

# INFINITESIMAL FOURIER TRANSFORMATION FOR THE SPACE OF FUNCTIONALS

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## Abstract

The purpose is to formulate a Fourier transformation for the space of functionals, as an infinitesimal meaning. We extend  $\mathbf{R}$  to  ${}^*(\mathbf{R})$  under the base of nonstandard methods for the construction. The domain of a functional is the set of all internal functions from a  ${}^*$ -finite lattice to a  ${}^*$ -finite lattice with a double meaning. Considering a  ${}^*$ -finite lattice with a double meaning, we find how to treat the domain for a functional in our theory of Fourier transformation, and calculate two typical examples.

## 0. Introduction

Recently many kinds of geometric invariants are defined on manifolds and they are used for studying low dimensional manifolds, for example, Donaldson's invariant, Chern-Simon's invariant and so on. They are originally defined as Feynman path integrals in physics. The Feynman path integral is in a sense an integral of a functional on an infinite dimensional space of functions. We would like to study the Feynman path integral and the originally defined invariants. For the purpose, we would be sure that it is necessary to construct a theory of Fourier transformation on the space of functionals. For it, as the later argument, we would need many stages of infinitesimals and infinites, that is, we need to put a concept of stage on the field of real numbers. We use nonstandard methods to develop a theory of Fourier transformation on the space of functionals.

Feynman([F-H]) used the concept of his path integral for physical quantizations. The word "physical quantizations" has two meanings : one is for quantum mechanics and the other is for quantum field theory. We usually use the same word "Feynman path integral". However the meanings included in "Feynman path integral" are two sides, according to the above. One is of quantum mechanics and the other is of quantum field theory. To understand the Feynman path integral of the first type, Fujiwara([F]) studied it as a fundamental solution, and

Hida([H]), Ichinose, Tamura([Ic],[I-T]) studied it from their stochastic interests and obtained deep results, using standard mathematics. In stochastic mathematics, Loeb([Loe]) constructed Loeb measure theory and investigated Brownian motion that relates to Itô integral([It]). Anderson([An]) deloped it. Kamae([Ka]) proved Ergodic theory using nonstandard analysis. From a nonstandard approach, Nelson([Ne]), Nakamura([Na1],[Na2]) studied Schrödinger equation, Dirac equation and Loo([Loo1],[Loo2]) calculated rigidly the quantum mechanics of harmonic oscillator. It corresponds to functional analysis on the space of functions in standard mathematics.

On the other hand, we would like to construct a frame of Feynman path integral of the second type, that is, a functional analysis on the space of functionals. Our idea is the following : in nonstandard analysis, model theory, especially non-well-founded set theory([N-O-T]), we can extend  $\mathbf{R}$  to  ${}^*\mathbf{R}$  furthermore a double extension  ${}^*({}^*\mathbf{R})$ , and so on. For formulation of Feynman path integral of the first type, it was necessary only one extension  ${}^*\mathbf{R}$  of  $\mathbf{R}$  in nonstandard analysis([A-F-HK-L]). In fact there exists an infinite in  ${}^*\mathbf{R}$ , however there are no elements in  ${}^*\mathbf{R}$ , that is greater than images of the infinite for any functions. The same situation occurs for infinitesimals. Hence we consider to need a further extension of  $\mathbf{R}$  to construct a formulation of Feynman path integral of the second type. If the further extension satisfies some condition, the extension  ${}^*({}^*\mathbf{R})$  has a higher degree of infinite and also infinitesimal. We use these to formulate the space of functionals. We would like to try to construct a theory of Fourier transformation on the space of functionals and calculate two typical examples of it.

Historically, for the theories of Fourier transformations in nonstandard analysis, in 1972, Luxemburg([Lu]) developed a theory of Fourier series with \*-finite summation on the basis of nonstandard analysis. The basic idea of his approach is to replace the usual  $\infty$  of the summation to an infinite natural number  $N$ . He approximated the Fourier transformation on the unit circle by the Fourier transformation on the group of  $N$ th roots of unity.

Gaishi Takeuti([T]) introduced an infinitesimal delta function  $\delta$ , and Kinoshita([Ki]) defined in 1988 a discrete Fourier transformation for each even \*-finite number  $H(\in {}^*\mathbf{R})$  :  $(F\varphi)(p) = \sum_{-\frac{H^2}{2} \leq z < \frac{H^2}{2}} \frac{1}{H} \exp(-2\pi ip \frac{1}{H}z) \varphi(\frac{1}{H}z)$ , called "infinitesimal Fourier transformation". He developed a theory for the infinitesimal Fourier transformation and studied the distribution space deeply, and proved the same properties hold as usual Fourier transformation of  $L^2(\mathbf{R})$ . Especially saying, the delta function  $\delta$  satisfies that  $\delta^2, \delta^3, \dots, \sqrt{\delta}, \dots$  are also hyperfunctions as their meaning, and  $F\delta = 1, F\delta^2 = H, F\delta^3 = H^2, \dots, F\sqrt{\delta} = \frac{1}{\sqrt{H}}, \dots$ .

In 1989, Gordon([G]) independently defined a generic, discrete Fourier transformation for each infinitesimal  $\Delta$  and \*-finite number  $M$ , defined by  $(F_{\Delta,M}\varphi)(p) = \sum_{-M \leq z \leq M} \Delta \exp(-2\pi ip \Delta z) \varphi(\Delta z)$ . He studied under which condition the discrete Fourier transformation  $F_{\Delta,M}$  approximates the usual Fourier transformation  $\mathcal{F}$  for  $L^2(\mathbf{R})$ . His proposed condition is (A') of his notation : let  $\Delta$  be an infinitely small and  $M$  an infinitely large natural number such that  $M \cdot \Delta$  is in-

finitely large. He showed that under the condition ( $A'$ ) the standard part of  $F_{\Delta, M} \varphi$  approximates the usual  $\mathcal{F}\varphi$  for  $\varphi \in L^2(\mathbf{R})$ . One of the different points between Kinoshita's and Gordon's is that there is the term  $\Delta \exp(-2\pi ip\Delta M)\varphi(\Delta M)$  in the summation of their two definitions or not. We mention that both definitions are same for the standard part of the discrete Fourier transformation for  $\varphi \in L^2(\mathbf{R})$  and Kinoshita's definition satisfies the condition ( $A'$ ) for an even infinite number  $H$  if  $\Delta = \frac{1}{H}$ ,  $M = \frac{H^2}{2}$ .

We shall extend their theory of Fourier transformation for the space of functions to a theory of Fourier transformation for the space of functionals. For the purpose of this, we shall represent a space of functions from  $\mathbf{R}$  to  $\mathbf{R}$  as a space of functions from a set of lattices in an infinite interval  $[-\frac{H}{2}, \frac{H}{2})$  to a set of lattices in an infinite interval  $[-\frac{H'}{2}, \frac{H'}{2})$ . We consider what  $H'$  is to treat any function from  $\mathbf{R}$  to  $\mathbf{R}$ . If we put a function  $a(x) = x^n (n \in \mathbf{Z}^+)$ , we need that  $\frac{H'}{2}$  is greater than  $(\frac{H}{2})^n$ , and if we choose a function  $a(x) = e^x$ , we need that  $\frac{H'}{2}$  is greater than  $e^{\frac{H}{2}}$ . If we choose any infinite number, there exists a function whose image is beyond the infinite number. Since we treat all functions from  $\mathbf{R}$  to  $\mathbf{R}$ , we need to put  $\frac{H'}{2}$  as an infinite number greater than any infinite number of  ${}^*\mathbf{R}$ . Hence we make  $[-\frac{H'}{2}, \frac{H'}{2})$  not in  ${}^*\mathbf{R}$  but in  ${}^*({}^*\mathbf{R})$ , where  ${}^*({}^*\mathbf{R})$  is a double extension of  $\mathbf{R}$ , that is,  $H'$  is an infinite number in  ${}^*({}^*\mathbf{R})$ . First we shall develop an infinitesimal Fourier transformation theory for the space of functionals, and secondly we calculate fundamental two examples for our infinitesimal Fourier transformation. In our case, we define an infinitesimal delta function  $\delta$  satisfies that  $F\delta = 1$ ,  $F\delta^2 = H'H^2$ ,  $F\delta^3 = H'^2H^2$ , ... ,  $F\sqrt{\delta} = H'^{-\frac{1}{2}H^2}$ , ... , that is,  $F\delta^2, F\delta^3, \dots$  are infinite and  $F\sqrt{\delta}, \dots$  are infinitesimal. These are a functional  $f$  and an infinite-dimensional Gaussian distribution  $g$  where  $\mathbf{st}(f(\alpha)) = \exp\left(\pi i \int_{-\infty}^{\infty} \alpha^2(t) dt\right)$ ,  $\mathbf{st}(g(\alpha)) = \exp\left(-\pi \int_{-\infty}^{\infty} \alpha^2(t) dt\right)$  for  $\alpha \in L^2(\mathbf{R})$ . We obtain the following results of standard meanings :  $(Ff)(b) = \overline{f(b)}$  or  $-f(b)$  and  $(Fg)(b) = C_2(b)g(b)$ ,  $\mathbf{st}(C_2(b)) = 1$  if  $b$  is finite valued. Our infinitesimal Fourier transformation of  $g$  is also  $g$  when the domain of  $g$  is standard.

### 1. Formulation (cf.[S],[T],[Ki]).

To explain our infinitesimal Fourier transformation for the space of functionals, we introduce Kinoshita's infinitesimal Fourier transformation for the space of functions. We fix an infinite set  $\Lambda$  and an ultrafilter  $F$  of  $\Lambda$  so that  $F$  includes the Fréchet filter  $F_0(\Lambda)$ . We remark that the set of natural numbers is naturally embedded in  $\Lambda$ . Let  $H$  be an even infinite number where the definition being even is the following : if  $H$  is written as  $[(H_\lambda, \lambda \in \Lambda)]$  then  $\{\lambda \in \Lambda \mid H_\lambda \text{ is even}\} \in F$ , where  $[ \ ]$  denotes the equivalence class with respect to the ultrafilter  $F$ . Let  $\varepsilon$  be  $\frac{1}{H}$ , that is, if  $\varepsilon$  is  $[(\varepsilon_\lambda, \lambda \in \Lambda)]$  then  $\varepsilon_\lambda$  is  $\frac{1}{H_\lambda}$ . Then we shall define a lattice space  $\mathbf{L}$ , a sublattice space  $L$  and a space of functions  $R(L)$  :

$$\begin{aligned} \mathbf{L} &:= \varepsilon {}^*\mathbf{Z} = \{\varepsilon z \mid z \in {}^*\mathbf{Z}\}, \\ L &:= \left\{ \varepsilon z \mid z \in {}^*\mathbf{Z}, -\frac{H}{2} \leq \varepsilon z < \frac{H}{2} \right\} \\ &= \{[(\varepsilon_\lambda z_\lambda), \lambda \in \Lambda] \mid \varepsilon_\lambda z_\lambda \in L_\lambda\} \quad (\subset \mathbf{L}) \end{aligned}$$

$$R(L) := \{\varphi \mid \varphi \text{ is an internal function from } L \text{ to } {}^*\mathbf{C}\} \\ = \{[(\varphi_\lambda, \lambda \in \Lambda)] \mid \varphi_\lambda \text{ is a function from } L_\lambda \text{ to } \mathbf{C}\},$$

where  $L_\lambda := \{\varepsilon_\lambda z_\lambda \mid z_\lambda \in \mathbf{Z}, -\frac{H_\lambda}{2} \leq \varepsilon_\lambda z_\lambda < \frac{H_\lambda}{2}\}$ .

Gaishi Takeuti([T]) introduced an infinitesimal delta function  $\delta(x) (\in R(L))$  and Kinoshita([Ki]) defined an infinitesimal Fourier transformation on  $R(L)$ . From now on, functions in  $R(L)$  are extended to periodic functions on  $\mathbf{L}$  with the period  $H$  and we denote them by the same notations. For  $\varphi (\in R(L))$ , the infinitesimal Fourier transformation  $F\varphi$ , the inverse infinitesimal Fourier transformation  $\overline{F}\varphi$ , and the convolution of  $\varphi, \psi (\in R(L))$  are defined as follows :

$$\delta(x) := \begin{cases} H & (x = 0), \\ 0 & (x \neq 0), \end{cases}$$

$$(F\varphi)(p) := \sum_{x \in L} \varepsilon \exp(-2\pi ipx) \varphi(x), \quad (\overline{F}\varphi)(p) := \sum_{x \in L} \varepsilon \exp(2\pi ipx) \varphi(x), \\ (\varphi * \psi)(x) := \sum_{y \in L} \varepsilon \varphi(x - y) \psi(y).$$

He obtained the following equalities as same as the usual Fourier analysis :

$$\delta = F1 = \overline{F}1, \quad F \text{ is unitary, } F^4 = 1, \quad \overline{F}F = F\overline{F} = 1, \\ \varphi * \delta = \delta * \varphi = \varphi, \quad \varphi * \psi = \psi * \varphi, \\ F(\varphi * \psi) = (F\varphi)(F\psi), \quad F(\varphi\psi) = (F\varphi) * (F\psi), \\ \overline{F}(\varphi * \psi) = (\overline{F}\varphi)(\overline{F}\psi), \quad \overline{F}(\varphi\psi) = (\overline{F}\varphi) * (\overline{F}\psi).$$

The most different point is that  $\delta^l (l \in \mathbf{R}^+)$  are also elements of  $R(L)$  and the Fourier transformation are able to be calculated as  $F\delta^l = H^{(l-1)}$ , by the above definition.

On the other hand, we obtain the following theorem from his result and an elementary calculation :

**Theorem 1.1.** *For an internal function with two variables  $f : L \times L \rightarrow {}^*\mathbf{C}$  and  $g (\in R(L))$ ,*

$$F_x \left( \sum_{y \in L} \varepsilon f(x - y, y) g(y) \right) (p) = \{F_y(F_u(f(u, y))(p)) * F_y(g(y))\} (p),$$

where  $F_x, F_y, F_u$  are Fourier transformations for  $x, y, u$ , and  $*$  is the convolution for the variable paired with  $y$  by the Fourier transformation.

*Proof.* By the above Kinoshita's result,  $F(\varphi\psi) = (F\varphi) * (F\psi)$ . We use it and obtain the following :

$$F_x \left( \sum_{y \in L} \varepsilon f(x - y, y) g(y) \right) (p) = \sum_{x, y \in L} \varepsilon \exp(-2\pi ipx) \varepsilon f(x - y, y) g(y) \\ = \sum_{y, u \in L} \varepsilon^2 \exp(-2\pi ip(y + u)) f(u, y) g(y) \quad (u := x - y) \\ = \sum_{y \in L} (\varepsilon \exp(-2\pi ipy) \left( \sum_{u \in L} \varepsilon \exp(-2\pi pu) f(u, y) \right) g(y)) \\ = F_y(F_u(f(u, y))(p) \cdot g(y))(p) = \{F_y(F_u(f(u, y))(p)) * F_y(g(y))\} (p).$$

To treat a \*-unbounded functional  $f$  in the nonstandard analysis, we need a second nonstandardization. Let  $F_2 := F$  be a nonprincipal ultrafilter on an infinite set  $\Lambda_2 := \Lambda$  as above. Denote the ultraproduct of a set  $S$  with respect to  $F_2$  by  ${}^*S$  as above. Let  $F_1$  be another nonprincipal ultrafilter on an infinite set  $\Lambda_1$ . Take the \*-ultrafilter  ${}^*F_1$  on  ${}^*\Lambda_1$ . For an internal set  $S$  in the sense of \*-nonstandardization,

let  $\star S$  be the  $\star$ -ultraproduct of  $S$  with respect to  $\star F_1$ . Thus, we define a double ultraproduct  $\star(\star \mathbf{R})$ ,  $\star(\star \mathbf{Z})$ , etc for the set  $\mathbf{R}$ ,  $\mathbf{Z}$ , etc. It is shown easily that

$$\star(\star \mathbf{S}) = S^{\Lambda_1 \times \Lambda_2} / F_1^{F_2},$$

where  $F_1^{F_2}$  denotes the ultrafilter on  $\Lambda_1 \times \Lambda_2$  such that for any  $A \subset \Lambda_1 \times \Lambda_2$ ,  $A \in F_1^{F_2}$  if and only if

$$\{\lambda \in \Lambda_1 \mid \{\mu \in \Lambda_2 \mid (\lambda, \mu) \in A\} \in F_2\} \in F_1.$$

We always work with this double nonstandardization. The natural imbedding  $\star S$  of an internal element  $S$  which is not considered as a set in  $\star$ -nonstandardization is often denoted simply by  $S$ .

DEFINITION 1.2 (cf.[N-O]). Let  $H(\in \star \mathbf{Z})$ ,  $H'(\in \star(\star \mathbf{Z}))$  be even positive numbers such that  $H'$  is larger than any element in  $\star \mathbf{Z}$ , and let  $\varepsilon(\in \star \mathbf{R})$ ,  $\varepsilon'(\in \star(\star \mathbf{R}))$  be infinitesimals satisfying  $\varepsilon H = 1$ ,  $\varepsilon' H' = 1$ . We define as follows :

$$\begin{aligned} \mathbf{L} &:= \varepsilon \star \mathbf{Z} = \{\varepsilon z \mid z \in \star \mathbf{Z}\}, \quad \mathbf{L}' := \varepsilon' \star(\star \mathbf{Z}) = \{\varepsilon' z' \mid z' \in \star(\star \mathbf{Z})\}, \\ L &:= \left\{ \varepsilon z \mid z \in \star \mathbf{Z}, -\frac{H}{2} \leq \varepsilon z < \frac{H}{2} \right\} (\subset \mathbf{L}), \\ L' &:= \left\{ \varepsilon' z' \mid z' \in \star(\star \mathbf{Z}), -\frac{H'}{2} \leq \varepsilon' z' < \frac{H'}{2} \right\} (\subset \mathbf{L}'). \end{aligned}$$

Here  $L$  is an ultraproduct of lattices

$$L_\mu := \left\{ \varepsilon_\mu z_\mu \mid z_\mu \in \mathbf{Z}, -\frac{H_\mu}{2} \leq \varepsilon_\mu z_\mu < \frac{H_\mu}{2} \right\} \quad (\mu \in \Lambda_2)$$

in  $\mathbf{R}$ , and  $L'$  is also an ultraproduct of lattices

$$L'_\lambda := \left\{ \varepsilon'_\lambda z'_\lambda \mid z'_\lambda \in \star \mathbf{Z}, -\frac{H'_\lambda}{2} \leq \varepsilon'_\lambda z'_\lambda < \frac{H'_\lambda}{2} \right\} \quad (\lambda \in \Lambda_1)$$

in  $\star \mathbf{R}$  that is an ultraproduct of

$$L'_{\lambda\mu} := \left\{ \varepsilon'_{\lambda\mu} z'_{\lambda\mu} \mid z'_{\lambda\mu} \in \mathbf{Z}, -\frac{H'_{\lambda\mu}}{2} \leq \varepsilon'_{\lambda\mu} z'_{\lambda\mu} < \frac{H'_{\lambda\mu}}{2} \right\} \quad (\mu \in \Lambda_2).$$

We define a latticed space of functions  $X$  as follows,

$$\begin{aligned} X &:= \{a \mid a \text{ is an internal function with double meanings, from } \star(L) \text{ to } L'\} \\ &= \{[(a_\lambda), \lambda \in \Lambda_1] \mid a_\lambda \text{ is an internal function from } L \text{ to } L'_\lambda\}, \end{aligned}$$

where  $a_\lambda : L \rightarrow L'_\lambda$  is  $a_\lambda = [(a_{\lambda\mu}), \mu \in \Lambda_2]$ ,  $a_{\lambda\mu} : L_\mu \rightarrow L'_{\lambda\mu}$ .

We define three equivarence relations  $\sim_H$ ,  $\sim_{\star(H)}$  and  $\sim_{H'}$  on  $\mathbf{L}$ ,  $\star(\mathbf{L})$  and  $\mathbf{L}'$  :

$$\begin{aligned} x \sim_H y &\iff x - y \in H \star \mathbf{Z}, \quad x \sim_{\star(H)} y \iff x - y \in \star(H) \star(\star \mathbf{Z}), \\ x \sim_{H'} y &\iff x - y \in H' \star(\star \mathbf{Z}). \end{aligned}$$

Then we identify  $\mathbf{L} / \sim_H$ ,  $\star(\mathbf{L}) / \sim_{\star(H)}$  and  $\mathbf{L}' / \sim_{H'}$  as  $L$ ,  $\star(L)$  and  $L'$ . Since  $\star(L)$  is identified with  $L$ , the set  $\star(\mathbf{L}) / \sim_{\star(H)}$  is identified with  $\mathbf{L} / \sim_H$ . Furthermore we represent  $X$  as the following internal set :

$\{a \mid a \text{ is an internal function with double meanings, from } \star(\mathbf{L}) / \sim_{\star(H)} \text{ to } \mathbf{L}' / \sim_{H'}\}$ .

We use the same notation as a function from  $\star(L)$  to  $L'$  to represent a function in the above internal set. We define the space  $A$  of functionals as follows :

$$A := \{f \mid f \text{ is an internal function with double meanings, from } X \text{ to } \star(\star \mathbf{C})\}.$$

Then  $f$  is written as  $f = [(f_\lambda), \lambda \in \Lambda_1]$ ,  $f_\lambda$  is an internal function from the set  $\{a_\lambda \mid a_\lambda \text{ is an internal function from } L \text{ to } L'_\lambda\}$  to  $\star \mathbf{C}$ , and  $f_\lambda$  is written as  $f_\lambda = [(f_{\lambda\mu}), \mu \in \Lambda_2]$ ,  $f_{\lambda\mu} : \{a_{\lambda\mu} : L_\mu \rightarrow L'_{\lambda\mu}\} \rightarrow \mathbf{C}$ .

We define an infinitesimal delta function  $\delta(a)(\in A)$ , an infinitesimal Fourier transformation of  $f(\in A)$ , an inverse infinitesimal Fourier transformation of  $f$  and a convolution of  $f, g(\in A)$ , by the following :

DEFINITION 1.3.

$$\delta(a) := \begin{cases} (H')^{(*H)^2} & (a = 0), \\ 0 & (a \neq 0), \end{cases}$$

$$\begin{aligned} \varepsilon_0 &:= (H')^{-(^*H)^2} \in ^*(^*\mathbf{R}), \\ (Ff)(b) &:= \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} a(k)b(k)\right) f(a), \\ (\overline{F}f)(b) &:= \sum_{a \in X} \varepsilon_0 \exp\left(2\pi i \sum_{k \in L} a(k)b(k)\right) f(a), \\ (f * g)(a) &:= \sum_{a' \in X} \varepsilon_0 f(a - a')g(a'). \end{aligned}$$

We define an inner product on  $A$  :  $(f, g) := \sum_{b \in X} \overline{f(b)}g(b)$ , where  $\overline{f(b)}$  is the complex conjugate of  $f(b)$ . Then we obtain the following theorem :

THEOREM 1.4.

$$\begin{aligned} (1) \quad & \delta = F1 = \overline{F}1, \quad (2) \quad F \text{ is unitary, } F^4 = 1, \overline{F}F = F\overline{F} = 1, \\ (3) \quad & f * \delta = \delta * f = f, \quad (4) \quad f * g = g * f, \\ (5) \quad & F(f * g) = (Ff)(Fg), \quad (6) \quad \overline{F}(f * g) = (\overline{F}f)(\overline{F}g), \\ (7) \quad & F(fg) = (Ff) * (Fg), \quad (8) \quad \overline{F}(fg) = (\overline{F}f) * (\overline{F}g). \end{aligned}$$

The definition implies the following proposition :

PROPOSITION 1.5. If  $l \in \mathbf{R}^+$ , then  $F\delta^l = (H')^{(l-1)(^*H)^2}$ .

We define two types of infinitesimal divided differences. Let  $f$  and  $a$  be elements of  $A$  and  $X$  respectively and let  $b(\in X)$  be an internal function whose image is in  $^*(^*\mathbf{Z}) \cap L'$ . We remark that  $\varepsilon'b$  is an element of  $X$ .

DEFINITION 1.6.

$$(D_{+,b}f)(a) := \frac{f(a+\varepsilon'b)-f(a)}{\varepsilon'}, \quad (D_{-,b}f)(a) := \frac{f(a)-f(a-\varepsilon'b)}{\varepsilon'}.$$

Let  $\lambda_b(a) := \frac{\exp(2\pi i \varepsilon' ab) - 1}{\varepsilon'}$ ,  $\overline{\lambda}_b(a) := \frac{\exp(-2\pi i \varepsilon' ab) - 1}{\varepsilon'}$ . Then we obtain the following theorem corresponding to Kinoshita's result for the relationship between the infinitesimal Fourier transformation and the infinitesimal divided differences :

THEOREM 1.7.

$$\begin{aligned} (1) \quad & (F(D_{+,b}f))(a) = \lambda_b(a)(Ff)(a), \quad (2) \quad (F(D_{-,b}f))(a) = -\overline{\lambda}_b(a)(Ff)(a), \\ (3) \quad & (F(\lambda_b f))(a) = -(D_{-,b}(Ff))(a), \quad (4) \quad (F(\overline{\lambda}_b f))(a) = (D_{+,b}(Ff))(a), \\ (5) \quad & (D_{+,b}(\overline{F}f))(a) = (\overline{F}(\lambda_b f))(a), \quad (6) \quad (D_{-,b}(\overline{F}f))(a) = -(\overline{F}(\overline{\lambda}_b f))(a), \\ (7) \quad & \lambda_b(a) = 2\pi i \left( \frac{\sin(\pi \varepsilon' ab)}{\pi \varepsilon'} \right) \exp(\pi i \varepsilon' ab). \end{aligned}$$

Theorem 1.7 implies the following Corollary :

COROLLARY 1.8. If  $\varepsilon'b$  is an element of  $X$ , then  $(f, D_{+,b}g) = -(D_{+,b}f, g)$  for  $f, g \in A$ .

Replacing the definitions of  $L', \delta, \varepsilon_0, F, \overline{F}$  in Definition 1.2 and Definition 1.3 by the following, we shall define another type of infinitesimal Fourier transformation.

The different point is only the definition of an inner product of the space of functions  $X$ . In Definition 1.3, the inner product of  $a, b(\in X)$  is  $\sum_{k \in L} a(k)b(k)$ , and in the following definition, it is  ${}^* \varepsilon \sum_{k \in L} a(k)b(k)$ .

DEFINITION 1.9.

$$L' := \left\{ \varepsilon' z' \mid z' \in {}^*(\mathbf{Z}), -{}^*H \frac{H'}{2} \leq \varepsilon' z' < {}^*H \frac{H'}{2} \right\},$$

$$\delta(a) := \begin{cases} ({}^*H)^{\frac{({}^*H)^2}{2}} H'^{({}^*H)^2} & (a = 0), \\ 0 & (a \neq 0), \end{cases}$$

$$\begin{aligned} \varepsilon_0 &:= ({}^*H)^{-\frac{({}^*H)^2}{2}} H'^{-({}^*H)^2} \\ (Ff)(b) &:= \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i {}^* \varepsilon \sum_{k \in L} a(k)b(k)\right) f(a), \\ (\overline{F}f)(b) &:= \sum_{a \in X} \varepsilon_0 \exp\left(2\pi i {}^* \varepsilon \sum_{k \in L} a(k)b(k)\right) f(a). \end{aligned}$$

In this case, we obtain the same theorems as Theorem 1.4 and Theorem 1.6, and the following theorem corresponding to Theorem 1.1 :

THEOREM 1.10. *For an internal function with two variables  $f : X \times X \rightarrow {}^*(\mathbf{C})$  and  $g(\in A)$ ,*

$F_a \left( \sum_{b \in X} \varepsilon_0 f(a-b, b)g(b) \right) (d) = \{F_b(F_c(f(c, b))(d)) * F_b(g(b))\} (d)$ ,  
where  $F_a, F_b, F_c$  are Fourier transformations for  $a, b, c$ , and  $*$  is the convolution for the variable pairing with  $b$  by the Fourier transformation.

## 2. Proofs of Theorems.

*Proof of Theorem 1.4.*

$$\begin{aligned} (1) \quad (F1)(0) &= \sum_{a \in X} \varepsilon_0 = \varepsilon_0 (H'^2)^{({}^*H)^2} = (H')^{({}^*H)^2}. \text{ If } b \neq 0, \text{ then} \\ (F1)(b) &= \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} a(k)b(k)\right) = \varepsilon_0 \prod_{k \in L} \sum_{a(k) \in L'} \exp(-2\pi i a(k)b(k)) \\ &= \varepsilon_0 \prod_{k \in L, b(k) \neq 0} \sum_{a(k) \in L'} \exp(-2\pi i a(k)b(k)) \cdot \prod_{k \in L, b(k) = 0} \sum_{a(k) \in L'} \exp(-2\pi i a(k)b(k)) \\ &= \prod_{k \in L, b(k) \neq 0} \varepsilon_0 \frac{\exp(-2\pi i \varepsilon' (-\frac{H'^2}{2})b(k))(1 - \exp(-2\pi i \varepsilon' H'^2 b(k)))}{1 - \exp(-2\pi i \varepsilon' b(k))} \\ &\quad \cdot \prod_{k \in L, b(k) = 0} \sum_{a(k) \in L'} \exp(-2\pi i a(k)b(k)) = 0. \end{aligned}$$

Hence  $F1 = \delta$ . The same argument implies that  $\overline{F}1 = \delta$ .

$$\begin{aligned} (2) \quad (Ff, Fg) &= \overline{\sum_{b \in X} \varepsilon_0 (Ff)(b)(Fg)(b)} \\ &= \sum_{b \in X} \varepsilon_0 \overline{\sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} a(k)b(k)\right) f(a)} \sum_{c \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} c(k)b(k)\right) \\ &\quad g(c) \\ &= \sum_{a \in X} \sum_{c \in X} \varepsilon_0^2 \overline{f(a)}g(c) \sum_{b \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} (c(k) - a(k))b(k)\right) \\ &= \sum_{a \in X} \sum_{c \in X} \varepsilon_0^2 \overline{f(a)}g(c) \delta(c - a) = \sum_{a \in X} \varepsilon_0 \overline{f(a)}g(a) = (f, g). \end{aligned}$$

Hence  $F$  is unitary. Since  $(F^2 f)(c) = (F(Ff))(c) = f(-c)$ ,  $F^4 = 1$ . Thus the eigenvalues of  $F$  are  $1, -1, -i, i$ . Furthermore,

$$\begin{aligned} (\overline{F}(Ff))(c) &= \sum_{b \in X} \varepsilon_0 \exp\left(2\pi i \sum_{k \in L} c(k)b(k)\right) \left(\sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} a(k)b(k)\right) f(a)\right) \\ &= \sum_{a \in X} \left(\sum_{b \in X} \varepsilon_0^2 \exp\left(-2\pi i \sum_{k \in L} b(k)(a(k) - c(k))\right)\right) f(a) \\ &= \sum_{a \in X} \varepsilon_0 \delta(a - c) f(a) = f(c). \end{aligned}$$

The same argument implies  $(F(\overline{F}f))(c) = f(c)$ .

$$(3) \quad (f * \delta)(a) = \sum_{b \in X} \varepsilon_0 f(a - b) \delta(b) = f(a),$$

- $(\delta * f)(a) = \sum_{b \in X} \varepsilon_0 \delta(a-b) f(b) = f(a)$ .  
(4)  $(f * g)(a) = \sum_{b \in X} \varepsilon_0 f(a-b) g(b) = \sum_{(a-b) \in X} \varepsilon_0 f(a-b) g(a-(a-b)) = (g * f)(a)$ .  
(5)  $(F(f * g))(c) = \sum_{a \in X} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} c(k) a(k)) \sum_{a \in X} \varepsilon_0 f(a-b) g(b)$   
 $= \sum_{a \in X} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} c(k) (b(k) + d(k))) \sum_{b \in X} \varepsilon_0 f(a-b) g(b)$ , where  $d(k) := a(k) - b(k)$ ,  
 $= \sum_{b \in X} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} c(k) b(k)) g(b) \sum_{d \in X-b} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} c(k) d(k)) f(d)$ ,  
where  $X - b := \{x - b \mid x \in X\}$ ,  
 $= \sum_{b \in X} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} c(k) b(k)) g(b) \sum_{d \in X} \varepsilon_0 \exp(-2\pi i \sum_{k \in L} c(k) d(k)) f(d)$   
 $= (Fg)(c)(Ff)(c) = (Ff)(c)(Fg)(c)$ .  
(6) Similarly,  $\overline{F}(f' * g') = (\overline{F}f')(\overline{F}g')$ .  
(7) The above (6) implies  $f' * g' = F((\overline{F}f')(\overline{F}g'))$ . We put  $f' = Ff$ ,  $g' = Fg$ . Then we obtain  $(Ff) * (Fg) = F(fg)$ .  
(8) Similarly,  $(\overline{F}f) * (\overline{F}g) = \overline{F}(fg)$ .

*Proof of Theorem 1.7.*

- (1)  $(F(D_{+,b}f))(a) = \sum_{c \in X} \varepsilon_0 \exp(-2\pi i ac) \frac{1}{\varepsilon'} (f(c + \varepsilon'b) - f(c))$   
 $= \sum_{c \in X} \varepsilon_0 (\frac{1}{\varepsilon'} (\exp(-2\pi i ac) f(c + \varepsilon'b) - \exp(-2\pi i ac) f(c)))$   
 $= \sum_{c \in X} \varepsilon_0 (\frac{1}{\varepsilon'} (\exp(2\pi i \varepsilon' ab) (\exp(-2\pi i a(c + \varepsilon'b)) f(c + \varepsilon'b) - \exp(-2\pi i ac) f(c))))$   
 $= \frac{1}{\varepsilon'} (\exp(2\pi i \varepsilon' ab) - 1) (Ff)(a) = \lambda_b(a) Ff(a)$ ,  
(2)  $(F(D_{-,b}f))(a) = \sum_{c \in X} \varepsilon_0 \exp(-2\pi i ac) \frac{1}{\varepsilon'} (f(c) - f(c - \varepsilon'b))$   
 $= \sum_{c \in X} \varepsilon_0 (\frac{1}{\varepsilon'} (\exp(-2\pi i ac) f(c) - \exp(-2\pi i \varepsilon' ab) \exp(-2\pi i a(c - \varepsilon'b)) f(c - \varepsilon'b)))$   
 $= \frac{1}{\varepsilon'} (1 - \exp(-2\pi i \varepsilon' ab)) (Ff)(a) = -\overline{\lambda}_b(a) Ff(a)$ ,  
(3)  $(F(\lambda_b f))(a) = \sum_{c \in X} \varepsilon_0 \exp(-2\pi i ac) (\lambda_b f)(c)$   
 $= \sum_{c \in X} \varepsilon_0 \exp(-2\pi i ac) \frac{1}{\varepsilon'} (\exp(2\pi i bc \varepsilon') - 1) f(c)$   
 $= \sum_{c \in X} \varepsilon_0 \frac{\exp(-2\pi i (a - b \varepsilon') c) - \exp(-2\pi i ac)}{\varepsilon'} f(c) = -D_{-,b}(Ff)(a)$ ,  
(4)  $(F(\overline{\lambda}_b f))(a) = \sum_{c \in X} \varepsilon_0 \exp(-2\pi i ac) (\overline{\lambda}_b f)(c)$   
 $= \sum_{c \in X} \varepsilon_0 \exp(-2\pi i ac) \frac{1}{\varepsilon'} (\exp(-2\pi i bc \varepsilon') - 1) f(c)$   
 $= \sum_{c \in X} \varepsilon_0 \frac{\exp(-2\pi i (a + b \varepsilon') c) - \exp(-2\pi i ac)}{\varepsilon'} f(c) = D_{+,b}(Ff)(a)$ .  
(1), (2) imply (5), (6).

*Proof of Theorem 1.8.*

$$\begin{aligned}
(f, D_{+,b}g) &= \sum_{a \in X} \varepsilon_0 \overline{f(a)} D_{+,b}g = \sum_{c \in X} \varepsilon_0 \overline{(Ff)(c)} (FD_{+,b})g(c) \\
&= \sum_{c \in X} \varepsilon_0 \overline{(Ff)(c)} \lambda_b(c) (Fg)(c) = \sum_{c \in X} \varepsilon_0 \overline{\lambda_b(c)} (Ff)(c) (Fg)(c) \\
&= - \sum_{c \in X} \varepsilon_0 \overline{F(D_{-,b}f)(c)} (Fg)(c) = - \sum_{a \in X} \varepsilon_0 \overline{D_{-,b}f(a)} g(a) = -(D_{-,b}f, g).
\end{aligned}$$

### 3. Examples.

We calculate two examples of the infinitesimal Fourier transformation for the space  $A$  of functionals. Let  $\star \circ \star : \mathbf{R} \rightarrow \star(\star\mathbf{R})$  be the natural elementary embedding and let  $\mathbf{st}(c)$  for  $c \in \star(\star\mathbf{R})$  be the standard part of  $c$  with respect to the natural elementary embedding  $\star \circ \star$ . The first is for  $\exp(i\pi \star \varepsilon \sum_{k \in L} a^2(k))$  and the second is for  $\exp(-\pi \star \varepsilon \sum_{k \in L} a^2(k))$ . We denote the two functionals by  $f(a), g(a)$ . If there is an  $L^2$ -function  $\alpha(t)$  on  $\mathbf{R}$  for  $a(k)$  so that  $a(k) = \star((\star\alpha)(k))$ , then  $\mathbf{st}(f(a)) =$



$\exp\left(i\pi \int_{-\infty}^{\infty} \alpha^2(t)dt\right)$ , and  $\mathbf{st}(g(a)) = \exp\left(-\pi \int_{-\infty}^{\infty} \alpha^2(t)dt\right)$ . Then we obtain the following results :

*Example 1.*  $(Ff)(b) = C_1 \overline{f(b)}$ , where  $C_1 = \sum_{a \in X} \varepsilon_0 \exp(i\pi * \varepsilon \sum_{k \in L} a^2(k))$ , it is just a standard number  $(-1)^{\frac{H}{2}}$ .

*Example 2.*  $(Fg)(b) = C_2(b)g(b)$ , where  $C_2(b) = \sum_{a \in X} \varepsilon_0 \exp(-\pi * \varepsilon \sum_{k \in L} (a(k) + ib(k))^2)$ , and if  $b$  is a finite valued function then it satisfies that  $\mathbf{st}(\mathbf{st}(C_2(b))) = 1$ .

For it, we calculate Kinoshita's infinitesimal Fourier transformation  $\varphi_1(x) = \exp(i\pi x^2)$ ,  $\varphi_2(x) = \exp(-\pi x^2)$  for the space  $R(L)$  of functions. We obtain :

$$(F\varphi_1)(p) = \exp\left(i\frac{\pi}{4}\right) \varphi_1(p),$$

$$(F\varphi_2)(p) = c(p)\varphi_2(p), \text{ where } \mathbf{st}(c(p)) = \int_{-\infty}^{\infty} \exp(-\pi t^2)dt, \text{ in the case of finite } p.$$

We denote the following :

$$R(\mathbf{L}) := \{\varphi' \mid \varphi' \text{ is an internal function from } \mathbf{L} \text{ to } * \mathbf{C}\},$$

$$R_H(\mathbf{L}) := \{\varphi' \in R(\mathbf{L}) \mid \varphi'(x+H) = \varphi'(x)\}.$$

Let  $e$  be a mapping from  $R(L)$  to  $R_H(\mathbf{L})$  defined by  $(e(\varphi))(x) = \varphi(\hat{x})$ , where  $\hat{x}$  is an element of  $L$  satisfying  $x \sim_H \hat{x}$ . Now  $\exp(i\pi x^2)$  is an element of  $R_H(\mathbf{L})$ , in fact, putting  $x \in \mathbf{L}$  :

$$\begin{aligned} \exp(i\pi(x+H)^2) &= \exp(i\pi(x^2 + 2xH + H^2)) \\ &= \exp(i\pi x^2) \exp(2\pi i \varepsilon z H) \exp(i\pi H^2) \quad (x = \varepsilon z (z \in * \mathbf{Z})) \\ &= \exp(i\pi x^2), \text{ as } \varepsilon H = 1 \text{ and } H \text{ is even.} \end{aligned}$$

Hence  $e(\exp(i\pi x^2)) = \exp(i\pi x^2)$ , that is,  $e(\varphi_1(x)) = \varphi_1(x)$ . We do the infinitesimal Fourier transformation of  $\varphi_1(x)$ ,

$$\begin{aligned} (F\varphi_1)(p) &= \sum_{x \in L} \varepsilon \exp(i\pi x^2) \exp(-2\pi i x p) = \sum_{x \in L} \varepsilon \exp(i\pi(x-p)^2) \exp(-i\pi p^2) \\ &= \sum_{x-y \in L} \varepsilon \exp(i\pi(x-p)^2) \exp(-i\pi p^2) = \sum_{x \in L} \varepsilon \exp(i\pi x^2) \exp(-i\pi p^2) \\ &= \left(\sum_{x \in L} \varepsilon \exp(i\pi x^2)\right) \overline{\exp(i\pi p^2)} = \left(\sum_{x \in L} \varepsilon \exp(i\pi x^2)\right) \overline{\varphi_1(p)}. \end{aligned}$$

By Gauss sums (cf.[R], p.409) :  $\sum_{n=0}^{N-1} \exp\left(\frac{2\pi i}{N} n^2\right) = \frac{1+(-i)^N}{1-i} \sqrt{N}$ , when  $N = 4m$  ( $m \in \mathbf{N}$ ),  $\sum_{n=0}^{N-1} \exp\left(\frac{2\pi i}{N} n^2\right) = (1+i)\sqrt{N}$ . Using it, we obtain the following :

$$\begin{aligned} \sum_{x \in L} \varepsilon \exp(i\pi x^2) &= \sum_{z=-\frac{H^2}{2}}^{\frac{H^2}{2}-1} \varepsilon \exp(i\pi(\varepsilon z)^2) = \sum_{z=-\frac{H^2}{2}}^{\frac{H^2}{2}-1} \varepsilon \exp\left(2\pi i \frac{z^2}{2H^2}\right) \\ &= \frac{1}{2} \varepsilon \sum_{z=0}^{2H^2-1} \exp\left(2\pi i \frac{z^2}{2H^2}\right) = \frac{1}{2} \varepsilon (1+i) \sqrt{2H^2} = \frac{1+i}{\sqrt{2}} = \exp\left(i\frac{\pi}{4}\right). \end{aligned}$$

Nextly we calculate the infinitesimal Fourier transformation of  $e(\varphi_2(x))$ .

$$\begin{aligned} (F(e(\varphi_2)))(p) &= \sum_{x \in L} \varepsilon e(\exp(-\pi x^2)) \exp(-2\pi i x p) \\ &= \left(\sum_{x \in L} \varepsilon \exp(-\pi(x+ip)^2)\right) \exp(-\pi p^2) = \left(\sum_{x \in L} \varepsilon \exp(-\pi(x+ip)^2)\right) \varphi_2(p). \end{aligned}$$

We assume that  $p \in L$  is finite. Since  $\exp(-\pi(x+ip)^2)$  is proved to be an  $S$ -integrable function directly, the term  $\sum_{x \in L} \varepsilon \exp(-\pi(x+ip)^2)$  satisfies the following (cf.[An], [Loe]) :

$$\mathbf{st}\left(\sum_{x \in L} \varepsilon \exp(-\pi(x+ip)^2)\right) = \int_{-\infty}^{\infty} \exp(-\pi(t+i \circ p)^2) dt,$$

where  $p \in L$ ,  $\circ p = \mathbf{st}(p) \in \mathbf{R}$ . We remark that the integral  $\int_{-\infty}^{\infty} \exp(-\pi(t+i \circ p)^2) dt$  is equal to  $\int_{-\infty}^{\infty} \exp(-\pi t^2) dt$ .

We define an equivalent relation  $\sim_{*HH'}$  in  $\mathbf{L}'$  by  $x \sim_{*HH'} y \Leftrightarrow x - y \in *HH'(*\mathbf{Z})$ . We identify  $\mathbf{L}' / \sim_{*HH'}$  with  $L'$ . Let

$X_{H, *HH'} := \{a' \mid a' \text{ is an internal function with double meanings, from } \star(\mathbf{L}) / \sim_{\star(H)} \text{ to } \mathbf{L}' / \sim_{*HH'}\}$ ,

and let  $\mathbf{e}$  is a mapping from  $X$  to  $X_{H, *HH'}$ , defined by  $(\mathbf{e}(a))([k]) = [a(\hat{k})]$ , where  $[ \ ]$  in left hand side represents the equivarent class for the equivarent relation  $\sim_{\star(H)}$  in  $\star(\mathbf{L})$ ,  $\hat{k}$  is a representative in  $\star(L)$  satisfying  $k \sim_{\star(H)} \hat{k}$ , and  $[ \ ]$  in right hand side represents the equivarent class for the equivarent relation  $\sim_{*HH'}$  in  $\mathbf{L}'$ . We consider an example  $f(a) = \exp(i\pi * \varepsilon \sum_{k \in L} a^2(k))$  in the space  $A$  of functionals, for  $a \in X$ .

$$\begin{aligned} & \exp(i\pi * \varepsilon \sum_{k \in L} (a(k) + *HH')^2) \\ &= \exp(i\pi * \varepsilon \sum_{k \in L} a^2(k)) \exp(2i\pi * \varepsilon \sum_{k \in L} *HH' a(k)) \exp(i\pi * \varepsilon \sum_{k \in L} *H^2 H'^2) \\ &= \exp(i\pi * \varepsilon \sum_{k \in L} a^2(k)). \end{aligned}$$

Hence if  $\mathbf{e}^\sharp(f)(a)$  is defined by  $f(\mathbf{e}(a))$ , then  $\mathbf{e}^\sharp(f) = f$ . We do the infinitesimal Fourier transformation of  $f(a)$ .

$$\begin{aligned} (Ff)(b) &= (F(\exp(i\pi * \varepsilon \sum_{k \in L} a^2(k))))(b) \\ &= \sum_{a \in X} \varepsilon_0 \exp(-2i\pi * \varepsilon \sum_{k \in L} a(k)b(k)) \exp(i\pi * \varepsilon \sum_{k \in L} a^2(k)) \\ &= (\sum_{a \in X} \varepsilon_0 \exp(i\pi * \varepsilon \sum_{k \in L} (a(k) - b(k))^2)) \exp(-i\pi * \varepsilon \sum_{k \in L} b^2(k)). \end{aligned}$$

The term  $\sum_{a \in X} \varepsilon_0 \exp(i\pi * \varepsilon \sum_{k \in L} (a(k) - b(k))^2)$  is represented as

$$\begin{aligned} & \sum_{a_{\lambda\mu} \in X_{\lambda\mu}} (\varepsilon_0)_{\lambda\mu} \exp(i\pi \varepsilon_\mu \sum_{k_\mu \in L_\mu} ((a(k))_{\lambda\mu} - (b(k))_{\lambda\mu})^2) \\ &= \prod_{k_\mu \in L_\mu} (\sum_{((a(k))_{\lambda\mu} \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(i\pi \varepsilon_\mu ((a(k))_{\lambda\mu} - (b(k))_{\lambda\mu})^2)) \end{aligned}$$

for  $\lambda\mu$  component, as  $\varepsilon_0 = (*\varepsilon)^{\frac{(*H)^2}{2}} \varepsilon'^{(*H)^2}$ . In the above,  $X_{\lambda\mu}$  is the set of functions from  $L_\mu$  to  $L'_{\lambda\mu}$ . If we put  $(a(k))_{\lambda\mu} = \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a$  ( $z_{\lambda\mu}^a \in \mathbf{Z}$ ) and  $(b(k))_{\lambda\mu} = \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b$  ( $z_{\lambda\mu}^b \in \mathbf{Z}$ ), we remark that  $z_{\lambda\mu}^a$  and  $z_{\lambda\mu}^b$  depend on  $k_\mu$ . The above is

$$\begin{aligned} & \prod_{k_\mu \in L_\mu} (\sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(i\pi \varepsilon_\mu (\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a - \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2)) \\ &= \prod_{k_\mu \in L_\mu} (\sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(i\pi (\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a - \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2)). \end{aligned}$$

Since  $\exp(i\pi \varepsilon x^2)$  is a periodic function with period  $*HH'$  on  $\mathbf{L}'$ , we obtain

$$\begin{aligned} & \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(i\pi (\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a - \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2) \\ &= \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(i\pi (\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a)^2). \end{aligned}$$

Hence

$$\sum_{a \in X} \varepsilon_0 \exp(i\pi * \varepsilon \sum_{k \in L} (a(k) - b(k))^2) = \sum_{a \in X} \varepsilon_0 \exp(i\pi * \varepsilon \sum_{k \in L} (a(k))^2) = C_1.$$

We calculate  $C_1$  as follows :

$$\begin{aligned} & \sum_{a \in X} \varepsilon_0 \exp(i\pi * \varepsilon \sum_{k \in L} (a(k))^2) = \varepsilon_0 \prod_{x \in L} \sum_{a(k) \in L'} \exp(i\pi * \varepsilon (a(k))^2) \\ &= \prod_{x \in L} (\sqrt{*HH'})^{-1} \sum_{z' = -\frac{*HH'}{2}}^{\frac{*HH'}{2}-1} \exp(i\pi * \varepsilon (\varepsilon' z')^2) \\ &= \prod_{x \in L} (\sqrt{*HH'})^{-1} \sum_{z' = -\frac{*HH'}{2}}^{\frac{*HH'}{2}-1} \exp(i\pi \frac{1}{*H} \frac{1}{H'^2} z'^2) \\ &= \prod_{x \in L} (\sqrt{*HH'})^{-1} \frac{1}{2} \sum_{z'=0}^{2*HH'-1} \exp\left(2\pi i \frac{z^2}{2*HH'^2}\right) \\ & \left( \text{Gauss sums (cf. [R], p.409)} \sum_{n=0}^{N-1} \exp\left(\frac{2\pi i}{N} n^2\right) = (1+i)\sqrt{N} \quad (N = 4m \quad (m \in \mathbf{N})) \right) \\ &= \prod_{x \in L} (\sqrt{*HH'})^{-1} \frac{1}{2} (1+i) \sqrt{2*HH'^2} = \prod_{x \in L} \frac{1+i}{\sqrt{2}} = \left(\frac{1+i}{\sqrt{2}}\right)^{H^2} = (\exp(i\frac{\pi}{4}))^{H^2} \\ &= (\exp(i\pi))^{\left(\frac{H}{2}\right)^2} = (-1)^{\left(\frac{H}{2}\right)^2} = (-1)^{\frac{H}{2}}. \end{aligned}$$

We do the infinitesimal Fourier transformation of  $g(a)$ .

$$\begin{aligned}
(Fg)(b) &= (F(\exp(-\pi^* \varepsilon \sum_{k \in L} a^2(k))))(b) \\
&= \sum_{a \in X} \varepsilon_0 \exp(-2i\pi^* \varepsilon \sum_{k \in L} a(k)b(k)) \exp\left(-\pi^* \varepsilon \sum_{k \in L} a^2(k)\right) \\
&= \left(\sum_{a \in X} \varepsilon_0 \exp(-\pi^* \varepsilon \sum_{k \in L} (a(k) + ib(k))^2)\right) \exp\left(-\pi^* \varepsilon \sum_{k \in L} b^2(k)\right).
\end{aligned}$$

We consider the term  $\sum_{a \in X} \varepsilon_0 \exp(-\pi^* \varepsilon \sum_{k \in L} (a(k) + ib(k))^2)$ . We write

$$(a(k))_{\lambda\mu} = \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a(k_\mu) \quad (z_{\lambda\mu}^a(k_\mu) \in \mathbf{Z}), \quad (b(k))_{\lambda\mu} = \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b(k_\mu) \quad (z_{\lambda\mu}^b(k_\mu) \in \mathbf{Z}).$$

From now on, we denote  $z_{\lambda\mu}^a(k_\mu)$ ,  $z_{\lambda\mu}^b(k_\mu)$  by  $z_{\lambda\mu}^a$ ,  $z_{\lambda\mu}^b$  for simplicity. Then the  $\lambda\mu$ -component of  $\sum_{a \in X} \varepsilon_0 \exp(-\pi^* \varepsilon \sum_{k \in L} (a(k) + ib(k))^2)$  is equal to

$$\begin{aligned}
&\sum_{a_{\lambda\mu} \in X_{\lambda\mu}} (\varepsilon_0)_{\lambda\mu} \exp\left(-\pi \varepsilon_\mu \sum_{k_\mu \in L_\mu} ((a(k))_{\lambda\mu} + i(b(k))_{\lambda\mu})^2\right) \\
&= \prod_{k_\mu \in L_\mu} \left(\sum_{(a(k))_{\lambda\mu} \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi \varepsilon_\mu ((a(k))_{\lambda\mu} + i(b(k))_{\lambda\mu})^2)\right) \\
&= \prod_{k_\mu \in L_\mu} \left(\sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi \varepsilon_\mu (\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a + i\varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2)\right) \\
&= \prod_{k_\mu \in L_\mu} \left(\sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a + i\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2)\right).
\end{aligned}$$

We assume that  $b \in X$  is finitely valued, that is,  $\exists b_0 \in \mathbf{R}$  s.t.  $k \in L \Rightarrow |b(k)| \leq \star(* (b_0))$ . The  $\lambda\mu$ -component of  $\frac{\sum_{a \in X} \varepsilon_0 \exp(-\pi^* \varepsilon \sum_{k \in L} (a(k) + ib(k))^2)}{\star\left(*\left(\int_{-\infty}^{\infty} \exp(-\pi x^2) dx\right)^{H^2}\right)}$  is equal to

$$\prod_{k_\mu \in L_\mu} \frac{\sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a + i\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2)}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx}.$$

We write  $B_{\lambda\mu}(k_\mu) := \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a + i\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2) - \int_{-\infty}^{\infty} \exp(-\pi(x + i\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2) dx$ .

It is equal to

$$\begin{aligned}
&-2 \int_{-\infty}^{-\sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}} \exp(-\pi(x + ib_{\lambda\mu})^2) dx \\
&+ \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a + i\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2) \\
&- \int_{-\sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}}^{\sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}} \exp(-\pi(x + ib_{\lambda\mu})^2) dx. \quad \dots (*_1)
\end{aligned}$$

Then the above is equal to

$$\begin{aligned}
&\prod_{k_\mu \in L_\mu} \frac{\sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a + i\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2)}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx} \\
&= \prod_{k_\mu \in L_\mu} \left(1 + \frac{B_{\lambda\mu}(k_\mu)}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx}\right) = \left(1 + \frac{B_{\lambda\mu}(k_\mu)}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx}\right)^{H_\mu^2} \\
&= \left(\left(1 + \frac{B_{\lambda\mu}(k_\mu)}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx}\right)^{\frac{1}{B_{\lambda\mu}(k_\mu)}}\right)^{B_{\lambda\mu}(k_\mu) H_\mu^2} \\
&= \left(\left(1 + \frac{1}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx \cdot \frac{1}{B_{\lambda\mu}(k_\mu)}}\right)^{\frac{1}{B_{\lambda\mu}(k_\mu)}}\right)^{B_{\lambda\mu}(k_\mu) H_\mu^2} \dots (*_2)
\end{aligned}$$

We show that  $[(B_{\lambda\mu}(k_\mu))]$  is infinitesimal in  $^*(^*\mathbf{C})$  with respect to  $\mathbf{C}$ . It implies that  $\left[\left(\frac{1}{B_{\lambda\mu}(k_\mu)}\right)\right]$  is infinite in  $^*(^*\mathbf{C})$ . For a sequence  $a_n$ , we remark that  $\lim_{n \rightarrow \infty} a_n = a \iff \forall N : \text{infinite with respect to } \mathbf{C} (\in ^*(^*\mathbf{C})), \mathbf{st}(^*(^*a_N)) = a$ . Hence

$$\mathbf{st}\left(\left[\left(1 + \frac{1}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx \cdot \frac{1}{B_{\lambda\mu}(k_\mu)}}\right)^{\frac{1}{B_{\lambda\mu}(k_\mu)}}\right]\right) = \exp\left(-\int_{-\infty}^{\infty} \exp(-\pi x^2) dx\right)$$

and  $\mathbf{st}([( * _1)]) = 1$ .

Since  $b_{\lambda\mu}$  is finite and  $\left[\left(\sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}\right)\right]$  is infinitesimal in  $^*(^*\mathbf{R})$  with respect to  $\mathbf{R}$ , the first term of  $(*_1)$  is infinitesimal in  $^*(^*\mathbf{C})$  with respect to  $\mathbf{C}$ . In order to show that  $[(B_{\lambda\mu}(k_\mu))]$  is infinitesimal in  $^*(^*\mathbf{C})$ , we consider the second and third terms in  $(*_1)$ , and we prove that it represents an infinitesimal number. First we calculate

$$\exp(-\pi(x + i\sqrt{\varepsilon_\mu}\varepsilon'_{\lambda\mu}z^b_{\lambda\mu})^2) - \exp(-\pi(\sqrt{\varepsilon_\mu}\varepsilon'_{\lambda\mu}z^a_{\lambda\mu} + i\sqrt{\varepsilon_\mu}\varepsilon'_{\lambda\mu}z^b_{\lambda\mu})^2).$$

For simplicity we write  $\sqrt{\varepsilon_\mu}\varepsilon'_{\lambda\mu}z^a_{\lambda\mu}$ ,  $\sqrt{\varepsilon_\mu}\varepsilon'_{\lambda\mu}z^b_{\lambda\mu}$  as  $a_{\lambda\mu}$ ,  $b_{\lambda\mu}$ . It is

$$\begin{aligned} & \exp(-\pi(x + ib_{\lambda\mu})^2) - \exp(-\pi(a_{\lambda\mu} + ib_{\lambda\mu})^2) \\ &= \exp(-\pi(x^2 - b_{\lambda\mu}^2)) \exp(-2i\pi b_{\lambda\mu}x) - \exp(-\pi(a_{\lambda\mu}^2 - b_{\lambda\mu}^2)) \exp(-2i\pi b_{\lambda\mu}a_{\lambda\mu}) \\ &= \{\exp(-\pi(x^2 - b_{\lambda\mu}^2)) \cos(2\pi b_{\lambda\mu}x) - \exp(-\pi(a_{\lambda\mu}^2 - b_{\lambda\mu}^2)) \cos(2\pi b_{\lambda\mu}a_{\lambda\mu})\} \\ & \quad - i\{\exp(-\pi(x^2 - b_{\lambda\mu}^2)) \sin(2\pi b_{\lambda\mu}x) - \exp(-\pi(a_{\lambda\mu}^2 - b_{\lambda\mu}^2)) \sin(2\pi b_{\lambda\mu}a_{\lambda\mu})\}. \quad \dots (*_3) \end{aligned}$$

We consider the first term of  $(*_3)$ . Then

$$\exp(-\pi(x^2 - b_{\lambda\mu}^2)) \cos(2\pi b_{\lambda\mu}x) = \exp(\pi b_{\lambda\mu}^2) \exp(-\pi x^2) \cos(2\pi b_{\lambda\mu}x).$$

We put  $f(x) = \exp(-\pi x^2) \cos(2\pi b_{\lambda\mu}x)$ . We assume that  $0 \leq b_{\lambda\mu}$ .

$$\begin{aligned} f'(x) &= -2\pi x \exp(-\pi x^2) \cos(2\pi b_{\lambda\mu}x) - \exp(-\pi x^2) 2\pi b_{\lambda\mu} \sin(2\pi b_{\lambda\mu}x) \\ &= -2\pi \sqrt{x^2 + b_{\lambda\mu}^2} \exp(-\pi x^2) \cos(2\pi b_{\lambda\mu}x + \alpha_x), \end{aligned}$$

where  $\cos \alpha_x = \frac{x}{\sqrt{x^2 + b_{\lambda\mu}^2}}$ ,  $-\sin \alpha_x = \frac{b_{\lambda\mu}}{\sqrt{x^2 + b_{\lambda\mu}^2}}$ . Since  $-\sin \alpha_x = \frac{b_{\lambda\mu}}{\sqrt{x^2 + b_{\lambda\mu}^2}}$ ,  $\alpha_x$  is

negative. There is a unique maximum of  $|f(x)|$  in

$\{x \in \mathbf{R} \mid \frac{\pi}{2}(2m-1) \leq 2\pi b_{\lambda\mu}x < \frac{\pi}{2}(2m+1)\}$  for each  $m \in \mathbf{Z}$ , that is,  $x$  satisfies

$$f'(x) = 0, \frac{\pi}{2}(2m-1) \leq 2\pi b_{\lambda\mu}x < \frac{\pi}{2}(2m+1) \iff 2\pi b_{\lambda\mu}x + \alpha_x = \frac{\pi}{2}(2m-1). \quad \dots (*_4)$$

We write the value of  $x$  having the maximum of  $|f(x)|$  in the interval as  $x_m$ . On

the other hand, we denote the value  $\alpha_x$  at  $x = A_{2m}$  by  $\alpha_{A_{2m}}$ . Then

$$A_{2m} = \frac{m - \frac{1}{2} - \frac{\alpha_{A_{2m}}}{\pi}}{2b_{\lambda\mu}}. \text{ The maximum of } f(x) \text{ is } f(A_{2m}) = \exp(-\pi x_m^2) \cos(m\pi - \frac{\pi}{2} - \alpha_{A_{2m}})$$

Hence  $|f(A_{2m})| \leq \exp\left(-\pi \left(\frac{m - \frac{1}{2} - \frac{\alpha_{A_{2m}}}{\pi}}{2b_{\lambda\mu}}\right)^2\right) \leq \exp\left(-\pi \left(\frac{m - \frac{1}{2}}{2b_{\lambda\mu}}\right)^2\right)$ . Then

$$\begin{aligned} & \left| \sum_{\varepsilon'_{\lambda\mu} z^a_{\lambda\mu} \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z^a_{\lambda\mu})^2 - (\varepsilon'_{\lambda\mu} z^b_{\lambda\mu})^2) \cos(2\pi \varepsilon'_{\lambda\mu} z^b_{\lambda\mu} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z^a_{\lambda\mu}) \right. \\ & \quad \left. - \int_{-\frac{\sqrt{\varepsilon_\mu} H'_{\lambda\mu}}{2}}^{\frac{\sqrt{\varepsilon_\mu} H'_{\lambda\mu}}{2}} \exp(-\pi(x^2 - b_{\lambda\mu}^2)) \cos(2\pi b_{\lambda\mu}x) dx \right| \\ &= \exp(\pi b_{\lambda\mu}^2) \left| \sum_{\varepsilon'_{\lambda\mu} z^a_{\lambda\mu} \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} f(\sqrt{\varepsilon_\mu} \varepsilon'_{\lambda\mu} z^a_{\lambda\mu})^2 - \int_{-\frac{\sqrt{\varepsilon_\mu} H'_{\lambda\mu}}{2}}^{\frac{\sqrt{\varepsilon_\mu} H'_{\lambda\mu}}{2}} f(x) dx \right|. \end{aligned}$$

We denote  $A_{2m+1} = \frac{1}{4b_{\lambda\mu}}(2m+1)$ . Since  $[(b(k)_{\lambda\mu})]$  is finite, there exists a positive real number  $c$  so that  $\star(*c) \leq \left[ \left( \left| \frac{1}{4b_{\lambda\mu}} \right| \right) \right]$ , that is,  $\left\{ \lambda \mid \left\{ \mu \mid c \leq \left| \frac{1}{4b_{\lambda\mu}} \right| \right\} \in F_2 \right\} \in F_1$ . Furthermore since  $\sqrt{\varepsilon\varepsilon'}$  is infinitesimal in  $\star(*\mathbf{R})$ ,  $k \in L \Rightarrow \sqrt{\varepsilon\varepsilon'} < \left| \frac{1}{4b(k)} \right|$ , that is,  $\left\{ \lambda \mid \left\{ \mu \mid \sqrt{\varepsilon_\mu\varepsilon'_{\lambda\mu}} < \left| \frac{1}{4b_{\lambda\mu}}(k_\mu) \right| \right\} \in F_2 \right\} \in F_1$ . We assume  $\lambda\mu$  satisfies the above condition. We denote  $\sqrt{\varepsilon_\mu\varepsilon'_{\lambda\mu}}$  by  $\Delta'$ . We shall show that the following term is infinitesimal in  $\star(*\mathbf{R})$ :  $\left| \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon_\mu\varepsilon'_{\lambda\mu}} f(\sqrt{\varepsilon_\mu\varepsilon'_{\lambda\mu}} z_{\lambda\mu}^a)^2 - \int_{-\sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}}^{\sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}} f(x) dx \right|$ .

We write the maximum of  $j$  as  $l$  so that  $A_j \in [0, \sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}]$ . Let  $x_k = \Delta'k$ ,  $x_{i_j} \leq A_j < x_{i_{j+1}}$  ( $1 \leq j \leq l$ ). Then we divide the interval  $[0, \sqrt{\varepsilon_\mu} \frac{H'_{\lambda\mu}}{2}]$  into suitable intervals and prove it.

CASE 1.

$$\begin{aligned}
& \left| \int_{-A_1}^{A_1} f(x) dx - \left( \sum_{i=-i_1}^{i_1} f(x_i) \right) \Delta' \right| \\
&= \left| \int_{-A_1}^{-x_{i_1}} f(x) dx + \sum_{i=-i_1}^{i_1} \left( \int_{x_i}^{x_{i+1}} (f(x) - f(x_i)) dx \right) \right. \\
&\quad \left. + \int_{x_{i_1}}^{A_1} (f(x) - f(x_{i_1})) dx - f(x_{i_1})(x_{i_1+1} - A_1) \right| \\
&= \left| \int_{-A_1}^{-x_{i_1}} f(x) dx + \int_{-x_{i_1}}^{-x_{i_1-1}} (f(x) - f(-x_{i_1})) dx + \cdots + \int_{-x_1}^{x_0} (f(x) - f(-x_1)) dx \right. \\
&\quad \left. + \int_{x_0}^{x_1} (f(x) - f(x_0)) dx + \cdots + \int_{x_{i_1-1}}^{x_{i_1}} (f(x) - f(x_{i_1-1})) dx \right. \\
&\quad \left. + \int_{x_{i_1}}^{A_1} (f(x) - f(x_{i_1})) dx - f(x_{i_1})(x_{i_1+1} - A_1) \right|, \text{ where } x_0 = 0, \\
&\leq \int_{-A_1}^{-x_{i_1}} f(x) dx + \int_{-x_{i_1}}^{-x_{i_1-1}} (f(x) - f(-x_{i_1})) dx + \cdots + \int_{-x_1}^{x_0} (f(x) - f(-x_1)) dx \\
&\quad + \int_{x_0}^{x_1} (f(x_0) - f(x)) dx + \cdots + \int_{x_{i_1-1}}^{x_{i_1}} (f(x_{i_1-1}) - f(x)) dx \\
&\quad + \int_{x_{i_1}}^{A_1} (f(x_{i_1}) - f(x)) dx + f(x_{i_1})(x_{i_1+1} - A_1) \\
&\text{, since } f \text{ is an even function, } f(-x) = f(x), \\
&= \int_{x_{i_1}}^{A_1} f(x) dx + \int_{x_{i_1-1}}^{x_{i_1}} (f(x) - f(x_{i_1})) dx + \cdots + \int_{-x_0}^{x_1} (f(x) - f(x_1)) dx \\
&\quad + \int_{x_0}^{x_1} (f(x_0) - f(x)) dx + \cdots + \int_{x_{i_1-1}}^{x_{i_1}} (f(x_{i_1-1}) - f(x)) dx \\
&\quad + \int_{x_{i_1}}^{A_1} (f(x_{i_1}) - f(x)) dx + f(x_{i_1})(x_{i_1+1} - A_1) \\
&= (f(x_0) - f(x_1) + f(x_1) - f(x_2) + \cdots + f(x_{i_1-1}) - f(x_{i_1}) + f(x_{i_1})) \Delta' \\
&= f(x_0) \Delta' = f(0) \Delta'.
\end{aligned}$$

CASE 2.

$$\begin{aligned}
& \left| \int_{-x_{i_2}}^{-A_1} f(x) dx + \int_{A_1}^{x_{i_2+1}} f(x) dx - \left( \sum_{i=-i_2}^{-(i_1+1)} f(x_i) + \sum_{i=i_1+1}^{i_2} f(x_i) \right) \Delta' \right| \\
&= \left| \int_{-x_{i_2}}^{-x_{i_2-1}} (f(x) - f(-x_{i_2})) dx + \cdots + \int_{-x_{i_1+2}}^{-x_{i_1+1}} (f(x) - f(-x_{i_1+2})) dx \right. \\
&\quad \left. + \int_{-x_{i_1+1}}^{-A_1} (f(x) - f(-x_{i_1+1})) dx - f(-x_{i_1+1})(A_1 - x_{i_1}) + \int_{A_1}^{x_{i_1+1}} f(x) dx \right. \\
&\quad \left. + \int_{x_{i_1+1}}^{x_{i_1+2}} (f(x) - f(x_{i_1+1})) dx + \cdots + \int_{x_{i_2}}^{x_{i_2+1}} (f(x) - f(x_{i_2})) dx \right| \\
&\leq \int_{-x_{i_2}}^{-x_{i_2-1}} (f(x) - f(-x_{i_2})) dx + \cdots + \int_{-x_{i_1+2}}^{-x_{i_1+1}} (f(x) - f(-x_{i_1+2})) dx \\
&\quad + \int_{-x_{i_1+1}}^{-A_1} (f(x) - f(-x_{i_1+1})) dx - f(-x_{i_1+1})(A_1 - x_{i_1}) + \int_{A_1}^{x_{i_1+1}} (-f(x)) dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{x_{i_1+1}}^{x_{i_1+2}} (f(x_{i_1+1}) - f(x)) dx + \cdots + \int_{x_{i_2}}^{x_{i_2+1}} (f(x_{i_2}) - f(x)) dx \\
& , \text{ since } f \text{ is an even function, } f(-x) = f(x), \\
& = \int_{x_{i_2-1}}^{x_{i_2}} (f(x) - f(x_{i_2})) dx + \cdots + \int_{x_{i_1+1}}^{x_{i_1+2}} (f(x) - f(x_{i_1+2})) dx \\
& + \int_{A_1}^{x_{i_1+1}} (f(x) - f(x_{i_1+1})) dx - f(-x_{i_1+1})(A_1 - x_{i_1}) + \int_{A_1}^{x_{i_1+1}} (-f(x)) dx \\
& + \int_{x_{i_1+1}}^{x_{i_1+2}} (f(x_{i_1+1}) - f(x)) dx + \cdots + \int_{x_{i_2}}^{x_{i_2+1}} (f(x_{i_2}) - f(x)) dx, \\
& , \text{ since } \int_{x_{i_2}}^{x_{i_2+1}} (f(x_{i_2}) - f(x)) dx \leq \int_{x_{i_2}}^{x_{i_2+1}} (f(x_{i_2}) - f(A_2)) dx, \\
& \leq (-f(x_{i_1+1}) + f(x_{i_1+1}) - f(x_{i_1+2}) + \cdots + f(x_{i_2-1}) - f(x_{i_2}) + f(x_{i_2}) - f(A_2)) \Delta' \\
& = (-f(A_2)) \Delta'.
\end{aligned}$$

Since the next steps are just same, we obtain the following :

$$\begin{aligned}
& \left| \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a \in L'_{\lambda\mu}} \sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} f(\sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a)^2 - \int_{-\sqrt{\varepsilon'_\mu} \frac{H'_{\lambda\mu}}{2}}^{\sqrt{\varepsilon'_\mu} \frac{H'_{\lambda\mu}}{2}} f(x) dx \right| \\
& \leq 2\sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} \sum_{m=0}^l \exp\left(-\pi \left(\frac{m-\frac{1}{2}}{2b_{\lambda\mu}}\right)^2\right).
\end{aligned}$$

Hence if  $0 \leq b_{\lambda\mu}$ ,

$$\begin{aligned}
& \left| \sum_{\varepsilon'_{\lambda\mu} z_{\lambda\mu}^a, \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b \in L'_{\lambda\mu}} \sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} \exp(-\pi(\sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a)^2 - (\varepsilon'_{\lambda\mu} z_{\lambda\mu}^b)^2) \cos(2\pi \varepsilon'_{\lambda\mu} z_{\lambda\mu}^b \sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} z_{\lambda\mu}^a) \right. \\
& \left. - \int_{-\sqrt{\varepsilon'_\mu} \frac{H'_{\lambda\mu}}{2}}^{\sqrt{\varepsilon'_\mu} \frac{H'_{\lambda\mu}}{2}} \exp(-\pi(x^2 - b_{\lambda\mu}^2)) \cos(2\pi b_{\lambda\mu} x) dx \right| \\
& \leq \exp(\pi b_{\lambda\mu}^2) \cdot 2\sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} \sum_{m=0}^l \exp\left(-\pi \left(\frac{m-\frac{1}{2}}{2b_{\lambda\mu}}\right)^2\right).
\end{aligned}$$

Since  $\left[\left(\sum_{m=0}^l \exp\left(-\pi \left(\frac{m-\frac{1}{2}}{2b_{\lambda\mu}}\right)^2\right)\right)\right] \leq \sum_{m=0}^l \exp\left(-\pi \left(\frac{m-\frac{1}{2}}{2b_0}\right)^2\right)$ , it is finite. Hence

$\left[\left(\exp(\pi b_{\lambda\mu}^2) \cdot 2\sqrt{\varepsilon'_\mu} \varepsilon'_{\lambda\mu} \sum_{m=0}^l \exp\left(-\pi \left(\frac{m-\frac{1}{2}}{2b_{\lambda\mu}}\right)^2\right)\right)\right]$  is infinitesimal in  $\ast(*\mathbf{R})$ . If  $b_{\lambda\mu} < 0$ , the argument is parallel, and also, for the term of sin in  $(\ast_3)$ , though sin is not an even function, the same argument holds. Hence  $[(B_{\lambda\mu}(k_\mu))]$  is infinitesimal in  $\ast(*\mathbf{C})$  with respect to  $\mathbf{C}$ . Put  $N_{\lambda\mu} = \frac{1}{B_{\lambda\mu}(k_\mu)}$ . Then it is infinite, and since  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{an}\right)^n = e^{-a}$ ,

$$\begin{aligned}
& \text{st} \left( \left[ \left( 1 + \frac{1}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx \cdot \frac{1}{B_{\lambda\mu}(k_\mu)}} \right)^{\frac{1}{B_{\lambda\mu}(k_\mu)}} \right] \right) \\
& = \text{st} \left( \left[ \left( 1 + \frac{1}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx \cdot N_{\lambda\mu}} \right)^{N_{\lambda\mu}} \right] \right) = \exp\left(-\int_{-\infty}^{\infty} \exp(-\pi x^2) dx\right).
\end{aligned}$$

There is an infinitesimal  $C (= [(C_{\lambda\mu})])$  in  $\ast(*\mathbf{C})$  so that

$$\left( 1 + \frac{1}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx \cdot N_{\lambda\mu}} \right)^{N_{\lambda\mu}} = \exp\left(-\int_{-\infty}^{\infty} \exp(-\pi x^2) dx\right) + C_{\lambda\mu}.$$

Then  $[((\ast_1))]$  is equal to

$$\left[ \left( \left( 1 + \frac{1}{\int_{-\infty}^{\infty} \exp(-\pi x^2) dx \cdot N_{\lambda\mu}} \right)^{N_{\lambda\mu}} \right)^{[(B_{\lambda\mu}(k_{\mu})H_{\mu}^2)]} \right]$$

$$= \left( \star \left( \star \left( \exp \left( - \int_{-\infty}^{\infty} \exp(-\pi x^2) dx \right) \right) \right) + C \right)^{[(B_{\lambda\mu}(k_{\mu})H_{\mu}^2)]}.$$

Since  $[(B_{\lambda\mu}(k_{\mu})H_{\mu}^2)]$  is infinitesimal,

$$\text{st} \left( \left( \star \left( \star \left( \exp \left( - \int_{-\infty}^{\infty} \exp(-\pi x^2) dx \right) \right) \right) + C \right)^{[(B_{\lambda\mu}(k_{\mu})H_{\mu}^2)]} \right) = 1.$$

Thus

$$\text{st} \left( \text{st} \left( \frac{\sum_{a \in X} \varepsilon_0 \exp \left( -\pi \star \varepsilon \sum_{k \in L} (a(k) + ib(k))^2 \right)}{\star \left( \star \left( \int_{-\infty}^{\infty} \exp(-\pi x^2) dx \right)^{H^2} \right)} \right) \right) = 1.$$

Since  $\int_{-\infty}^{\infty} \exp(-\pi x^2) dx = 1$ , then  $\text{st}(\text{st}(C_2(b))) = 1$ .

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## References

- [An] R.M. Anderson, A non-standard representation for Brownian motion and Itô integration, Israel J. Math. **25** (1976), 15-46.
- [A-F-HK-L] S. Albeverio, J.E. Fenstad, R. Høegh-Krohn, T. Lindstroöm, Non-standard methods in stochastic analysis and mathematical physics, Academic Press (1986).
- [F] D. Fujiwara, A construction of the fundamental solution for the Schrödinger equation, J. D'Analyse Math. **35** (1979), 41-96.
- [F-H] R.P. Feynman, A.R. Hibbs, Quantum mechanics and path integrals, McGraw-Hill Inc. All rights (1965).
- [G] E.I. Gordon, Nonstandard methods in commutative harmonic analysis, Translations of mathematical monographs **164** American mathematical society, 1997.
- [H] T. Hida, Brownian motion, in Japanese, Iwanami shoten, 1975.
- [Ic] T. Ichinose, Path integral for the Dirac equation in two space-time dimensions, Proc. Japan Acad. Ser. A. Math. Sci. **58** (1982), 290-293.
- [It] T. Itô, Differential equations determining a Markoff process (original Japanese: Zenkoku Sizyo Sugaku Danwakai-si), Journ. Pan-Japan Math. Coll. No. **1077**, 1942.
- [I-T] T. Ichinose, H. Tamura, Path integral approach to relativistic quantum mechanics-Two-dimensional Dirac equation, Suppl. Prog. Theor. Phys. **92** (1987), 144-175.
- [Ka] T. Kamae, A simple proof of the Ergodic theorem using non-standard analysis, Israel J. Math. **42** (1982), 284-290.
- [Ki] M. Kinoshita, Nonstandard representation of distribution I, Osaka J. Math. **25** (1988), 805-824.

- [Loe] P.A. Loeb, Conversion from nonstandard to standard measure spaces and application in probability theory, *Trans. Amer. Math. Soc.* **211** (1975), 113-122.
- [Loo1] K. Loo, Nonstandard Feynman path integral for harmonic oscillator, *J. Math. Phys.*, **40** (1999), 5511-5521.
- [Loo2] K. Loo, A rigorous real-time Feynman path integral and propagator, *J. Phys.*, **A33** (2000), 9215-9239.
- [Lu] W.A. Luxemburg, A Nonstandard approach to Fourier analysis, *Contributions to Nonstandard Analysis*, North-Holland, Amsterdam, pp.16-39, 1972.
- [Na1] T. Nakamura, A nonstandard representation of Feynman's Path integrals, *J. Math. Phys.*, **32** (1991), 457-463.
- [Na2] T. Nakamura, Path space measure for the 3+1-dimensional Dirac equation in momentum space, *J. Math. Phys.*, **41** (2000), 5209-5222.
- [Ne] E. Nelson, Feynman integrals and the Schrödinger equation, *J. Math. Phys.*, **5** (1964), 332-343.
- [N-O] T. Nitta and T. Okada, Double infinitesimal Fourier transformation for the space of functionals and reformulation of Feynman path integral, *Lecture Note Series in Mathematics*, Osaka University Vol.7 (2002), 255-298 in Japanese.
- [N-O-T] T. Nitta, T. Okada and A. Tzouvaras, Classification of non-well-founded sets and an application, *Math. Log. Quart.* **49** (2003), 187-200.
- [R] R. Remmert, *Theory of complex functions*, Graduate Texts in Mathematics **122**, Springer, Berlin-Heidelberg-New York, 1992.
- [S] M. Saito, *Ultraproduct and non-standard analysis*, in Japanese, Tokyo tosho, 1976.
- [T] G. Takeuti, Dirac space, *Proc. Japan Acad.* **38** (1962), 414-418.



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