctly the antisymmetric algebra: $A_{ii} = (1 + \hat{q})x_i \otimes x_i$, where $\hat{q} = q_{ij}q_{ji}$.)

**Remark 3.1.** Each scaling automorphism $\sigma$ of the quantum hyperplane algebra is defined by its action on the generators, for each $i = 1, \ldots, N$: $\sigma(x_i) = p_i x_i$, $p_i \in \mathbb{C}, p_i \neq 0$. By $S_Q^\sigma(V)$ we shall denote $S_Q(V)$ with the bimodule structure (1).

Following Wambst, we define:

**Definition 3.2.** The Koszul resolution of $S_Q(V)$ is the following complex:

$$K_n = S_Q(V) \otimes \Lambda_Q^n(V) \otimes S_Q(V),$$

with the differential $d_n : K_n \mapsto K_{n-1}$:

$$
\begin{aligned}
d_n(a \otimes x_{i_1} &\wedge x_{i_2} \wedge \cdots \wedge x_{i_n} \otimes b) = \\
&= \sum_{k=1}^{n} (-1)^{k+1} \left( (\prod_{s=1}^{k-1} q_{i_s i_k}) a x_{i_k} \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \widehat{x_{i_k}} \cdots \wedge x_{i_n} \otimes b \right. \\
&+ \left. (\prod_{s=k+1}^{n} q_{i_k i_s}) a \otimes x_{i_1} \wedge x_{i_2} \wedge \cdots \widehat{x_{i_k}} \cdots \wedge x_{i_n} \otimes x_{i_k} b \right).
\end{aligned}
$$

In the formulae above $i_1 < i_2 < \cdots < i_n$ and $\widehat{x_{i_k}}$ denotes that this element falls out from the product in $\Lambda_Q^*(V)$.
We denote this complex $K(S_Q(V), S_Q(V) \otimes S_Q(V)^{op})$.

In his work Wambst proved ([7], Proposition 4.1) that this is a projective resolution over identity of $S_Q(V)$ as a module over $S_Q(V) \otimes S_Q(V)^{op}$. Moreover, he constructed explicitly an $S_Q(V) - S_Q(V)$ linear morphism $\gamma'$ between this Koszul resolution and the standard Hochschild resolution $C_*(S_Q(V))$, thus enabling the direct calculation of the Hochschild homology groups. We skip here detailed presentation of the morphism referring to the mentioned work (see [7], Lemma 5.3). Similarly as he did in the theorem 5.4, by applying $S_Q^\sigma(V) \otimes_{S_Q(V) \otimes S_Q(V)^{op}}$ to both resolutions, we get a quasi-isomorphism between the "twisted" versions of Koszul and Hoschschild complexes. Therefore the task of calculating the Hochschild homology groups $HH(S_Q(V), S_Q^\sigma(V))$ is reduced to calculation of the homology of the Koszul complex $K_*(S_Q(V), S_Q^\sigma(V))$.
Before we proceed with the calculation of respective homology groups let us notice that the Hochschild chain complex and the Koszul chain complex for the quantum hyperplane and the scaling isomorphism $\sigma$ satisfy the assumptions of the lemma (2.1):

**Lemma 3.3.** *For the Koszul chain complex of the quantum hyperplane the condition (5) is valid for the scaling automorphism $\sigma$.*

*Proof.* We shall prove a slightly more general statement, keeping in mind that the Koszul chain complex has also a basis, which is scaled by the automorphism $\sigma$.

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Let $\{k_i\}_{i \in I}$ be such a basis of $K_*$, that the automorphism $\sigma$ is scaling, $\sigma(k_i) = \lambda_i k_i$, $\lambda_i \in \mathbb{C}$, for each $i \in I$. Then for every $k \in K_{n+1}$ we have:

$$k = \sum_i \alpha_i k_i,$$

where only finitely many of $\alpha_i$ are different from $0$, so only finitely many possible scaling coefficients $\lambda_i$ appear. Name the set of this coefficients $\Lambda$ and set $k_\lambda$ the part of $k$, which has the scaling coefficient $\lambda$. Then:

$$k = \sum_{\lambda \in \Lambda} k_\lambda. \tag{6}$$

Next, assume that $bk \in K_n^{inv}$. Since $b$ commutes with the action of $\sigma$, the action of $\sigma$ on each component $bk_\lambda$ shall have the same scaling coefficient $\lambda$. However, since $bk \in K_n^{inv}$, its scaling coefficient must be $1$. On the other hand $bk$ has parts $bk_\lambda$, with scaling coefficients $\lambda \neq 1$, so if their sum vanishes then all such elements must be zero. Thus we obtain that in the sum (6) $k_1$ might be arbitrary but for all $\Lambda \ni \lambda \neq 1$ we must have $bk_\lambda = 0$. Therefore if $bk \in K_n^{inv}$ then $k = k_1 + \sum_{\lambda \neq 1} k_\lambda$, $k_1 \in K_{n+1}^{inv}$ and $k_\lambda \in Z(K_{n+1})$. $\qquad\square$

From the explicit construction of the quasi-isomorphism $\gamma$ in [7] we directly conclude:

**Remark 3.4.** The quasi-isomorphism $\gamma$ is equivariant with respect to the action of $\mathbb{Z}$ set by the scaling automorphism $\sigma$,

It is the obvious conclusion from the construction of $\gamma$, which involves only operations of (deformed) antisymmetrisations.
Finally we obtain:

**Corollary 3.5.** *The homology groups of the Hochschild complex of $S_Q(V)$ with values in $S(Q)_\sigma$ are isomorphic with the corresponding homology groups of the Koszul complex. Similarly, invariant twisted Hochschild homology is isomorphic to the homology of the invariant subcomplex of the Koszul complex.*

## 3.1  Homology groups

We begin with natural twisted Hochschild homology of the quantum hyperplanes $HH(S_Q(V), S_Q(V)_\sigma)$.

We use the notation of [7]: let $\alpha \in \mathbb{N}^N$ denote a natural multiindex and $\beta \in \{0,1\}^N$ a multiindex with values $0$ and $1$. We use the following shorthand notation:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N},$$
$$x^\beta = x_1^{\beta_1} \wedge x_2^{\beta_2} \wedge \cdots \wedge x_N^{\beta_N}.$$

The length of $\alpha$ and $\beta$ defined as sum of all subindices is denoted by $|\alpha|$, $|\beta|$, respectively. We introduce also $\kappa_i \in \{0,1\}^N$, such that $\kappa_i(j)$ is zero for all $j \neq i$ and $\kappa_i(i) = 1$.

Then, in the complex $S_Q(V)_\sigma \otimes_{S_Q(V) \otimes S_Q(V)^{op}} K(S_Q(V), S_Q(V) \otimes S_Q(V)^{op})$ the differential on homogeneous elements of the type $x^\alpha \otimes x^\beta$ is:

$$d(x^\alpha \otimes x^\beta) = \sum_{i=1}^{N} \delta_1^{\beta(i)} \Omega(\alpha, \beta, i) x^{\alpha+\kappa_i} \otimes x^{\beta-\kappa_i}.$$

The first factor $\delta_1^{\beta(i)}$ assures that the sum is only over such indices $i$ for which $\beta(i)$ does not vanish and:

$$\Omega(\alpha, \beta, i) =$$
$$= (-1)^{\sum_{s=1}^{i-1} \beta(s)} \left( (\prod_{s=1}^{i-1} q_{si}^{\beta(s)})(\prod_{r=i+1}^{N} q_{ir}^{-\alpha(r)}) - p_i (\prod_{s=i+1}^{N} q_{is}^{\beta(s)})(\prod_{r=1}^{i-1} q_{ri}^{-\alpha(r)}) \right)$$

By direct calculation it could be shown that $\Omega(\alpha, \beta, i) = 0$ if and only if $x^{\alpha+\beta} x^i = \sigma(x^i) x^{\alpha+\beta}$.

Let us denote by $C_\sigma$ the set of all multiindices $\gamma \in \mathbb{N}^N$, such that either $\gamma(i) = 0$ or $x^\gamma x^i = \sigma(x^i) x^\gamma$. From the earlier remark we see that if $\alpha + \beta \in C_\sigma$ then $d(x^\alpha \otimes x^\beta) = 0$. Since the differential $d$ preserves the total multi-grade, $\alpha + \beta$, of the element $x^\alpha \otimes x^\beta$, to calculate the homology of the complex we might restrict ourselves to the elements of a fixed multi-grade.

We have:

**Proposition 3.6 (compare theorem 6.1 [7]).** *The Hochschild homology of the quantum multiparameter hyperplane $S_Q(V)$, with values in $S_Q^\sigma(V)$ is:*

$$HH_n(S_Q(V), S_Q^\sigma(V)) = \bigoplus_{\substack{\beta \in \{0,1\}^N \\ |\beta|=n}} \bigoplus_{\substack{\alpha \in \mathbb{N}^N \\ \alpha+\beta \in C_\sigma}} \mathbb{C} x^\alpha \otimes x^\beta.$$

Proof follows exactly the proof of theorem 6.1 in [7]. First, it is clear that all homogeneous elements $x^\alpha \otimes x^\beta$ for $\alpha + \beta \in C_\sigma$ are in the homology groups of the complex. It remains to show that for $\alpha + \beta \notin C_\sigma$ we might construct homotopy $h$ showing that this part of the complex is acyclic.

We define:

$$h(x^\alpha \otimes x^\beta) = \frac{1}{||\alpha + \beta||} \sum_{i=1}^{N} \omega(\alpha, \beta, i) x^{\alpha - \kappa_i} \otimes x^{\beta + \kappa_i},$$

where:

$$\omega(\alpha, \beta, i) = \begin{cases} 0 & \text{if } \alpha + \beta \in C_\sigma, \\ 0 & \text{if } \beta(i) = 1, \\ 0 & \text{if } \alpha(i) = 0, \\ \Omega(\alpha - \kappa_i, \beta + \kappa_i, i)^{-1} & \text{otherwise.} \end{cases}$$

Then one can calculate that:

$$dh + hd = \text{id}.$$

For details of the proof we refer again to [7].

Let us recall here the notion of *generic* deformation parameters $q_{ij}$. We say that the parameters are not generic if there exists such $\gamma \in \mathbb{N}^N$, $\gamma \neq 0$ which is not $\kappa_j$ for some $j = 1, \ldots, N$ and for every $i$ such that $\gamma(i) > 0$ we have:

$$\prod_{k=1}^{N} (q_{ki})^{\gamma(k)} = 1.$$

Note that this is equivalent to stating that the center of any subalgebra generated within $S_Q(V)$ by more than two of its generators is nontrivial.

We are ready now to prove the main theorem:

**Proposition 3.7.** *For each multiparameter quantum hyperplane $S_Q(V)$ with generic deformation parameters $q_{ij}$ there exist a scaling automorphism $\sigma$ such that the top natural twisted Hochschild homology group of $S_Q(V)$ is of dimension $N$, in particular, there exists a unique choice of these parameters such that the top class in the twisted Hochschild homology is given by $1 \otimes x^1 \wedge x^2 \wedge \cdots \wedge x^N$ in the Koszul complex.*

*Proof.* Clearly $HH_n(S_Q(V), S_Q^\sigma(V)) = 0$ for $n > N$. From the previous proposition we need to find such automorphism $\sigma$ that the space of elements $x^\alpha \otimes x^\beta$

with $|\beta| = N, \alpha + \beta \in C_\sigma$ were not empty. Since $\beta(i) = 1$ for all $i = 1, 2, \ldots, N$ (we denote it by $1_N$) we see that the condition becomes:

$$p_i x_i x^{\alpha+1_N} = x^{\alpha+1_N} x_i, \quad \forall i = 1, 2, \ldots, N.$$

Using the commutation rules for the $S_Q(V)$ we obtain:

$$p_i \left( \prod_{j=1}^{i-1} (q_{ji})^{-\alpha(j)-1} \right) = \left( \prod_{j=i+1}^{N} (q_{ij})^{-\alpha(j)-1} \right).$$

and we might rewrite it as:

$$p_i = \left( \prod_{j=1}^{i-1} (q_{ji})^{\alpha(j)+1} \right) \left( \prod_{j=i+1}^{N} (q_{ij})^{-\alpha(j)-1} \right) = \left( \prod_{j=1}^{N} (q_{ji})^{\alpha(j)+1} \right).$$

Therefore for any choice of $\alpha$ we might set the numbers $p_i$ so that the highest twisted Hochschild homology group does not vanish. In particular for the canonical choice $\alpha = 0$ we have:

$$p_i = \left( \prod_{j=1}^{N} (q_{ji}) \right). \tag{7}$$

Then $HH_N(S_Q(V), S_Q^\sigma(V)) = \mathbb{C}$ unless the family of parameters $q_{ij}$ is not generic. We shall call such scaling automorphisms as set by (7) *canonical automorphisms*. $\qquad\square$

Finally, we obtain:

**Proposition 3.8.** *The twisted Hochschild homology of the quantum hyperplane for the canonical scaling automorphism (7) is, in dimension $n$:*

$$HH_n^\sigma(S_Q(V)) = \bigoplus_{\substack{\beta \in \{0,1\}^N \\ |\beta|=n}} \bigoplus_{\substack{\alpha \in \mathbb{N}^N \\ \alpha+\beta \in C_\sigma}} \mathbb{C} x^\alpha \otimes x^\beta.$$

*where $C_\sigma$ is the set of all multiindices $\gamma \in \mathbb{N}^N$ such that for each $i = 1, 2, \ldots N$ either $\gamma(i) = 0$ or*

$$\prod_j q_{ji}^{\gamma(j)-1} = 1. \tag{8}$$

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*Proof.* The proof is a direct consequence of the twisted Hochschild homology construction, Proposition (3.6) and the formula (7) for the coefficients of the canonical scaling automorphism.

First, notice that for the canonical scaling automorphism the condition that $\gamma \in C_\sigma$ could be rewritten (for $\gamma(i) \neq 0$) exactly as in (8).

It remains only to show that the elements of the Koszul complex of the form $x^\alpha \otimes x^\beta$, with $\alpha + \beta \in C_\sigma$ are invariant with respect to the canonical automorphism. Let us verify explicitly the action of the canonical scaling automorphism $\sigma$ on $x^\gamma$ for $\gamma \in C_\sigma$:

$$\sigma(x^\gamma) = \prod_i (p_i)^{\gamma(i)} = \prod_i \prod_j (q_{ji})^{(\gamma(i))} = \prod_i \prod_j (q_{ji})^{(\gamma(i)\gamma(j))} = 1.$$

where we have used first formula (7), then (8) and finally $q_{ij}q_{ji} = 1$.

Since we have shown already that for scaling automorphisms the twisted Hochschild homology is equal to the invariant twisted Hochschild homology we obtain the desired result. $\square$

## 3.2 The two-dimensional quantum plane

In the simplest possible case of the generic two-dimensional quantum plane $\mathbb{A}_q^2$ given by the relation $xy = qyx$, ($q$ not a root of unity) we have the canonical scaling automorphism:

$$\sigma(x) = \frac{1}{q}x, \quad \sigma(y) = qy.$$

The twisted Hochschild homology groups of the quantum plane are:

$$HH_0^\sigma(\mathbb{A}_q^2) = \mathbb{C} \oplus \mathbb{C}xy$$
$$HH_1^\sigma(\mathbb{A}_q^2) = (\mathbb{C}x \otimes y) \oplus (\mathbb{C}y \otimes x),$$
$$HH_2^\sigma(\mathbb{A}_q^2) = \mathbb{C}1 \otimes x \wedge y,$$
$$HH_n^\sigma(\mathbb{A}_q^2) = 0, \quad n > 3.$$

where we have written the representatives from each of the homology class from the (invariant) Koszul chain complex.

To show it, we use the Proposition (3.8). In dimension $0$ we have $\beta \equiv 0$ and two possibilities for $\alpha$, which belong to $C_\sigma^{inv}$: either $\alpha(i) = 0, i = 1, 2$; or $\alpha(i) = 1, i = 1, 2$. In dimension $1$ we proceed likewise.

Finally, consider the example of the multidimensional generalization of the quantum plane, the one-parameter quantum hyperplane, which we use to illustrate that the explicit calculations get quite technical in concrete examples. The algebra, $\mathbb{A}_q^N$, is defined as an algebra generated by $N$ generators $x_i$, $i = 1, \ldots, N$ and relations (we again assume that $q$ is not a root of unity):

$$x_i x_j = q x_j x_i, \quad N \geq i > j \geq 1, \tag{9}$$

The canonical scaling transformations (7) for $\mathbb{A}_q^N$ are:

$$\sigma(x_i) = q^{N-2i+1} x_i, \quad i = 1, \ldots, N. \tag{10}$$

The space $C_\sigma$ consists of all natural solutions to the set of linear equations for $\gamma$:

$$P_k \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 1 & \cdots & 1 \\ \vdots & & & & \vdots \\ -1 & \cdots & -1 & 0 & 1 \\ -1 & \cdots & -1 & -1 & 0 \end{pmatrix} \gamma = P_k \begin{pmatrix} N - 2 + 1 \\ N - 4 + 1 \\ \vdots \\ 2(N-1) - N + 1 \\ 2N - N + 1 \end{pmatrix}, \tag{11}$$

where $P_k$ is any diagonal matrix with $0, 1$ on the diagonal (the matrix, which projects the equation on the non-zero values of $\gamma$.

Using this technical description we have the tools to calculate the twisted Hochschild homology in any dimension. In particular, in the top dimension $n = N$, we have the only solution $\gamma = (1, 1, \ldots, 1)$.

## 4   Conclusions

We have demonstrated that for quantum hyperplanes there exist automorphisms such that the top non-vanishing twisted Hochschild homology group is of "classical" dimension. This merely confirms that the twisted version of Hochschild homology seems to be more adapted to the case of quantum deformations. Note that in the discussed case we did not start with a given automorphism but rather found them (among some special type of scaling automorphisms) using the requirement for the non-vanishing of the top dimension.

Our results could be, of course, used also for the calculations of the twisted cyclic homology. In fact, from the Connes long exact sequence linking cyclic and Hochschild homology, which is applicable also to the twisted case, we might immediately get some results. In particular for $N$-dimensional quantum hyperplanes

the twisted cyclic homology stabilizes starting from the dimension $N - 1$. More detailed analysis of this issue as well as of the case of non-generic deformation parameters shall be treated elsewhere.

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