

Twisted homology of quantum $SL(2)$

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Abstract

We calculate the twisted Hochschild and cyclic homology (in the sense of Kustermans, Murphy and Tuset) of the coordinate algebra of the quantum $SL(2)$ group relative to twisting automorphisms acting by rescaling the standard generators a, b, c, d . We discover a family of automorphisms for which the “twisted” Hochschild dimension coincides with the classical dimension of $SL(2, \mathbb{C})$, thus avoiding the “dimension drop” in Hochschild homology seen for many quantum deformations. Strikingly, the simplest such automorphism is the canonical modular automorphism arising from the Haar functional. In addition, we identify the twisted cyclic cohomology classes corresponding to the three covariant differential calculi over quantum $SU(2)$ discovered by Woronowicz.

1 Introduction

Cyclic homology and cohomology were independently discovered by Alain Connes [1] and Boris Tsygan [21] in the early 1980’s, and should be thought of as extensions of de Rham (co)homology to various categories of noncommutative algebras. Quantum groups also appeared in the same period, with the first example of a “compact quantum group” in the C^* -algebraic setting being Woronowicz’s “quantum $SU(2)$ ” [24].

The noncommutative differential geometry (in the sense of Connes) of quantum $SU(2)$ was thoroughly investigated by Masuda, Nakagami

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and Watanabe [15]. They calculated the Hochschild and cyclic homology of the coordinate algebra $\mathcal{A}(SL_q(2))$ of quantum $SL(2)$ as well as the K-theory and K-homology of the C*-algebra of the compact quantum $SU(2)$ group. This work was extended by Feng and Tsygan [5], who computed the Hochschild and cyclic homology of the standard quantized coordinate algebra $\mathcal{A}(G_q)$ associated to an arbitrary complex semisimple Lie group G . The homologies are roughly speaking those of a classical space labelling the symplectic leaves of the Poisson-Lie group G (the semiclassical limit of $\mathcal{A}(G_q)$). In particular, the Hochschild dimension of $\mathcal{A}(G_q)$ equals the rank of G . This “dimension drop” had already been observed for other quantizations of Poisson algebras. Many authors regarded it as an unpleasant feature and asked for generalizations of cyclic homology which detect the quantized parts of quantum groups as well.

One candidate is twisted Hochschild and cyclic (co)homology defined by Kustermans, Murphy and Tuset [12], relative to a pair of an algebra \mathcal{A} and automorphism σ . This reduces to ordinary Hochschild and cyclic (co)homology of \mathcal{A} on taking σ to be the identity. The standard theory is intimately related with the idea of considering tracial functionals on noncommutative algebras as analogues of integrals, whereas the twisted theory arises naturally from functionals whose tracial properties are of the form $h(ab) = h(\sigma(b)a)$. Noncommutative spaces equipped with such functionals include duals of nonunimodular groups, type III von Neumann algebras and compact quantum groups. The aim of [12] was to adapt Connes’ constructions relating cyclic cohomology and differential calculi to covariant differential calculi in the sense of Woronowicz, since the volume forms of such calculi define in general twisted cocycles rather than usual ones [17]. The possibility of pairing twisted cyclic cocycles (e.g. over quantum homogeneous spaces) with equivariant K-theory was demonstrated in [16], and it seems an interesting problem to investigate how far this original motivation of cyclic cohomology extends to the twisted setting.

In this paper we compute the twisted Hochschild and cyclic homologies $HH_*^\sigma(\mathcal{A})$, $HC_*^\sigma(\mathcal{A})$ for the coordinate algebra $\mathcal{A} = \mathcal{A}(SL_q(2))$ of the quantum $SL(2)$ group, with generic deformation parameter q . We consider all automorphisms σ of the form $a, b, c, d \mapsto \lambda a, \mu b, \mu^{-1}c, \lambda^{-1}d$, where a, b, c, d are the standard generators, and λ, μ are nonzero elements of k . As an overview we collect the dimensions of $HH_n^\sigma(\mathcal{A})$ as a k -vector space, see the main text for explicit formulas for generators:

THEOREM 1.1 *We have*

$$\begin{aligned} \dim HH_n^\sigma(\mathcal{A}) &= 0, \quad n > 3, \\ \dim HH_3^\sigma(\mathcal{A}) &= \begin{cases} N+1 & \lambda = q^{-(N+2)}, \mu = 1, \\ 0 & \text{otherwise,} \end{cases} \\ \dim HH_2^\sigma(\mathcal{A}) &= \begin{cases} N+1 & \lambda = q^{-(N+2)}, \mu = 1, \\ 2 & \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)}, \\ 0 & \text{otherwise,} \end{cases} \\ \dim HH_1^\sigma(\mathcal{A}) &= \begin{cases} 0 & \lambda \notin q^{-\mathbb{N}}, \mu = q^{\pm(M+1)}, \\ 0 & \lambda \neq 1, \mu \notin q^{\mathbb{Z}}, \\ 4 & \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)}, \\ \infty & \text{otherwise,} \end{cases} \\ \dim HH_0^\sigma(\mathcal{A}) &= \begin{cases} \infty & \mu = 1, \\ 2 & \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for $M, N \in \mathbb{N}$.

Strikingly (Theorem 4.12), there exists a family of automorphisms for which the twisted Hochschild dimension takes the classical value three (note also that the homological dimension of $\mathcal{A}(SL_q(2))$ is three [13]) - the twisted theory avoids the “dimension drop”. Remarkably, the simplest such automorphism ($\lambda = q^{-2}, \mu = 1$) is the canonical modular automorphism associated to the Haar functional on \mathcal{A} . Similar results were obtained for Podleś quantum spheres [7] and quantum hyperplanes [20].

In [5], Feng and Tsygan considered formal quantizations, with $\mathcal{A}(G_q)$ a Hopf algebra over $\mathbb{C}[[\hbar]]$ with $q = e^\hbar$. They showed that for a Hopf algebra \mathcal{A} over a field k , with coproduct Δ , counit ε and antipode S , and an \mathcal{A} -bimodule \mathcal{M} , there is an isomorphism

$$H_n(\mathcal{A}, \mathcal{M}) \simeq \text{Tor}_n^{\mathcal{A}}(\mathcal{M}', k) \quad (1)$$

Here, \mathcal{M}' is \mathcal{M} as a linear space with right action given by

$$m \blacktriangleleft a := \sum S(a_{(2)})ma_{(1)} \quad (2)$$

using Sweedler’s notation for the coproduct, and $k = \mathcal{A}/\ker \varepsilon$ is the trivial left \mathcal{A} -module. Then they computed these Tor-groups using the spectral sequence associated to the filtration induced by \hbar .

In this paper we compute $\text{Tor}_n^{\mathcal{A}}(\mathcal{M}', k)$ from a Koszul-type free resolution

$$0 \rightarrow \mathcal{A} \rightarrow \mathcal{A}^3 \rightarrow \mathcal{A}^3 \rightarrow \mathcal{A} \rightarrow k \rightarrow 0 \quad (3)$$

of k . Noncommutative Koszul resolutions were studied by several authors, in particular Wambst [22], but as far as we know were not applied to quantum groups. In our opinion this resolution shows very clearly the geometric mechanisms behind the computations. We will see that the maps of the resulting complex computing the twisted Hochschild homology become zero for $q = 1$, so one obtains the Hochschild-Kostant-Rosenberg theorem for $SL(2)$ (the algebraic cotangent bundle of $SL(2)$ is trivial). However, for $q \neq 1$ this does not happen for any twisting automorphism.

A summary of this paper is as follows. In section 2 we recall how the twisted theory was discovered [12], then give the definitions of $HH_*^\sigma(\mathcal{A})$ and $HC_*^\sigma(\mathcal{A})$, and the underlying cyclic object. We specialize to Hopf algebras and explain the methods adapted from [5]. We then present the general scheme of the noncommutative Koszul complexes used here. In section 3 we introduce the quantum $SL(2)$ group. In section 4 we present our calculations of $HH_*^\sigma(\mathcal{A})$ for $\mathcal{A} = \mathcal{A}(SL_q(2))$.

Twisted cyclic homology is defined as the total homology of Connes’ mixed (b, B) -bicomplex coming from the underlying cyclic object, as in [14]. In section 5 we compute this homology via a spectral sequence.

Finally, in section 6 we discuss the relation of our results to previously known twisted cyclic cocycles coming from the three covariant differential calculi over $\mathcal{A}(SL_q(2))$ discovered by Woronowicz. The twisted cyclic 3-cocycle arising from the three dimensional left covariant calculus was given explicitly in [12] and [17]. We show (Theorem 6.1) that this 3-cocycle is a trivial element of twisted cyclic cohomology. Further, the twisted 4-cocycles arising from the two bicovariant four dimensional calculi both correspond to the twisted 0-cocycle coming from the Haar functional (as elements of even periodic twisted cyclic cohomology).

2 Twisted cyclic homology

2.1 MOTIVATION

Twisted cyclic (co)homology arose from the study of covariant differential calculi over quantum groups [12].

Let \mathcal{A} be an algebra over \mathbb{C} . Given a differential calculus (Ω, d) over \mathcal{A} , with $\Omega = \bigoplus_{n=0}^N \Omega_n$, Connes [3] considered linear functionals $f : \Omega_N \rightarrow \mathbb{C}$, which are closed and graded traces on Ω , meaning

$$\int d\omega = 0 \quad \forall \omega \in \Omega_{N-1}$$

$$\int \omega_m \omega_n = (-1)^{mn} \int \omega_n \omega_m \quad \forall \omega_m \in \Omega_m, \omega_n \in \Omega_n \quad (4)$$

Connes found that such linear functionals are in one to one correspondence with cyclic N -cocycles τ on the algebra, via

$$\tau(a_0, a_1, \dots, a_N) = \int a_0 da_1 da_2 \dots da_N \quad (5)$$

which led directly to his simplest formulation of cyclic cohomology [3].

If \mathcal{A} is the coordinate algebra of a quantum group, then Woronowicz proposed to study covariant differential calculi, for which the left coaction of \mathcal{A} on \mathcal{A} given by the coproduct $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ extends to a coaction $\Delta_L : \Omega \rightarrow \mathcal{A} \otimes \Omega$ compatible with the differential d [24], [25]. For such calculi the natural linear functionals $f : \Omega_N \rightarrow \mathbb{C}$ are no longer graded traces, but twisted graded traces, meaning that

$$\int \omega_m \omega_n = (-1)^{mn} \int \sigma(\omega_n) \omega_m \quad \forall \omega_m \in \Omega_m, \omega_n \in \Omega_n \quad (6)$$

for some degree zero automorphism σ of Ω . In particular, σ restricts to an automorphism of \mathcal{A} , and, for any $a \in \mathcal{A}$, $\omega_N \in \Omega_N$ we have

$$\int \omega_N a = \int \sigma(a) \omega_N \quad (7)$$

Hence for each covariant calculus there is a natural automorphism of \mathcal{A} . Motivated by this observation, Kustermans, Murphy and Tuset defined ‘‘twisted’’ Hochschild and cyclic cohomology for any pair of an algebra \mathcal{A} and automorphism σ , and showed that the one-to-one correspondence between graded traces and cyclic cocycles generalizes to this setting. The next section recalls their definitions, transposed to homology.

2.2 TWISTED HOCHSCHILD AND CYCLIC HOMOLOGY

Let \mathcal{A} be a unital, associative algebra over a field k (assumed to be of characteristic zero) and σ an automorphism. We define the cyclic object [2], [14] underlying twisted cyclic homology $HC_*^\sigma(\mathcal{A})$ of \mathcal{A} relative to σ . Set $C_n := \mathcal{A}^{\otimes(n+1)}$. For clarity, we will denote $a_0 \otimes a_1 \otimes \dots \otimes a_n \in C_n$ by (a_0, a_1, \dots, a_n) . Define

$$d_{n,i}(a_0, a_1, \dots, a_n) = (a_0, \dots, a_i a_{i+1}, \dots, a_n) \quad 0 \leq i \leq n-1$$

$$d_{n,n}(a_0, a_1, \dots, a_n) = (\sigma(a_n) a_0, a_1, \dots, a_{n-1})$$

$$s_{n,i}(a_0, a_1, \dots, a_n) = (a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n) \quad 0 \leq i \leq n$$

$$\tau_n(a_0, a_1, \dots, a_n) = (\sigma(a_n), a_0, \dots, a_{n-1}) \quad (8)$$

For $\sigma = \text{id}$ these are the face, degeneracy and cyclic operators of the standard cyclic object associated to \mathcal{A} [14]. For general σ the operator $T_n := \tau_n^{n+1}$ is not equal to the identity, but all other relations of the cyclic category are fulfilled. Hence C_* becomes what is called a paracyclic object [6]. To obtain a cyclic object, we pass to the cokernels $C_n^\sigma := C_n/C_n^1$, $C_n^1 := \text{im}(\text{id} - T_n)$. Dualizing [12], we call the cyclic homology of this cyclic object the σ -twisted cyclic homology $HC_*^\sigma(\mathcal{A})$ of \mathcal{A} . Hence $HC_*^\sigma(\mathcal{A})$ is the total homology of Connes' mixed (b, B) -bicomplex

$$\begin{array}{ccccccc} & & & & & & \\ & & & & & & \\ & & & & & & \\ b_4 \downarrow & & b_3 \downarrow & & b_2 \downarrow & & b_1 \downarrow \\ C_3^\sigma & \xleftarrow{B_2} & C_2^\sigma & \xleftarrow{B_1} & C_1^\sigma & \xleftarrow{B_0} & C_0^\sigma \\ b_3 \downarrow & & b_2 \downarrow & & b_1 \downarrow & & \\ C_2^\sigma & \xleftarrow{B_1} & C_1^\sigma & \xleftarrow{B_0} & C_0^\sigma & & \\ b_2 \downarrow & & b_1 \downarrow & & & & \\ C_1^\sigma & \xleftarrow{B_0} & C_0^\sigma & & & & \\ b_1 \downarrow & & & & & & \\ C_0^\sigma & & & & & & \end{array} \quad (9)$$

The maps b_n and B_n are given by

$$b_n = \sum_{i=0}^n (-1)^i d_{n,i}, \quad B_n = (1 + (-1)^n \tau_{n+1}) s_n N_n, \quad (10)$$

with $N_n = \sum_{j=0}^n (-1)^{nj} \tau_n^j$, and $s_n : C_n^\sigma \rightarrow C_{n+1}^\sigma$ the "extra degeneracy"

$$s_n(a_0, a_1, \dots, a_n) = (1, a_0, a_1, \dots, a_n) \quad (11)$$

We calculate $HC_*^\sigma(\mathcal{A})$ via the spectral sequence associated to the mixed complex. Let $HH_*^\sigma(\mathcal{A})$ denote the entries of its first page, that is, $HH_n^\sigma(\mathcal{A}) := H_n(C_*^\sigma, b_*)$ (the homologies of the columns). For $\sigma = \text{id}$ these are the Hochschild homologies $HH_*(\mathcal{A}) = H_*(\mathcal{A}, \mathcal{A})$. Hence we call $HH_*^\sigma(\mathcal{A})$ as in [12] the σ -twisted Hochschild homology of \mathcal{A} .

To compute $HH_*^\sigma(\mathcal{A})$ consider the mixed complex (9) with C_n^σ replaced by the original C_n . This is not a bicomplex: the commutation relations in a paracyclic object imply that the (lifts of the) operators b_* and B_* anticommute according to (see [6], Theorem 2.3)

$$b_{n+1} B_n + B_{n-1} b_n = \text{id} - T_n. \quad (12)$$

But the columns form the complex (C_*, b_*) which computes the Hochschild homology $H_*(\mathcal{A}, \sigma\mathcal{A})$ of \mathcal{A} with coefficients in the bimodule $\sigma\mathcal{A}$ which is \mathcal{A} as a vector space with bimodule structure

$$a \triangleright b \triangleleft c := \sigma(a)bc \quad (13)$$

In many cases $C_n = C_n^0 \oplus C_n^1$, $C_n^0 := \ker(\text{id} - T_n)$, for example when σ is diagonalizable. In this case, $T_n = \sigma^{\otimes(n+1)}$ is also diagonalizable, and C_n^0 and C_n^1 are the eigenspace of T_n corresponding to the eigenvalue 1 and the direct sum of all other eigenspaces, respectively. Then:

PROPOSITION 2.1 *If $C_n = C_n^0 \oplus C_n^1$, then $H_*(\mathcal{A}, \sigma\mathcal{A}) \cong HH_*^\sigma(\mathcal{A})$.*

Proof. Note that (12) implies that b_* commutes with $\text{id} - T_*$, so the decomposition $C_n = C_n^0 \oplus C_n^1$ defines a decomposition of complexes, and we can identify $HH_*^\sigma(\mathcal{A})$ with the homologies of the subcomplex $(C_*^0, b_*) \subset (C_*, b_*)$. Hence $H_*(\mathcal{A}, \sigma\mathcal{A})$ is the direct sum of $HH_*^\sigma(\mathcal{A})$ and the homologies of (C_*^1, b_*) . But $(\text{id} - T_n)|_{C_n^1}$ is a bijection under these assumptions, and we have on C_n^1 again by (12) the relation

$$b_{n+1}(1 - T_n)^{-1}B_n + (1 - T_n)^{-1}B_{n-1}b_n = \text{id}.$$

So $(\text{id} - T_n)^{-1}B_*$ is a contracting homotopy for (C_*^1, b_*) and the claim follows. \square

This will allow us to calculate $HH_*^\sigma(\mathcal{A})$ using standard techniques of homological algebra.

The spectral sequence calculation is most efficiently done by passing to the normalized mixed complex (see for example [23], Application 9.8.4). This leaves the first page unchanged. The second step is to calculate the horizontal homology of the rows relative to the maps B_n which in the normalized complex are given explicitly by

$$B_n(a_0, \dots, a_n) = \sum_{i=0}^n (-1)^{ni} (1, \sigma(a_i), \dots, \sigma(a_n), a_0, \dots, a_{i-1}). \quad (14)$$

For quantum $SL(2)$, we find that everything stabilises at the second page, and we can then read off the twisted cyclic homology.

For later use we note that by using the Hochschild-Kostant-Rosenberg theorem applied to an appropriate subalgebra, we obtain:

LEMMA 2.2 *If x, y are commuting elements of \mathcal{A} , with $\sigma(x) = x$, $\sigma(y) = y$, then for any $s, t \geq 0$ we have*

$$B_0[x^s y^t] = t[(x^s y^{t-1}, y)] + s[(x^{s-1} y^t, x)] \in HH_1^\sigma(\mathcal{A})$$

From now on, we will drop the suffices and write b_n as b .

2.3 HOCHSCHILD HOMOLOGY OF HOPF ALGEBRAS

For arbitrary algebras, the Hochschild homologies are derived functors in the category of $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$ -modules, and working with explicit resolutions usually involves lengthy calculations. But if \mathcal{A} is a Hopf algebra then we can describe $H_*(\mathcal{A}, \mathcal{M})$ for an arbitrary \mathcal{A} -bimodule \mathcal{M} as a derived functor in the category of \mathcal{A} -modules. Define a right \mathcal{A} -module \mathcal{M}' which is \mathcal{M} as a vector space with right action given by

$$m \blacktriangleleft a := \sum S(a_{(2)}) \triangleright m \triangleleft a_{(1)}, \quad a \in \mathcal{A}, m \in \mathcal{M}. \quad (15)$$

Consider k as the trivial \mathcal{A} -module $\mathcal{A}/\ker \varepsilon$. Feng and Tsygan proved:

PROPOSITION 2.3 [5] *There is an isomorphism of vector spaces*

$$H_n(\mathcal{A}, \mathcal{M}) \simeq \text{Tor}_n^{\mathcal{A}}(\mathcal{M}', k).$$

Proof. The $\text{Tor}_n^{\mathcal{A}}(\mathcal{M}', k)$ are computed from the complex (C_*, d) (with zeroth tensor component now being \mathcal{M}') with boundary map d given by

$$d = \tilde{d}_0 + \sum_{i=1}^{n-1} (-1)^i d_i + (-1)^n \tilde{d}_n, \quad (16)$$

where the d_i are as above and

$$\begin{aligned}\tilde{d}_0(a_0, a_1, \dots, a_n) &:= (a_0 \blacktriangleleft a_1, a_2, \dots, a_n), \\ \tilde{d}_n(a_0, a_1, \dots, a_n) &:= (\varepsilon(a_n)a_0, a_1, \dots, a_{n-1}).\end{aligned}\quad (17)$$

We define two linear maps $\xi, \xi' : C_n \rightarrow C_n$ by

$$\begin{aligned}\xi(a_0, a_1, \dots, a_n) &:= (S((a_1 \dots a_n)_{(2)}) \triangleright a_0, (a_1)_{(1)}, \dots, (a_n)_{(1)}) \\ \xi'(a_0, \dots, a_n) &:= ((a_1 \dots a_n)_{(2)}) \triangleright a_0, (a_1)_{(1)}, \dots, (a_n)_{(1)}.\end{aligned}\quad (18)$$

Then $\xi \circ \xi' = \xi' \circ \xi = \text{id}_{C_n}$. It is easily checked that ξ commutes with d_i for $1 \leq i \leq n-1$ and that $\xi \circ \tilde{d}_i = d_i \circ \xi$, $i = 0, n$. Hence $\xi \circ d = b \circ \xi$ and ξ is an isomorphism of complexes of k -vector spaces. \square

Let $\pi : \mathcal{M}' \rightarrow H_0(\mathcal{A}, \mathcal{M})$ be the canonical projection. Then we have $\pi(m \blacktriangleleft a) = \varepsilon(a)\pi(m)$, and if we consider $H_0(\mathcal{A}, \mathcal{M})$ as trivial right \mathcal{A} -module, then $\pi \otimes \text{id}_{\mathcal{A}^{\otimes n}}$ induces a morphism

$$H_n(\mathcal{A}, \mathcal{M}) \rightarrow H_0(\mathcal{A}, \mathcal{M}) \otimes_k \text{Tor}_n^{\mathcal{A}}(k, k).$$

If \mathcal{A} is commutative and $\mathcal{M} = \mathcal{A}$ with the standard bimodule structure, then $H_0(\mathcal{A}, \mathcal{M}) = \mathcal{A}$, π is the identity, and the above map is the isomorphism of the Hochschild-Kostant-Rosenberg theorem. For $\mathcal{M} = {}_\sigma \mathcal{A}$ the map defines a ‘‘classical shadow’’ of twisted Hochschild homology.

2.4 NONCOMMUTATIVE KOSZUL RESOLUTIONS

Propositions 2.1 and 2.3 allow us to compute $HH_*^{\sigma}(\mathcal{A})$ for Hopf algebras \mathcal{A} and diagonalizable σ from a resolution of the trivial \mathcal{A} -module k . In the commutative case, such a resolution can be constructed in form of a Koszul complex associated to a minimal set of generators of $\ker \varepsilon$. We will see that one can proceed in the same way for quantum $SL(2)$. The general scheme of the construction of the resolution is as follows.

Let \mathcal{A} be an algebra and d be a positive integer. Let $x_{i,j}$, $1 \leq i, j \leq d$ be elements of \mathcal{A} satisfying

$$x_{i,j}x_{i-1,k} = x_{i,k}x_{i-1,j}. \quad (19)$$

In the commutative case one can take $x_{i,j} = x_{1,j}$, and in many examples the $x_{i,j}$ will be uniquely determined by the $x_{1,j}$.

For $0 \leq n \leq d$, let $K_n(x_{i,j})$ be the \mathcal{A} -bimodule $\mathcal{A}^{\binom{d}{n}}$, which we identify for $n > 0$ with the submodule of $\mathcal{A}^d \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} \mathcal{A}^d$ (n factors) spanned over \mathcal{A} by

$$e_{i_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} e_{i_n}, \quad 1 \leq i_1 < \dots < i_n \leq d,$$

where e_i is a basis of \mathcal{A}^d . For $n > d$ we set $K_n(x_{i,j}) := 0$. For an \mathcal{A} -bimodule \mathcal{N} , set $K_n(x_{i,j}, \mathcal{N}) := K_n(x_{i,j}) \otimes_{\mathcal{A}} \mathcal{N}$ and define \mathcal{A} -module maps

$$k_m : K_n(x_{i,j}, \mathcal{N}) \rightarrow K_{n-1}(x_{i,j}, \mathcal{N}), \quad m = 1, \dots, n$$

(we suppress the index n at the k_m) by

$$k_m : e_{i_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} e_{i_n} \otimes_{\mathcal{A}} y \mapsto e_{i_1} \otimes_{\mathcal{A}} \dots \hat{e}_{i_m} \dots \otimes_{\mathcal{A}} e_{i_n} \otimes_{\mathcal{A}} y \triangleleft x_{n, i_m}.$$

Then for $r < s$:

$$\begin{aligned}& (k_r k_s - k_{s-1} k_r)(e_{i_1} \otimes_{\mathcal{A}} \dots \otimes_{\mathcal{A}} e_{i_n} \otimes_{\mathcal{A}} y) \\ &= e_{i_1} \otimes_{\mathcal{A}} \dots \hat{e}_{i_r} \dots \hat{e}_{i_s} \dots \otimes_{\mathcal{A}} e_{i_n} \otimes_{\mathcal{A}} y \triangleleft (x_{n, i_s} x_{n-1, i_r} - x_{n, i_r} x_{n-1, i_s}).\end{aligned}$$

The last bracket vanishes by (19). Thus we get

PROPOSITION 2.4 *The map $k := \sum_{r=1}^n (-1)^r k_r$ makes $K_*(x_{i,j}, \mathcal{N})$ into a complex which we call the Koszul complex associated to $x_{i,j}$ and \mathcal{N} .*

The zeroth homology of this complex is obviously the quotient of \mathcal{N} by the submodule generated by all elements of the form $y \triangleleft x_{1,i}$, $y \in \mathcal{N}$, $i = 1, \dots, d$. The classical application of Koszul complexes is to produce resolutions of this quotient, but the Koszul complex is not always acyclic (see [19] for the commutative case). In our application we will take $\mathcal{N} = \mathcal{A}$ to be a Hopf algebra with the standard bimodule structure, and the $x_{1,j}$ ($1 \leq j \leq d$) will generate $\ker \varepsilon$ as a (left or right) \mathcal{A} -module. The associated Koszul complex will be checked by hand to be acyclic (see Proposition 4.1 below), so it provides a resolution of $\mathcal{A}/\ker \varepsilon$, and $\text{Tor}_*^{\mathcal{A}}(\mathcal{M}', k)$ equals the homologies of the complex $\mathcal{M}' \otimes_{\mathcal{A}} K_*(x_{i,j}, \mathcal{A})$. The quasi-isomorphism from this complex to the standard complex (C_*, d) calculating the Tor-groups described in the proof of Proposition 2.3 is then given by

$$e_{i_1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} e_{i_n} \mapsto x_{1,i_1} \wedge \cdots \wedge x_{1,i_n} := \sum_{s \in S_n} (-1)^{|s|} x_{n,i_{s(n)}} \otimes \cdots \otimes x_{1,i_{s(1)}}. \quad (20)$$

3 Quantum $SL(2)$

In this section, we introduce the main facts on the standard quantized coordinate ring $\mathcal{A} = \mathcal{A}(SL_q(2))$ that will be used below.

3.1 THE HOPF ALGEBRA $\mathcal{A}(SL_q(2))$

Let k be a field of characteristic zero, and $q \in k$ some nonzero parameter, which we assume is not a root of unity. The coordinate algebra $\mathcal{A} = \mathcal{A}(SL_q(2))$ of the quantum group $SL_q(2)$ over k is the k -algebra generated by symbols a, b, c, d with relations

$$\begin{aligned} ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb \\ ad - qbc = 1, \quad da - q^{-1}bc = 1 \end{aligned} \quad (21)$$

There is a unique Hopf algebra structure on \mathcal{A} such that

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \\ \varepsilon(a) &= \varepsilon(d) = 1, & \varepsilon(b) &= \varepsilon(c) = 0, \\ S(a) &= d, & S(b) &= -q^{-1}b, & S(c) &= -qc, & S(d) &= a. \end{aligned} \quad (22)$$

A vector space basis of \mathcal{A} is given by the monomials

$$e_{i,j,k} := a^i b^j c^k, \quad i \in \mathbb{Z}, j, k \in \mathbb{N}, \quad a^i := d^{-i} \quad \text{for } i < 0, \quad (23)$$

(we use the convention that $x^0 = 1$, for $x \in \mathcal{A}$, $x \neq 0$). We have

$$e_{i,j,k} e_{l,m,n} = q^{-l(j+k)} e_{i+l,j+m,k+n} + \sum_{r>0} \lambda_{i,j,k,l,m,n}(r) e_{i+l,j+m+r,k+n+r}$$

for some constants $\lambda_{i,j,k,l,m,n}(r)$. It follows that \mathcal{A} admits a \mathbb{Z} -grading and three separating decreasing \mathbb{N} -filtrations

$$\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i^a, \quad \mathcal{A} = \mathcal{A}_0^x \supset \mathcal{A}_1^x \supset \dots, \quad x = b, c, bc, \quad (24)$$

where $\mathcal{A}_i^q = \text{span}\{e_{i,j,k}\}_{j,k}$ and \mathcal{A}_n^x is the span of $e_{i,j,k}$ with $j, k, j+k \geq n$ for $x = b, c, bc$, respectively. For $x \in \mathcal{A}$, let x_i be its component in \mathcal{A}_i^q . Set $\mathcal{A}_{i,n}^x := \mathcal{A}_n^x \cap \mathcal{A}_i^q$. Then $\mathcal{A}_{i,n}^x \mathcal{A}_{j,m}^x = \mathcal{A}_{i+j,n+m}^x$. Define a Hermitian inner product on \mathcal{A} by requiring that $e_{i,j,k}$ are orthonormal and let $\pi_x, \pi_{i,j,k}$, denote the orthogonal projections onto $(\mathcal{A}_1^x)^\perp, e_{i,j,k}$. We freely consider $\pi_{i,j,k}$ as a map $\mathcal{A} \rightarrow k$. Note that $\pi_x(y)_i = \pi_x(y_i)$ for all $y \in \mathcal{A}$.

Finally, \mathcal{A} has a \mathbb{Z}^2 -grading given by

$$\text{deg}(e_{i,j,k}) = (i, j - k) \quad (25)$$

This grading extends to $\mathcal{A}^{\otimes(n+1)}$ and is preserved by the Hochschild boundary and the maps B_n (14). Hence $HH_*^\sigma(\mathcal{A})$ and $HC_*^\sigma(\mathcal{A})$ are naturally \mathbb{Z}^2 -graded.

3.2 THE HAAR FUNCTIONAL

The Hopf algebra \mathcal{A} is cosemisimple [11], that is, there is a unique linear functional $h : \mathcal{A} \rightarrow k$ satisfying $h(1) = 1$ and

$$(h \otimes \text{id})\Delta(x) = h(x)1 = (\text{id} \otimes h)\Delta(x) \quad \forall x \in \mathcal{A} \quad (26)$$

If $k = \mathbb{C}$ and $q \in \mathbb{R}$, then \mathcal{A} can be made into a Hopf *-algebra whose C*-algebraic completion is Woronowicz's quantum $SU(2)$ group [24]. The functional h extends to the Haar state of this compact quantum group. Hence (with slight abuse of terminology) we also in the general case call h the Haar functional of \mathcal{A} . For any $x, y \in \mathcal{A}$, we have

$$h(xy) = h(y\sigma_{mod}(x)) \quad (27)$$

where σ_{mod} is the so-called modular automorphism of \mathcal{A} given by

$$\sigma_{mod}(a) = q^{-2}a, \quad \sigma_{mod}(d) = q^2d, \quad \sigma_{mod}(b) = b, \quad \sigma_{mod}(c) = c \quad (28)$$

So h is a σ_{mod}^{-1} -twisted cyclic 0-cocycle.

3.3 THE AUTOMORPHISM GROUP OF $\mathcal{A}(SL_q(2))$

For $\lambda, \mu \in k^\times$ there are unique automorphisms $\sigma_{\lambda,\mu}, \tau_{\lambda,\mu}$ of \mathcal{A} with

$$\begin{aligned} \sigma_{\lambda,\mu}(a) &= \lambda a, & \sigma_{\lambda,\mu}(b) &= \mu b, & \sigma_{\lambda,\mu}(c) &= \mu^{-1}c, & \sigma_{\lambda,\mu}(d) &= \lambda^{-1}d, \\ \tau_{\lambda,\mu}(a) &= \lambda a, & \tau_{\lambda,\mu}(b) &= \mu^{-1}c, & \tau_{\lambda,\mu}(c) &= \mu b, & \tau_{\lambda,\mu}(d) &= \lambda^{-1}d. \end{aligned} \quad (29)$$

It is easy to check that this list is complete, although we do not know a reference where this was pointed out explicitly:

PROPOSITION 3.1 *If σ is an automorphism of $\mathcal{A}(SL_q(2))$, then either $\sigma = \sigma_{\lambda,\mu}$ or $\sigma = \tau_{\lambda,\mu}$ for some λ, μ .*

Proof. Using the \mathbb{Z} -grading and the \mathbb{N} -filtrations mentioned above it is a straightforward calculation to check that up to rescaling and exchanging b and c the original generators are the only elements of the algebra that fulfill the defining relations. \square

The $\sigma_{\lambda,\mu}$ act diagonally with respect to the generators a, b, c, d . The $\tau_{\lambda,\mu}$ are also diagonalizable. For fixed λ, μ define $x_\pm = c \pm \mu b$. Then $\tau_{\lambda,\mu}(x_\pm) = \pm x_\pm$, and a, x_+, x_-, d generate \mathcal{A} . So by Proposition 2.1:

COROLLARY 3.2 *For $\mathcal{A} = \mathcal{A}(SL_q(2))$, and for each n and every automorphism σ , we have $HH_n^\sigma(\mathcal{A}) \cong H_n(\mathcal{A}, \sigma\mathcal{A})$.*

4 Twisted Hochschild homology of $\mathcal{A}(SL_q(2))$

4.1 A KOSZUL RESOLUTION OF $\mathcal{A}/\ker \varepsilon$

Using the above facts it is easy to see that $\ker \varepsilon$ is generated as both a left and right \mathcal{A} -module by $x_{1,1} := a - 1$, $x_{1,2} := b$, $x_{1,3} := c$. For these elements there exists a Koszul complex (K_*, k) , $K_n := K_n(x_{i,j}, \mathcal{A})$, in the sense of section 2.4 with the $x_{i,j}$ given by

$$\begin{pmatrix} a-1 & b & c \\ q^{-1}a-1 & b & c \\ q^{-2}a-1 & b & c \end{pmatrix}. \quad (30)$$

We check by explicit calculation that this Koszul complex is acyclic:

PROPOSITION 4.1 *The left \mathcal{A} -module $\mathcal{A}/\ker \varepsilon$ possesses a resolution (K_*, k) of the form*

$$0 \rightarrow \mathcal{A} \xrightarrow{k_3} \mathcal{A}^3 \xrightarrow{k_2} \mathcal{A}^3 \xrightarrow{k_1} \mathcal{A} \rightarrow \mathcal{A}/\ker \varepsilon \rightarrow 0.$$

The augmentation map $K_0 = \mathcal{A} \rightarrow k = \mathcal{A}/\ker \varepsilon$ is given by the counit ε . The left \mathcal{A} -module morphisms $k_n : K_n \rightarrow K_{n-1}$, $n = 1, 2, 3$, are given by

$$k_1 : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto a-1, b, c$$

$$k_2 : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} b \\ 1-q^{-1}a \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \\ 1-q^{-1}a \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix},$$

$$k_3 : 1 \mapsto \begin{pmatrix} c \\ -b \\ q^{-2}a-1 \end{pmatrix}.$$

Proof. It follows from Proposition 2.4 (or directly) that this is a complex. Let $(x, y, z)^t \in \ker(k_1)$, i.e. $x(a-1) + yb + zc = 0$. Then $\pi_{bc}(x(a-1)) = 0$. Using the \mathbb{Z} -grading we have $\pi_{bc}(x) = 0$, so $x = x'b + x''c$. Subtracting $k_2(x', x'', 0)^t$ from $(x, y, z)^t$ we get a new element of $\ker(k_1)$ with $x = 0$. Hence $\pi_c(y) = \pi_b(z) = 0$, so $y = y'c$, $z = z'b$, $z' = -y'$ and this element is a multiple of $k_2(0, 0, 1)^t$. In a similar manner

$$x \begin{pmatrix} b \\ 1-q^{-1}a \\ 0 \end{pmatrix} + y \begin{pmatrix} c \\ 0 \\ 1-q^{-1}a \end{pmatrix} + z \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix} = 0$$

implies $x = x'c$, $y = -x'b$, $x'(q^{-2}a - 1) = z$ for some $x' \in \mathcal{A}$. \square

COROLLARY 4.2 *If $n > 3$, then $HH_n^\sigma(\mathcal{A}) = 0$ for all automorphisms σ .*

The morphism between the resulting short complex ${}_\sigma \mathcal{A}' \otimes_{\mathcal{A}} K_*$ and the standard complex for $\text{Tor}_n^{\mathcal{A}}({}_\sigma \mathcal{A}', k)$ yielding an isomorphism in homology is given explicitly by:

1. The map $\varphi_0 : \mathcal{A} \rightarrow C_0 = \mathcal{A}$ is the identity.

2. The map $\varphi_1 : \mathcal{A}^3 \rightarrow C_1 = \mathcal{A}^{\otimes 2}$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (x, a-1) + (y, b) + (z, c). \quad (31)$$

Since $d(x, 1, 1) = (x, 1)$ for any x , we have $[(x, a-1)] = [(x, a)]$ in $\text{Tor}_1^{\mathcal{A}}(\sigma\mathcal{A}', k)$.

3. The map $\varphi_2 : \mathcal{A}^3 \rightarrow C_2 = \mathcal{A}^{\otimes 3}$ is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (x, b, a-1) - (x, q^{-1}a-1, b) + \\ + (y, c, a-1) - (y, q^{-1}a-1, c) + (z, c, b) - (z, b, c) \quad (32)$$

4. Finally, the map $\varphi_3 : \mathcal{A} \rightarrow C_3 = \mathcal{A}^{\otimes 4}$ in the complex for the Tor-groups is given by $x \mapsto x \otimes v$, where

$$v = -(q^{-2}a-1, b, c) + (q^{-2}a-1, c, b) - (c, q^{-1}a-1, b) + \\ + (c, b, a-1) - (b, c, a-1) + (b, q^{-1}a-1, c). \quad (33)$$

One sees by direct computation that this is a morphism of complexes, and by the comparison theorem (see [23], Theorem 2.2.6) this is a quasi-isomorphism.

4.2 COMPUTATION OF $HH_n^\sigma(\mathcal{A})$, $n \leq 3$

All automorphisms arising from finite-dimensional calculi are of the form $\sigma = \sigma_{\lambda, \mu}$, and from now on we will only consider automorphisms of this type. In fact, they are of the form $\sigma(x) = \sigma_{mod}^{-1}(f * x)$, where f is a functional in the dual Hopf algebra \mathcal{A}° acting on x by $f * x = \sum f(x_{(2)})x_{(1)}$ (see Theorems 4.1, 4.3 and 4.8 in [12]). It is clear that such automorphisms do not exchange b and c . By Corollary 3.2 we have $HH_n^\sigma(\mathcal{A}) \cong H_n(\mathcal{A}, \sigma\mathcal{A})$, and the homologies $H_*(\mathcal{A}, \sigma\mathcal{A})$ can be calculated via our noncommutative Koszul resolution.

So let $\lambda, \mu \in k^\times$ and $\sigma = \sigma_{\lambda, \mu}$. We apply $\sigma\mathcal{A}' \otimes_{\mathcal{A}} \cdot$ to our resolution and obtain the complex (F_*, f) of vector spaces

$$0 \rightarrow \mathcal{A} \xrightarrow{f_3} \mathcal{A}^3 \xrightarrow{f_2} \mathcal{A}^3 \xrightarrow{f_1} \mathcal{A} \rightarrow 0, \quad (34)$$

with morphisms f_n given by

$$f_1 \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = x_1 \blacktriangleleft a - x_1 + y_1 \blacktriangleleft b + z_1 \blacktriangleleft c, \\ f_2 \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_2 \blacktriangleleft b + y_2 \blacktriangleleft c \\ x_2 - q^{-1}x_2 \blacktriangleleft a + z_2 \blacktriangleleft c \\ y_2 - q^{-1}y_2 \blacktriangleleft a - z_2 \blacktriangleleft b \end{pmatrix}, \\ f_3(x_3) = \begin{pmatrix} x_3 \blacktriangleleft c \\ -x_3 \blacktriangleleft b \\ q^{-2}x_3 \blacktriangleleft a - x_3 \end{pmatrix}. \quad (35)$$

Writing $\varepsilon_{i,j,k} := q^{i+j+k+2}\lambda\mu^{-1}$ we have

$$\begin{aligned}
\lambda q^{j+k} e_{i,j,k} \triangleleft a &= e_{i,j,k} + q^{-1-i-|i|} (1 - \varepsilon_{|i|,j,k}) e_{i,j+1,k+1}, \\
\lambda^{-1} e_{i,j,k} \triangleleft b &= (1 - \varepsilon_{i,j,k}^{-1}) e_{i+1,j+1,k} \\
&\quad + \begin{cases} 0 & : i \geq 0 \\ q^{-2i-1} (1 - \varepsilon_{-i,j,k}^{-1}) e_{i+1,j+2,k+1} & : i < 0 \end{cases}, \\
\lambda e_{i,j,k} \triangleleft c &= (1 - \varepsilon_{-i,j,k}) e_{i-1,j,k+1} \\
&\quad + \begin{cases} q^{-2i+1} (1 - \varepsilon_{i,j,k}) e_{i-1,j+1,k+2} & : i > 0 \\ 0 & : i \leq 0 \end{cases} \quad (36)
\end{aligned}$$

For $q = \lambda = \mu = 1$ we have $f_n = 0$ and we recover the Hochschild-Kostant-Rosenberg theorem for $SL(2, k)$. The cotangent bundle of an algebraic group is trivial, so in the classical case $HH_n(\mathcal{A}) = \mathcal{A} \otimes \Lambda^n k^3$. It is clear, however, that for $q \neq 1$ there is no twisting automorphism for which this happens.

The calculations lead to five distinct cases:

1. $\mu = 1$, $\lambda \notin \{q^{-(N+2)}\}_{N \geq 0}$, and $\mu \neq 1$, $\lambda = 1$.
2. $\mu = 1$, $\lambda = q^{-(N+2)}$, $N \geq 0$.
3. $\mu = q^{M+1}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$.
4. $\mu = q^{-(M+1)}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$.
5. $\mu = q^{\pm(M+1)}$, $M \geq 0$, $\lambda \notin q^{-\mathbb{N}}$, and $\mu \notin q^{\mathbb{Z}}$, $\lambda \neq 1$.

The computation of $HH_0^\sigma(\mathcal{A})$ and $HH_1^\sigma(\mathcal{A})$ is done most easily “by hand” using the original Hochschild complex, but for $HH_2^\sigma(\mathcal{A})$ and $HH_3^\sigma(\mathcal{A})$ the calculations are done via the Koszul resolution.

4.3 $HH_0^\sigma(\mathcal{A})$

We calculate from first principles the twisted Hochschild homology $HH_0^\sigma(\mathcal{A})$ for all automorphism $\sigma = \sigma_{\lambda,\mu}$. We start with the observation that:

$$b(a_1, a_2 a_3) = b(a_1 a_2, a_3) + b(\sigma(a_3) a_1, a_2) \quad \forall a_1, a_2, a_3 \in \mathcal{A}$$

So for any $a_1, a_2 \in \mathcal{A}$, there exist $x_a, x_b, x_c, x_d \in \mathcal{A}$ such that

$$b(a_1, a_2) = b[(x_a, a) + (x_b, b) + (x_c, c) + (x_d, d)]$$

Hence the image of the twisted Hochschild boundary is spanned by

$$\begin{aligned}
A_{i,j,k} &:= e_{i,j,k} a - \lambda a e_{i,j,k} \\
&= (q^{-(j+k)} - \lambda) e_{i+1,j,k} \\
&\quad + \begin{cases} 0 & : i \geq 0 \\ (q^{-(j+k+1)} - \lambda q^{-2i-1}) e_{i+1,j+1,k+1} & : i < 0 \end{cases} \\
B_{i,j,k} &:= e_{i,j,k} b - \mu b e_{i,j,k} = (1 - \mu q^{-i}) e_{i,j+1,k}, \\
C_{i,j,k} &:= e_{i,j,k} c - \mu^{-1} c e_{i,j,k} = (1 - \mu^{-1} q^{-i}) e_{i,j,k+1}, \\
D_{i,j,k} &:= e_{i,j,k} d - \lambda^{-1} d e_{i,j,k} \\
&= (q^{j+k} - \lambda^{-1}) e_{i-1,j,k} \\
&\quad + \begin{cases} 0 & : i \leq 0 \\ (q^{j+k+1} - \lambda^{-1} q^{-2i+1}) e_{i-1,j+1,k+1} & : i > 0 \end{cases}.
\end{aligned}$$

For given (i, j, k) , the elements $B_{i,j,k}$ and $C_{i,j,k}$ both vanish if and only if $i = 0$ and $\mu = 1$. Therefore, for all λ, μ , $\text{im } b$ contains the basis elements

$$e_{i,j,k}, \quad i \neq 0, j, k > 0. \quad (37)$$

Omitting the span of these terms from the above list of generators we see that $\text{im } b$ is spanned by (37) together with

$$\begin{aligned} A_{-1,j,k} &= (q^{-(j+k)} - \lambda)e_{0,j,k} + (q^{-(j+k+1)} - \lambda q)e_{0,j+1,k+1} \\ \tilde{A}_{i,j,k} &= (q^{-(j+k)} - \lambda)e_{i+1,j,k}, \quad i \neq -1 \\ B_{i,j,0} &= (1 - \mu q^{-i})e_{i,j+1,0}, \quad B_{0,j,k} = (1 - \mu)e_{0,j+1,k}, \\ C_{i,0,k} &= (1 - \mu^{-1}q^{-i})e_{i,0,k+1}, \quad C_{0,j,k} = (1 - \mu^{-1})e_{0,j,k+1}, \\ D_{1,j,k} &= (q^{j+k} - \lambda^{-1})e_{0,j,k} + (q^{j+k+1} - \lambda^{-1}q^{-1})e_{0,j+1,k+1} \\ \tilde{D}_{i,j,k} &= (q^{j+k} - \lambda^{-1})e_{i-1,j,k}, \quad i \neq 1. \end{aligned}$$

Since $\tilde{D}_{i+2,j,k}$ is proportional to $\tilde{A}_{i,j,k}$ and both vanish if and only if $\lambda = q^{-(j+k)}$, we can omit $\tilde{D}_{i,j,k}$ from this list. We also have

$$A_{-1,j,k} = -\lambda q^{-(j+k)} D_{1,j,k},$$

so the $D_{1,j,k}$ can be omitted as well. Finally, $C_{0,j,k}$ is for $j > 0$ a nonzero multiple of $B_{0,j-1,k+1}$ and can be omitted. Thus the degree 0 part (with respect to the \mathbb{Z} -grading) of $\text{im } b$ is spanned by

$$\begin{aligned} (1 - \lambda q^{j+k})e_{0,j,k} + (q^{-1} - \lambda q^{j+k+1})e_{0,j+1,k+1}, \\ (1 - \mu)e_{0,r,s}, \quad r + s > 0 \end{aligned}$$

and the nonzero degrees by (37) together with

$$(1 - \lambda q^{j+k})e_{i,j,k}, \quad (1 - \mu q^{-i})e_{i,j+1,0}, \quad (1 - \mu^{-1}q^{-i})e_{i,0,k+1}.$$

where $i \neq 0$, and $j, k \geq 0$. Dually,

$$HH_\sigma^0(\mathcal{A}) = \{ \text{linear } h : \mathcal{A} \rightarrow k : h(a_1 a_2) = h(\sigma(a_2) a_1) \}$$

For $\lambda \notin q^{-\mathbb{N}}$ we have $h(e_{i,j,k}) = 0$ for $i \neq 0$, and

$$h(b^j c^k) = \begin{cases} (-q)^{-k} \frac{f(j-k)}{f(j+k)} h(b^{j-k}) : & j \geq k \\ (-q)^{-j} \frac{f(k-j)}{f(j+k)} h(c^{k-j}) : & j \leq k \end{cases} \quad (38)$$

where $f(n) = \lambda - q^{-n}$.

We now present the generating twisted 0-cycles, together with dual twisted 0-cycles. Our calculations now break down into five cases:

Case 1: $\mu = 1, \lambda \notin \{q^{-(N+2)}\}_{N \geq 0}$ and $\mu \neq 1, \lambda = 1$. Then

$$HH_0^\sigma(\mathcal{A}) = k[1] \oplus \bigoplus_{x \in \{a,b,c,d\}, \sigma(x)=x} \left(\sum_{r \geq 0}^{\oplus} k[x^{r+1}] \right) \quad (39)$$

For $\mu = 1 = \lambda$ (i.e. $\sigma = \text{id}$) this agrees with [15]. The dual σ -twisted 0-cycles are defined on basis elements $x = e_{i,j,k}$ with $\sigma(x) = x$ as follows:

$$h_{[1]}(x) = \begin{cases} 1 & : x = 1 \\ (-q)^{s+1} \frac{f(0)}{f(2j+2)} & : x = (bc)^{j+1} \\ 0 & : \text{otherwise} \end{cases} \quad (40)$$

(if $\lambda = 1$, obviously $f(0) = 0$). For $y = [a^{r+1}]$, $[d^{r+1}]$ define

$$h_{[y]}(x) = \begin{cases} 1 & : x = y \\ 0 & : \text{otherwise} \end{cases} \quad (41)$$

For $y = [b^{s+1}]$, $[c^{t+1}]$ define

$$h_{[b^{s+1}]}(x) = \begin{cases} (-q)^k \frac{f(s+1)}{f(s+1+2k)} & : x = b^{s+1}(bc)^k \\ 0 & : \text{otherwise} \end{cases} \quad (42)$$

$$h_{[c^{t+1}]}(x) = \begin{cases} (-q)^j \frac{f(t+1)}{f(t+1+2j)} & : x = (bc)^j c^{t+1} \\ 0 & : \text{otherwise} \end{cases} \quad (43)$$

These all satisfy (38). For any $[x]$, $[y]$ in (39), we have $h_{[y]}(x) = \delta_{[x],[y]}$, so the 0-cycles given in (39) are linearly independent, hence a basis.

Case 2: $\mu = 1$, $\lambda = q^{-(N+2)}$, $N \geq 0$. We have

$$HH_0^\sigma(\mathcal{A}) = \left(\sum_{s \in S} k[b^s] \right) \oplus \left(\sum_{t \in S} k[c^t] \right) \oplus \left(\sum_{0 \leq i \leq N+2} k[b^i c^{N+2-i}] \right) \quad (44)$$

where $S = \{\text{integers} \geq N+3\} \cup \{N+1, N-1, N-3, \dots \geq 0\}$, with the convention that if $0 \in S$, we include only one copy of $k[1]$. Dual 0-cocycles are $h_{[y]}$, defined for $[y] = [b^i c^{N+2-i}]$ on the basis $e_{i,j,k}$ by

$$h_{[y]}(x) = \begin{cases} 1 & : x = y \\ 0 & : \text{otherwise} \end{cases} \quad (45)$$

and for $[y] = [b^s]$, $[c^t]$, $s, t \in S$ by

$$h_{[b^s]}(x) = \begin{cases} (-q)^k \frac{f(s)}{f(s+2k)} & : x = b^s (bc)^k \\ 0 & : \text{otherwise} \end{cases}$$

$$h_{[c^t]}(x) = \begin{cases} (-q)^j \frac{f(t)}{f(t+2j)} & : x = (bc)^j c^t \\ 0 & : \text{otherwise} \end{cases} \quad (46)$$

So for each pair $[x]$, $[y]$ appearing in (44), we have $h_{[y]}(x) = \delta_{[x],[y]}$.

Case 3: $\mu = q^{M+1}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$.

$$HH_0^\sigma(\mathcal{A}) \cong k^2 = k[d^{M+1}c^{N+1}] \oplus k[a^{M+1}b^{N+1}] \quad (47)$$

Also $HH_\sigma^0(\mathcal{A}) \cong k^2$, with basis the twisted 0-cocycles $h_{[y]}$, $[y] = [d^{M+1}c^{N+1}]$, $[a^{M+1}b^{N+1}]$, defined on elements $x = e_{i,j,k}$, with $\sigma(x) = x$ by

$$h_{[y]}(x) = \begin{cases} 1 & : x = y \\ 0 & : \text{otherwise} \end{cases} \quad (48)$$

Case 4: $\mu = q^{-(M+1)}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$. We have

$$HH_0^\sigma(\mathcal{A}) \cong k^2 = k[d^{M+1}b^{N+1}] \oplus k[a^{M+1}c^{N+1}] \quad (49)$$

with $HH_\sigma^0(\mathcal{A}) \cong k^2$ with basis $h_{[y]}$, $[y] = [d^{M+1}b^{N+1}]$, $[a^{M+1}c^{N+1}]$, defined as in (48).

Case 5: $\mu = q^{\pm(M+1)}$, $M \geq 0$, $\lambda \notin q^{-\mathbb{N}}$, and $\mu \notin q^{\mathbb{Z}}$, $\lambda \neq 1$. Then

$$HH_0^\sigma(\mathcal{A}) = 0 = HH_\sigma^0(\mathcal{A})$$

4.4 Twisted cocycles defined by derivations

Before proceeding with $HH_1^\sigma(\mathcal{A})$, we present a general construction of σ -twisted Hochschild n -cocycles. It is essentially a variant of the characteristic map of [4, 10].

Let \mathcal{A} be a k -algebra and σ be an automorphism. A σ -derivation of \mathcal{A} is a k -linear map $\partial : \mathcal{A} \rightarrow \mathcal{A}$, such that $\partial(a_0 a_1) = a_0 \partial(a_1) + \partial(a_0) \sigma(a_1)$. The following is straightforward:

PROPOSITION 4.3 *If h is a σ_1 -twisted 0-cocycle, $\partial_1, \dots, \partial_{n-1}$ derivations of \mathcal{A} , and ∂_n is a σ_2 -derivation, then*

$$\phi_n(a_0, a_1, \dots, a_n) = h(a_0 \partial_1(a_1) \dots \partial_n(a_n)) \quad (50)$$

is a $\sigma_1 \circ \sigma_2$ -twisted Hochschild n -cocycle

In general there is no reason for such a cocycle to represent a nontrivial element of Hochschild cohomology, nor for it also to be cyclic. However:

LEMMA 4.4 *Suppose \mathcal{A} is a unital algebra, h a σ_1 -twisted 0-cocycle and ∂ a σ_2 -derivation of \mathcal{A} . Defining ϕ_1 by $\phi_1(x, y) = h(x\partial(y))$, then ϕ_1 is a $\sigma_1 \circ \sigma_2$ -twisted cyclic cocycle if and only if $h(\partial(a)) = 0$ for all $a \in \mathcal{A}$.*

For $\mathcal{A} = \mathcal{A}(SL_q(2))$ there are obvious derivations ∂_a, ∂_b defined by

$$\begin{aligned} \partial_a(a) &= a, & \partial_a(b) &= 0, & \partial_a(c) &= 0, & \partial_a(d) &= -d \\ \partial_b(a) &= 0, & \partial_b(b) &= b, & \partial_b(c) &= -c, & \partial_b(d) &= 0 \end{aligned} \quad (51)$$

and extended via the Leibniz rule. For any $x \in \mathcal{A}$ define an inner derivation ∂'_x by $\partial'_x(y) = [x, y] = xy - yx$. The following is straightforward:

PROPOSITION 4.5 *The vector space of all derivations of $\mathcal{A}(SL_q(2))$ is spanned by ∂_a, ∂_b together with the inner derivations.*

In the sequel we will use the derivation $\partial_0 = \partial_a + \partial_b$, which satisfies

$$\partial_0(a) = a, \quad \partial_0(b) = b, \quad \partial_0(c) = -c, \quad \partial_0(d) = -d \quad (52)$$

and also the $\sigma_{\lambda,1}$ -derivation defined by

$$\partial(a) = a, \quad \partial(b) = 0 = \partial(c), \quad \partial(d) = -\lambda^{-1}d. \quad (53)$$

4.5 $HH_1^\sigma(\mathcal{A})$

The second twisted Hochschild boundary is given by

$$b(a_1, a_2, a_3) = (a_1 a_2, a_3) - (a_1, a_2 a_3) + (\sigma(a_3) a_1, a_2).$$

In particular (take $a_2 = a_3 = 1$) the image of b contains all elementary tensors of the form $(a_1, 1)$, and the residue classes of

$$(e_{i,j,k}, a), \quad (e_{i,j,k}, b), \quad (e_{i,j,k}, c), \quad (e_{i,j,k}, d)$$

generate $\mathcal{A} \otimes \mathcal{A} / \text{im } b$. Now, $HH_1^\sigma(\mathcal{A})$ is the kernel of the map $\mathcal{A} \otimes \mathcal{A} / \text{im } b \rightarrow \mathcal{A}$ induced by the first twisted Hochschild boundary. This sends the classes of the above elements to $A_{i,j,k}, B_{i,j,k}, C_{i,j,k}, D_{i,j,k}$ from the previous section. It is straightforward to check for triviality and linear dependence.

We now present generators of $HH_1^\sigma(\mathcal{A})$ and dual twisted 1-cocycles. From Proposition 4.3, for any σ -twisted 0-cocycle h and derivation ∂ , defining $\phi_1(x, y) = h(x\partial(y))$ gives a σ -twisted Hochschild 1-cocycle. For

$\mathcal{A}(SL_q(2))$ all automorphisms $\sigma_{\lambda,\mu}$ commute with the derivations ∂_a, ∂_b (51), and the $e_{i,j,k}$ are eigenvectors for these derivations. By Lemma 4.4, ϕ_1 is cyclic if and only if $h \circ \partial = 0$. We take $\partial_0 = \partial_a + \partial_b$ defined by (52).

Case 1: $\mu = 1, \lambda \notin \{q^{-(N+2)}\}_{N \geq 0}$ and $\mu \neq 1, \lambda = 1$. Then

$$HH_1^\sigma(\mathcal{A}) = k[\omega_1] \oplus \bigoplus_{x \in \{a,b,c,d\}, \sigma(x)=x} \left(\sum_{r \geq 0}^{\oplus} k[(x^r, x)] \right) \quad (54)$$

where $\omega_1 = (\mu^{-1} - 1)(d, a) + (q - q^{-1})(b, c)$. We note that, for all μ and for $\lambda \neq q^{-2}$, we have $[(c, b)] = -\mu[(b, c)]$, $[(d, a)] = -\lambda[(a, d)]$. For $\mu = 1 = \lambda$ this is in agreement with [15], apart from the sign change in ω_1 . Now recall the 0-cocycles $h_{[x]}$ defined in (40)-(43). Given such an x , define a Hochschild 1-cocycle $\phi_{[x]}$ by

$$\phi_{[x]}(y, z) = h_{[x]}(y\partial_0(z)) \quad (55)$$

Then the Hochschild 1-cocycles dual to the generators of $HH_1^\sigma(\mathcal{A})$ are:

$$\phi_{[x^{r+1}]} \leftrightarrow (x^r, x), \quad x \in \{a, b, c, d\}, \quad \sigma(x) = x.$$

Dual to ω_1 we have

$$\phi_{\omega_1}(x, y) = h_{[1]}(x\partial_0(y)) \quad (56)$$

with $h_{[1]}$ defined in (40) and ∂_0 defined in (52). Then for $\lambda = 1$ we have $\phi_{\omega_1}(d, a) = 1$, and for $\lambda \neq q^{-2}$ we have $\phi_{\omega_1}(b, c) = \frac{q(1-\lambda)}{\lambda q^2 - 1}$. Since $h_{[1]} \circ \partial_0 = 0$, by Lemma 4.4 ϕ_{ω_1} is in fact a σ -twisted cyclic 1-cocycle.

Case 2. $\mu = 1, \lambda = q^{-(N+2)}, N \geq 0$. Then

$$\begin{aligned} HH_1^\sigma(\mathcal{A}) &= \left(\sum_{s \in S'}^{\oplus} k[(b^s, b)] \right) \oplus \left(\sum_{t \in S'}^{\oplus} k[(c^t, c)] \right) \oplus \\ &\left(\sum_{0 \leq i \leq N}^{\oplus} k[(b^i c^{N+1-i}, b)] \right) \oplus \left(\sum_{0 \leq i \leq N}^{\oplus} k[(b^{i+1} c^{N-i}, c)] \right) \oplus k[\omega_1] \end{aligned} \quad (57)$$

Here $S' = \{\text{integers} \geq N\} \cup \{N-2, N-4, \dots \geq 0\}$, and for $N \geq 1$ we have $[\omega_1] = [(b, c)] = -[(c, b)]$ for N odd, $[\omega_1] = 0$ for N even. For $\lambda = q^{-2}$, $[(b, c)]$ and $[(c, b)]$ are linearly independent.

Recall the 0-cocycles $h_{[x]}$, with $[x] = [b^s], [c^t]$, defined in (42), (43). The dual Hochschild 1-cocycles (55) are

$$\phi_{[b^{s+1}]} \leftrightarrow (b^s, b), \quad \phi_{[c^{t+1}]} \leftrightarrow (c^t, c)$$

together with the twisted cyclic 1-cocycle ϕ_{ω_1} (56) dual to ω_1 . To define twisted 1-cocycles dual to $[(b^i c^{N+1-i}, b)], [(b^{i+1} c^{N-i}, c)]$ we need to work a little harder. It is straightforward to show that any σ -twisted Hochschild 1-cocycle is uniquely defined by its values $\phi(y, t)$, for $t = a, b, c, d$ and basis elements $y = e_{i,j,k}$.

LEMMA 4.6 For $\lambda = q^{-(N+2)}, \mu = 1$ defining ϕ_1, ϕ_2 on basis elements $y = e_{i,j,k}$ by

$$\begin{aligned} \phi_1(b^i c^{N+1-i}, b) &= \beta_{N,i}, & \phi_1(db^i c^{N-i}, a) &= q^{-(N+1)} \beta_{N,i}, \\ \phi_1(y, b) = 0 = \phi_1(y, a) & \text{ otherwise,} & \phi_1(y, c) = 0 = \phi_1(y, d) & \quad \forall y \\ \phi_2(ab^i c^{N-i}, d) &= q^{N+1} \gamma_{N,i}, & \phi_2(b^{i+1} c^{N-i}, c) &= \gamma_{N,i} \\ \phi_2(y, d) = 0 = \phi_2(y, c) & \text{ otherwise,} & \phi_2(y, a) = 0 = \phi_1(y, b) & \quad \forall y \end{aligned}$$

for arbitrary $\beta_{N,i}, \gamma_{N,i}, 0 \leq i \leq N$, gives well-defined σ -twisted Hochschild 1-cocycles.

Setting each $\beta_{N,i}, \gamma_{N,i}$ to 1 in turn, and all others to zero, we see that the twisted Hochschild 1-cocycles $[(b^i c^{N+1-i}, b)], [(b^{i+1} c^{N-i}, c)]$ are nontrivial and linearly independent. This extends to $N+1$ linearly independent twisted cyclic 1-cocycles. Define $\phi = \phi_1 + \phi_2$. Cyclicity requires

$$\phi(xc, b) = -\phi(b, xc) = -i\phi(xc, b) - (N+1-i)\phi(bx, c)$$

where $x = b^i c^{N-i}$. Hence for ϕ to be cyclic, we need

$$(i+1)\beta_{N,i} = -(N+1-i)\gamma_{N,i} \quad 0 \leq i \leq N$$

i.e. $\gamma_{N,i} = \frac{-(i+1)}{N+1-i}\beta_{N,i}$ for $0 \leq i \leq N$. Then:

LEMMA 4.7 For each $0 \leq i \leq N$, defining $\phi_{N,i} = \phi_1 + \phi_2$ with

$$\beta_{N,i} = N+1-i, \quad \gamma_{N,i} = -(i+1)$$

gives a well-defined twisted cyclic 1-cocycle satisfying

$$\begin{aligned} \phi_{N,i}(xc, b) &= N+1-i, & \phi_{N,i}(bx, c) &= -(i+1) \\ \phi_{N,i}(ax, d) &= -q^{N+1}(i+1), & \phi_{N,i}(dx, a) &= q^{-(N+1)}(N+1-i) \end{aligned}$$

for $x = b^i c^{N-i}$, and $\phi_{N,i}(y, t) = 0$ for all basis elements y and $t = a, b, c, d$ otherwise.

Case 3. $\mu = q^{M+1}, \lambda = q^{-(N+1)}, M, N \geq 0$. Then $HH_1^\sigma(\mathcal{A}) \cong k^4$, with basis given by the Hochschild cycles

$$(a^M b^{N+1}, a), \quad (a^{M+1} b^N, b), \quad (d^{M+1} c^N, c), \quad (d^M c^{N+1}, d) \quad (58)$$

The dual basis for $HH_\sigma^1(\mathcal{A}) \cong k^4$ is given by the 1-cocycles $\phi_{[x],t}$, for $[x] = [a^{M+1} b^{N+1}], [d^{M+1} c^{N+1}]$, and $t = a, b$ defined by

$$\phi_{[x],t}(y, z) = h_{[x]}(y\partial_t(z)) \quad (59)$$

where the $h_{[x]}$ were defined in (48) and ∂_a, ∂_b in (51). We have

$$\begin{aligned} \phi_{[a^{M+1} b^{N+1}],a}(a^M b^{N+1}, a) &= q^{-(N+1)} \\ \phi_{[a^{M+1} b^{N+1}],b}(a^{M+1} b^N, b) &= 1 \\ \phi_{[d^{M+1} c^{N+1}],b}(d^{M+1} c^N, c) &= -1 \\ \phi_{[d^{M+1} c^{N+1}],a}(d^M c^{N+1}, d) &= -q^{N+1} \end{aligned}$$

with all other pairings being zero.

Case 4. $\mu = q^{-(M+1)}, \lambda = q^{-(N+1)}, M, N \geq 0$. $HH_1^\sigma(\mathcal{A}) \cong k^4$, with basis given by the Hochschild cycles

$$(a^M c^{N+1}, a), \quad (d^{M+1} b^N, b), \quad (a^{M+1} c^N, c), \quad (d^M b^{N+1}, d) \quad (60)$$

Analogously to Case 3, the dual basis for $HH_\sigma^1(\mathcal{A}) \cong k^4$ is given by the 1-cocycles $\phi_{[x],t}$, (59) for $[x] = [a^{M+1} c^{N+1}], [d^{M+1} b^{N+1}]$, and $t = a, b$.

Case 5. For $\mu = q^{\pm(M+1)}, M \geq 0, \lambda \notin q^{-\mathbb{N}}$, and $\mu \notin q^{\mathbb{Z}}, \lambda \neq 1$, $HH_1^\sigma(\mathcal{A}) = 0$ and $HH_\sigma^1(\mathcal{A}) = 0$.

4.6 $HH_2^\sigma(\mathcal{A})$

We now compute $HH_2^\sigma(\mathcal{A})$. The first step is to describe the kernel of

$$\psi : \mathcal{A}^2 \rightarrow \mathcal{A}, \quad (x, y) \mapsto \psi_+(x) + \psi_-(y)$$

If $\ker \psi_\pm^c \subset \mathcal{A}$ are fixed complements to $\ker \psi_\pm$ and

$$\phi_\pm : \text{im } \psi_\pm \rightarrow \ker \psi_\pm^c$$

are the inverses of $\psi_\pm|_{\ker \psi_\pm^c}$, then the linear map

$$\begin{aligned} \ker \psi_+ \oplus (\text{im } \psi_+ \cap \text{im } \psi_-) \oplus \ker \psi_- &\rightarrow \ker \psi, \\ (x, y, z) &\mapsto (x + \phi_+(y), z - \phi_-(y)) \end{aligned}$$

is an isomorphism of vector spaces. Using (36) one determines a basis of $\text{im } \psi_+ \cap \text{im } \psi_-$ and obtains:

PROPOSITION 4.8 *The set*

$$\begin{aligned} \mathcal{B}_{\ker \psi} &:= \{(x_+, 0), (0, x_-) \mid x_\pm \in \mathcal{B}_{\ker \psi_\pm}\} \\ &\cup \left\{ (-\lambda^{-2} \left(\frac{1 - \varepsilon_{-i,j,k}}{1 - \varepsilon_{i-2,j,k}} e_{i-2,j-1,k+1} - \varepsilon_{-i,j,k+1} e_{i-2,j,k+2} \right), e_{i,j,k}), \right. \\ &\quad \left. (e_{-i,k,j}, -\lambda^2 \left(\frac{1 - \varepsilon_{-i,j,k}}{1 - \varepsilon_{i-2,j,k}} e_{-i+2,k+1,j-1} - \varepsilon_{-i,j,k+1} e_{-i+2,k+2,j} \right)) \mid \right. \\ &\quad \left. i \geq 2, j \geq 1, k \geq 0, \varepsilon_{i-2,j,k} \neq 1 \right\} \\ &\cup \{(\lambda^{-2} q e_{i-2,0,j+2}, e_{i,0,j}), (e_{-i,j,0}, \lambda^2 q^{-1} e_{-i+2,j+2,0}) \mid i \geq 2, j \geq 0, \varepsilon_{-i,0,j} = 1\} \\ &\cup \{(e_{-1,j,k}, \lambda^2 \varepsilon_{-1,j,k}^{-1} e_{1,j+1,k-1}) \mid j \geq 0, k \geq 1\}. \end{aligned}$$

is a vector space basis of $\ker \psi$.

Now we can compute which of these remain nontrivial and linearly independent modulo the image of the map $\varphi : x \mapsto (\psi_-(x), -\psi_+(x))$.

PROPOSITION 4.9 *The classes of*

$$\begin{aligned} &\{(e_{i,j,0}, 0), (0, e_{-i,0,j}) \mid i \geq 0, j \geq 0, \varepsilon_{i,0,j} = 1\} \\ &\cup \{(\lambda^{-2} q e_{i-2,0,k+2}, e_{i,0,k}) \mid i \geq 2, k \geq 0, \varepsilon_{-i,0,k} = 1\} \\ &\cup \{(e_{i,j,0}, \lambda^2 q^{-1} e_{i+2,j+2,0}) \mid i \leq -2, j \geq 0, \varepsilon_{i,j,0} = 1\} \\ &\cup \{(e_{-1,j,k}, \lambda^2 e_{1,j+1,k-1}) \mid j \geq 0, k \geq 1, \varepsilon_{-1,j,k} = 1\} \end{aligned}$$

form a vector space basis of $\ker \psi / \text{im } \varphi$.

Next we check for which linear combinations $(x, y) \neq (0, 0)$ of these there exists z with $(x, y, z) \in \ker f_2$, and determine those z with $(0, 0, z) \in \ker f_2$, giving a generating set for $HH_2^\sigma(\mathcal{A})$:

PROPOSITION 4.10 *The classes of*

$$\begin{aligned} &\{(0, 0, e_{0,j,k}) \mid j, k \geq 0, \varepsilon_{0,j,k} = 1\} \\ &\cup \{(e_{i,j,0}, 0, 0), (0, e_{-i,0,j}, 0) \mid i \geq 0, j \geq 0, \lambda = q^{-j-1}, \mu = q^{i+1}\} \\ &\cup \{(\lambda^{-2} q e_{i-2,0,k+2}, e_{i,0,k}, \lambda^{-1} q^{-1} e_{i-1,0,k+1}) \mid i \geq 2, k \geq 0, \lambda = q^{-k-1}, \mu = q^{-i+1}\} \\ &\cup \{(e_{i,j,0}, \lambda^2 q^{-1} e_{i+2,j+2,0}, \lambda q^{-1} e_{i+1,j+1,0}) \mid i \leq -2, j \geq 0, \lambda = q^{-j-1}, \mu = q^{i+1}\} \\ &\cup \{(e_{-1,j,k}, \lambda^2 e_{1,j+1,k-1}, \lambda q^{-1} e_{0,j+1,k}) \mid j \geq 0, k \geq 1, \lambda = q^{-j-k-1}, \mu = 1\} \end{aligned}$$

generate $HH_2^\sigma(\mathcal{A})$.

The classes of the elements in the first line are trivial for $\mu \neq 1$, and for $\mu = 1$ they contain those from the last line. It follows directly from the definition of f_3 that the remaining classes are independent. Hence:

$$HH_2^\sigma(\mathcal{A}) \cong \begin{cases} k^{N+1} & : \lambda = q^{-(N+2)}, N \geq 0, \mu = 1, \\ k^2 & : \lambda = q^{-(N+1)}, \mu = q^{\pm(M+1)}, M, N \geq 0, \\ 0 & : \text{otherwise.} \end{cases} \quad (61)$$

To calculate $HC_*^\sigma(\mathcal{A})$ we need generators in the original Hochschild complex. In Case 2 we compute generators from the above using φ_2 (32) and ξ (18). In Cases 3 and 4 we use simpler generators that are directly verified to be homologous to those obtained from the above ones:

Case 2. $\mu = 1, \lambda = q^{-(N+2)}, N \geq 0$. Then $HH_2^\sigma(\mathcal{A}) \cong k^{N+1}$. Taking $x = b^i c^{N-i}$ ($0 \leq i \leq N$), a basis is given by

$$\begin{aligned} \omega_2(N, i) &= (bcx, a, d) - (bcx, d, a) - q(dbx, a, c) + q(bdx, c, a) \\ &+ (dax, b, c) - (adx, c, b) - q^{-1}(cax, b, d) + q^{-1}(acx, d, b) \end{aligned} \quad (62)$$

Case 3. $\mu = q^{M+1}, \lambda = q^{-(N+1)}, M, N \geq 0$. Then $HH_2^\sigma(\mathcal{A}) \cong k^2$, with basis given by the Hochschild cycles

$$\begin{aligned} \omega_2 &= (a^M b^N, b, a) - q^{-1}(a^M b^N, a, b) \\ \omega_2' &= (d^M c^N, c, d) - q(d^M c^N, d, c) \end{aligned} \quad (63)$$

Case 4. $\mu = q^{-(M+1)}, \lambda = q^{-(N+1)}, M, N \geq 0$. $HH_2^\sigma(\mathcal{A}) \cong k^2$, with basis given by the Hochschild cycles

$$\begin{aligned} \omega_2 &= (a^M c^N, c, a) - q^{-1}(a^M c^N, a, c) \\ \omega_2' &= (d^M b^N, b, d) - q(d^M b^N, d, b) \end{aligned} \quad (64)$$

Finally, $HH_2^\sigma(\mathcal{A}) = 0$ for all other $\sigma = \sigma_{\lambda, \mu}$.

4.7 $HH_3^\sigma(\mathcal{A})$

The third homology $HH_3^\sigma(\mathcal{A})$ can be determined easily using the Koszul resolution. We abbreviate:

$$\psi_+ : \mathcal{A} \rightarrow \mathcal{A}, \quad x \mapsto x \blacktriangleleft b, \quad \psi_- : \mathcal{A} \rightarrow \mathcal{A}, \quad x \mapsto x \blacktriangleleft c.$$

From (36) we obtain in a straightforward way:

PROPOSITION 4.11 *The sets*

$$\mathcal{B}_{\ker \psi_\pm} := \{ e_{i,j,k} \mid \pm i \geq 0, \varepsilon_{\pm i,j,k} = 1 \}$$

are vector space bases of $\ker \psi_\pm$. Hence the sets

$$\mathcal{B}_{\text{im } \psi_\pm} := \{ E_{i,j,k}^\pm \mid e_{i,j,k} \notin \ker \psi_\pm \}, \quad E_{i,j,k}^\pm := \psi_\pm(e_{i,j,k})$$

are vector space bases of $\text{im } \psi_\pm$.

If $x \in HH_3^\sigma(\mathcal{A}) = \ker f_3$, then $x \in \ker \psi_+ \cap \ker \psi_-$, so by Proposition 4.11, $\pi_{i,j,k}(x) \neq 0$ implies $i = 0$. Insertion in $q^{-2}x \triangleleft a = x$ gives

$$x \in \begin{cases} \text{span} \{b^i c^{N-i}\}_{i=0, \dots, N} & : \lambda = q^{-N-2}, \mu = 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Conversely, all these monomials are elements of $\ker f_3$. Hence:

Case 2: For $\mu = 1$, $\lambda = q^{-(N+2)}$, $N \geq 0$ we have $HH_3^\sigma(\mathcal{A}) \cong k^{N+1}$.

Cases 1, 3, 4, 5: $HH_3^\sigma(\mathcal{A}) = 0$.

It is also straightforward that $HH_3^\sigma(\mathcal{A}) = 0$ for all $\sigma = \tau_{\lambda, \mu}$. Therefore:

THEOREM 4.12 *For any automorphism σ , we have*

$$HH_3^\sigma(\mathcal{A}) = \begin{cases} k^{N+1} & : \sigma = \sigma_{\lambda, \mu}, \lambda = q^{-(N+2)}, N \geq 0, \mu = 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Note that the $N = 0$ case ($\lambda = q^{-2}$, $\mu = 1$) is precisely the modular automorphism (28). For $\lambda = q^{-(N+2)}$ we translate the generators back to the original Hochschild complex using the maps φ_3 (33) and ξ (18), giving

$$\omega_3(N, i) = A(N, i) - B(N, i), \quad 0 \leq i \leq N, \quad (65)$$

$$\begin{aligned} A(N, i) &= dx \otimes (a \wedge b \wedge c) + cx \otimes (b \wedge a \wedge d) \\ B(N, i) &= -qdbx \otimes (1 \wedge a \wedge c) - q^{-1}cax \otimes (1 \wedge b \wedge d) \\ &\quad + dax \otimes (1 \wedge b \wedge c) + bcx \otimes (1 \wedge a \wedge d) \\ &\quad + (q - q^{-1})bcx \otimes ((c, b, 1) - (1, c, b) + (c, 1, b)) \end{aligned}$$

with $x = b^i c^{N-i}$, and the terms “ $a_0 \wedge a_1 \wedge a_2$ ” are given by:

$$\begin{aligned} a \wedge b \wedge c &= (a, b, c) - (a, c, b) + q(c, a, b) - q^2(c, b, a) + q^2(b, c, a) - q(b, a, c) \\ b \wedge a \wedge d &= (b, a, d) - (b, d, a) + q(d, b, a) - (d, a, b) + (a, d, b) - q^{-1}(a, b, d) \\ 1 \wedge a \wedge c &= (1, a, c) - q(1, c, a) + q(c, 1, a) - q(c, a, 1) + (a, c, 1) - (a, 1, c) \\ 1 \wedge b \wedge d &= (1, b, d) - q(1, d, b) - (b, 1, d) + (b, d, 1) - q(d, b, 1) + q(d, 1, b) \\ 1 \wedge b \wedge c &= (1, b, c) - (1, c, b) - (b, 1, c) + (b, c, 1) + (c, 1, b) - (c, b, 1) \\ 1 \wedge a \wedge d &= (1, a, d) - (1, d, a) + (d, 1, a) - (d, a, 1) + (a, d, 1) - (a, 1, d) \end{aligned}$$

and throughout we denote $a_0 \otimes a_1 \otimes a_2$ by (a_0, a_1, a_2) .

In the normalized complex this becomes $\omega_3(N, i) = A(N, i)$ since $B(N, i)$ is degenerate.

5 Twisted cyclic homology of $\mathcal{A}(SL_q(2))$

We calculate the twisted cyclic homology of $\mathcal{A}(SL_q(2))$ as the total homology of Connes’ mixed (b, B) -bicomplex (9) coming from the underlying cyclic object, as in section 2.2. Having found the twisted Hochschild homology, we can now complete the spectral sequence calculation. We remind the reader that throughout we are working with the normalized mixed complex.

5.1 Case 1

PROPOSITION 5.1 *In case 1, $\mu = 1$, $\lambda \notin \{q^{-(N+2)}\}_{N \geq 0}$, and $\mu \neq 1$, $\lambda = 1$, $HC_0^\sigma(\mathcal{A})$ is infinite-dimensional, $HC_{2n+1}^\sigma(\mathcal{A}) = k[\omega_1]$, and $HC_{2n+2}^\sigma(\mathcal{A}) = k[1]$, where $[\omega_1]$ is the distinguished generator of $HH_1^\sigma(\mathcal{A})$.*

Proof. By definition, $HC_0^\sigma(\mathcal{A}) = HH_0^\sigma(\mathcal{A})$, generated by $[1]$, together with $[x^{r+1}]$ ($r \geq 0$), for those $x \in \{a, b, c, d\}$ with $\sigma(x) = x$, while $HH_1^\sigma(\mathcal{A})$ is generated by $[(x^r, x)]$, ($r \geq 0$) for the same set of x , together with the distinguished generator $[\omega_1] = (\mu^{-1} - 1)[(d, a)] + (q - q^{-1})[(b, c)]$. We have

$$\begin{aligned} B_0[1] &= [(1, 1)] = [b(1, 1, 1)] = 0, \\ B_0[x^{r+1}] &= [(1, x^{r+1})] = (r+1)[(x^r, x)] \end{aligned}$$

by Lemma 2.2. Hence $\ker(B_0) = k[1]$, and $HH_1^\sigma(\mathcal{A}) = \text{im}(B_0) \oplus k[\omega_1]$. Further $HH_n^\sigma(\mathcal{A}) = 0$ for $n \geq 2$ in each case. Hence the spectral sequence stabilizes at the second page:

$$\begin{array}{ccccc} & & \downarrow & & \downarrow \\ & & 0 & \longleftarrow & k[\omega_1] & \longleftarrow & k[1] \\ & & \downarrow & & \downarrow & & \downarrow \\ & & k[\omega_1] & \longleftarrow & k[1] & & \\ & & \downarrow & & & & \\ & & HH_0^\sigma(\mathcal{A}) & & & & \end{array}$$

with all further maps being zero. The result follows. \square

5.2 Case 2

PROPOSITION 5.2 *In case 2, $\mu = 1$, $\lambda = q^{-(N+2)}$, $N \geq 0$, we have $HC_0^\sigma(\mathcal{A})$ infinite dimensional, while*

$$HC_1^\sigma(\mathcal{A}) \cong \begin{cases} k^{N+1} & : N \text{ even} \\ k^{N+2} & : N \text{ odd} \end{cases}, \quad HC_2^\sigma(\mathcal{A}) \cong k^{N+2}, \quad N \text{ odd}$$

Proof. Recall from (44) that

$$HH_0^\sigma(\mathcal{A}) = \left(\sum_{s \in S} k[b^s] \right) \oplus \left(\sum_{t \in S} k[c^t] \right) \oplus \left(\sum_{0 \leq i \leq N+2} k[b^i c^{N+2-i}] \right)$$

where $S = \{\text{integers} \geq N+3\} \cup \{N+1, N-1, N-3, \dots \geq 0\}$, with the convention that if $0 \in S$, we include only one copy of $k[1]$. From (57)

$$\begin{aligned} HH_1^\sigma(\mathcal{A}) &= \left(\sum_{s \in S'} k[(b^s, b)] \right) \oplus \left(\sum_{t \in S'} k[(c^t, c)] \right) \oplus \\ &\left(\sum_{0 \leq i \leq N} k[(b^i c^{N+1-i}, b)] \right) \oplus \left(\sum_{0 \leq i \leq N} k[(b^{i+1} c^{N-i}, c)] \right) \oplus k[\omega_1] \end{aligned}$$

Here $S' = \{\text{integers} \geq N\} \cup \{N-2, N-4, \dots \geq 0\}$, and $[\omega_1] = [(b, c)] = -[(c, b)]$ for N odd, $[\omega_1] = 0$ for N even. Now, for $s, t \geq 0$,

$$B_0[b^{s+1}] = (s+1)[(b^s, b)], \quad B_0[c^{t+1}] = (t+1)[(c^t, c)]$$

Note that, for $s \geq 0$, $s+1 \in S'$ if and only if $s \in S'$. We also have

$$B_0[1] = [(1, 1)] = [b(1, 1, 1)] = 0$$

By Lemma 2.2, for $0 \leq i \leq N$

$$B_0[b^{i+1}c^{N+1-i}] = (N+1-i)[(b^{i+1}c^{N-i}, c)] + (i+1)[(b^i c^{N+1-i}, b)]$$

Hence $\ker(B_0) = k[1]$ if N is odd, 0 if N is even. Further,

$$HH_1^\sigma(\mathcal{A})/\text{im}(B_0) \cong \begin{cases} k^{N+1} = \sum_{0 \leq i \leq N}^\oplus k[\omega_1(N, i)] & : N \text{ even} \\ k^{N+2} = \sum_{0 \leq i \leq N}^\oplus k[\omega_1(N, i)] \oplus k[\omega_1] & : N \text{ odd} \end{cases} \quad (66)$$

with generators

$$[\omega_1(N, i)] = [(xc, b)] = [(bx, c)], \quad x = b^i c^{N-i}, \quad 0 \leq i \leq N \quad (67)$$

together with (if N is odd) $[\omega_1] = [(c, b)] = -[(b, c)]$.

PROPOSITION 5.3 For N odd, $B_1 = 0$. For N even, $\text{im}(B_1)$ is at most one-dimensional, spanned by $[B_1(\omega_1(N, \frac{1}{2}N))]$.

Proof. Recall that $HH_2^\sigma(\mathcal{A}) \cong k^{N+1}$, with generators $\omega_2(N, i)$, $0 \leq i \leq N$ given in (62). We use the construction of σ -twisted Hochschild n -cocycles of Proposition 4.3. Using (40)-(43), for each $n \in \mathbb{Z}$ define a trace h_n by

$$h_n(a^{i+1}) = 0 = h_n(d^{i+1}), \quad h_n(b^j) = \delta_{n,j}, \quad h_n(c^j) = \delta_{-n,j} \quad (68)$$

for $i, j \geq 0$. For the derivation ∂_b (51) and σ -derivation ∂ (53) define a σ -twisted Hochschild 2-cocycle $\phi_{2,n}$ by

$$\phi_{2,n}(x, y, z) = h_n(x\partial_b(y)\partial(z)) \quad (69)$$

LEMMA 5.4 $\langle \phi_{2,n}, \omega_2(N, i) \rangle = 0$, unless $n = 2i - N$. For $i \neq \frac{1}{2}N$, we have $\langle \phi_{2,2i-N}, \omega_2(N, i) \rangle \neq 0$.

Proof. Directly, $\langle \phi_{2,n}, \omega_2(N, i) \rangle = \langle \phi_{2,n}, q(bdx, c, a) - q^{-1}(cax, b, d) \rangle$ (considering only potentially nonzero terms)

$$\begin{aligned} &= h_n(qbdx(-c)a - q^{-1}caxb(-\lambda^{-1}d)) = h_n(q^{-1}\lambda^{-1}caxbd - qbdxca) \\ &= h_n(q^N\lambda^{-1}bcxad - q^{-N}bcxda) = h_n(q^{-N}bcx(q^{3N+2}ad - da)) \\ &= q^{N+2}(q^N - q^{-N})h_n(bcxad) = q^{N+2}(q^N - q^{-N})h_n(bcx(1 + qbc)) \\ &= \frac{q^2(q^{2N} - 1)(q^2 - 1)}{(q^{N+4} - 1)}h_n(bcx) \end{aligned}$$

Since $bcx = b^{i+1}c^{N+1-i}$, it's clear from (42), (43) that $h_n(bcx) = 0$ unless $n = (i+1) - (N-i+1) = 2i - N$, and $h_{2i-N}(bcx) \neq 0$ unless $2i = N$. \square

To find $\text{im}(B_1)$, since $B_1 \circ B_0 = 0$, we need only consider B_1 applied to ω_1 and the $\omega_1(N, i)$. For $[\omega_1] = [(c, b)] = -[(b, c)]$, which is nonzero if and only if N is odd, we have $\text{deg}(\omega_1) = (0, 0)$ for the \mathbb{Z}^2 -grading (25).

The maps B_n preserve the grading, so $\deg(B_1(\omega_1)) = (0, 0)$ also. Now, $\deg(\omega_2(N, i)) = (0, 2i - N) \neq (0, 0)$ for any N, i since N is odd. The Hochschild boundary maps b also preserve the grading, so $B_1(\omega_1)$ cannot be cohomologous to any nontrivial element of $HH_2^\sigma(\mathcal{A})$.

Now consider the generators $\omega_1(N, i)$ (67). These contain only b 's and c 's, so combining this with (14), it is immediate that each $\phi_{2,n}$ vanishes on $\text{im}(B_1)$. So for $i \neq \frac{1}{2}N$, we have $[B_1(\omega_1(N, i))] = 0$. \square

The second page of the spectral sequence reads:

$$\begin{array}{ccccccc}
& & & \downarrow & & \downarrow & \\
& & & 0 & \longleftarrow & HH_3^\sigma(\mathcal{A})/\text{im}(B_2) & \longleftarrow & \ker(B_2)/\text{im}(B_1) & \longleftarrow & \\
& & & \downarrow & & \downarrow & & \downarrow & & \\
HH_3^\sigma(\mathcal{A})/\text{im}(B_2) & \longleftarrow & \ker(B_2)/\text{im}(B_1) & \longleftarrow & \ker(B_1)/\text{im}(B_0) & \longleftarrow & \\
& & \downarrow & & \downarrow & & \\
HH_2^\sigma(\mathcal{A})/\text{im}(B_1) & \longleftarrow & \ker(B_1)/\text{im}(B_0) & \longleftarrow & \ker(B_0) & & \\
& & \downarrow & & & & \\
HH_1^\sigma(\mathcal{A})/\text{im}(B_0) & \longleftarrow & \ker(B_0) & & & & \\
& & \downarrow & & & & \\
& & HH_0^\sigma(\mathcal{A}) & & & &
\end{array}$$

The only potentially nonzero differential is $f : \ker(B_0) \rightarrow HH_3^\sigma(\mathcal{A})/\text{im}(B_2)$, and after this step the spectral sequence stabilises, giving

$$\begin{aligned}
HC_0^\sigma(\mathcal{A}) &= HH_0^\sigma(\mathcal{A}), & HC_1^\sigma(\mathcal{A}) &= HH_1^\sigma(\mathcal{A})/\text{im}(B_0), \\
HC_2^\sigma(\mathcal{A}) &= (HH_2^\sigma(\mathcal{A})/\text{im}(B_1)) \oplus \ker(B_0) \\
HC_{2n+3}^\sigma(\mathcal{A}) &= ((HH_3^\sigma(\mathcal{A})/\text{im}(B_2))/\text{im}f) \oplus (\ker(B_1)/\text{im}(B_0)), \\
HC_{2n+4}^\sigma(\mathcal{A}) &= (\ker(B_2)/\text{im}(B_1)) \oplus \ker(f)
\end{aligned}$$

Hence $HC_0^\sigma(\mathcal{A})$ is infinite-dimensional, given by (44), whilst $HC_1^\sigma(\mathcal{A})$ is given by (66). Now, $\ker(B_0) = k[1]$, which is nonzero if and only if N is odd. So for N even, $\ker(f) = 0 = \text{im}(f)$. For N odd, $\text{im}(B_1) = 0$, hence $HC_2^\sigma(\mathcal{A}) = HH_2^\sigma(\mathcal{A}) \oplus k[1] \cong k^{N+2}$. This completes the proof of Proposition 5.2. \square

Recall that $HH_2^\sigma(\mathcal{A})$ and $HH_3^\sigma(\mathcal{A})$ are both isomorphic to k^{N+1} with generators $\omega_2(N, i)$ (62), $\omega_3(N, i)$ (65) ($0 \leq i \leq N$) respectively. By symmetry and the \mathbb{Z}^2 -grading ($\deg(\omega_2(N, i)) = (0, 2i - N) = \deg(\omega_3(N, i))$) we expect, but do not have a proof for:

CONJECTURE 5.5 1. For N even, $[B_1(\omega_1(N, \frac{1}{2}N))] \neq 0 \in HH_2^\sigma(\mathcal{A})$, and is proportional to $[\omega_2(N, \frac{1}{2}N)]$.

2. For all N , $B_2 : HH_2^\sigma(\mathcal{A})/\text{im}(B_1) \rightarrow HH_3^\sigma(\mathcal{A})$ is injective. It follows that for N odd, $f : \ker(B_0) \rightarrow HH_3^\sigma(\mathcal{A})$ is the zero map.

From this it would follow that, for N even, $HC_2^\sigma(\mathcal{A}) \cong k^N$, generated by $[\omega_2(N, i)]$ for $i \neq \frac{1}{2}N$, $HC_{2n+3}^\sigma(\mathcal{A}) \cong k^{N+2}$, generated by $[\omega_3(N, \frac{1}{2}N)]$ together with (66), and $HC_{2n+4}^\sigma(\mathcal{A}) \cong k[\omega_2(N, \frac{1}{2}N)]$. For N odd we would have $HC_{2n+3}^\sigma(\mathcal{A}) \cong k^{N+2}$, given by (66), and $HC_{2n+4}^\sigma(\mathcal{A}) \cong k[1]$.

5.3 Cases 3, 4 and 5

PROPOSITION 5.6 For case 3, $\mu = q^{M+1}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$,

1. $HC_0^\sigma(\mathcal{A}) \cong k^2$, with generators $[d^{M+1}c^{N+1}]$, $[a^{M+1}b^{N+1}]$.
2. $HC_1^\sigma(\mathcal{A}) \cong k^2$, with generators $[(a^{M+1}b^N, b)]$, $[(d^{M+1}c^N, c)]$,
equivalently $[(d^M c^{N+1}, d)]$, $[(a^M b^{N+1}, a)]$.
3. $HC_n^\sigma(\mathcal{A}) = 0$, $n \geq 2$

Proof. Recall that $HH_0^\sigma(\mathcal{A}) \cong k^2$, generated by $[d^{M+1}c^{N+1}]$, $[a^{M+1}b^{N+1}]$,
 $HH_1^\sigma(\mathcal{A}) \cong k^4$, generated by $[(a^{M+1}b^N, b)]$, $[(d^{M+1}c^N, c)]$, $[(d^M c^{N+1}, d)]$
and $[(a^M b^{N+1}, a)]$, and $HH_2^\sigma(\mathcal{A}) \cong k^2$ with generators ω_2 and ω_2' (63).

LEMMA 5.7 For $B_0 : HH_0^\sigma(\mathcal{A}) \rightarrow HH_1^\sigma(\mathcal{A})$ we have:

$$\begin{aligned} B_0[d^{M+1}c^{N+1}] &= (N+1)[(d^{M+1}c^N, c)] + (M+1)q^{-(N+1)}[(d^M c^{N+1}, d)] \\ B_0[a^{M+1}b^{N+1}] &= (N+1)[(a^{M+1}b^N, b)] + (M+1)q^{N+1}[(a^M b^{N+1}, b)] \end{aligned}$$

Proof. We treat only $[d^{M+1}c^{N+1}]$, the calculations for $[a^{M+1}b^{N+1}]$ are completely analogous. By considering $b(1, d^{M+1}c^N, c)$, we find that

$$B_0[d^{M+1}c^{N+1}] = [(1, d^{M+1}c^{N+1})] = [(d^{M+1}c^N, c)] + q^{-(M+1)}[(c, d^{M+1}c^N)]$$

Now, for any $x, y \in \mathcal{A}$, and for all $r \geq 0$, a simple induction shows that:

$$[(x, y^{r+1})] = \sum_{j=0}^r [(\sigma(y^j)xy^{r-j}, y)] \quad (70)$$

Also, $[(c, d^{M+1}c^N)] = q^{M+1}[(d^{M+1}c, c^N)] + q^{-N(M+1)}[(c^{N+1}, d^{M+1})]$. It follows from (70) that $[(d^{M+1}c, c^N)] = N[(d^{M+1}c^N, c)]$ and $[(c^{N+1}, d^{M+1})] = (M+1)q^{M(N+1)}[(d^M c^{N+1}, d)]$. Hence the result. \square

COROLLARY 5.8 It follows that:

1. $\ker(B_0) = \{0\}$.
2. $HH_1^\sigma(\mathcal{A})/\text{im}(B_0) \cong k^2$, generated by $[(a^{M+1}b^N, b)]$, $[(d^{M+1}c^N, c)]$,
equivalently $[(d^M c^{N+1}, d)]$, $[(a^M b^{N+1}, a)]$.

LEMMA 5.9 $B_1 : HH_1^\sigma(\mathcal{A}) \rightarrow HH_2^\sigma(\mathcal{A})$ is surjective.

Proof. Using (14) we have

$$\begin{aligned} B_1(a^M b^{N+1}, a) &= (1, a^M b^{N+1}, a) - q^{-(N+1)}(1, a, a^M b^{N+1}) \\ B_1(d^M c^{N+1}, d) &= (1, d^M c^{N+1}, d) - q^{N+1}(1, d, d^M c^{N+1}) \end{aligned}$$

Consider the twisted Hochschild 2-cocycles given by

$$\phi_2(x, y, z) = h_{[a^{M+1}b^{N+1}]}(x\partial_a(y)\partial_b(z))$$

$$\phi_2'(x, y, z) = h_{[d^{M+1}c^{N+1}]}(x\partial_a(y)\partial_b(z))$$

with $h_{[a^{M+1}b^{N+1}]}$, $h_{[d^{M+1}c^{N+1}]}$ defined in (48). Then

$$\begin{aligned} \phi_2(B_1(a^M b^{N+1}, a)) &= -(N+1)q^{-(N+1)} \\ \phi_2'(B_1(a^M b^{N+1}, a)) &= 0 = \phi_2(B_1(d^M c^{N+1}, d)) \\ \phi_2'(B_1(d^M c^{N+1}, d)) &= -q^{N+1}(N+1) \end{aligned}$$

It follows that $B_1(a^M b^{N+1}, a)$ and $B_1(d^M c^{N+1}, d)$ are nontrivial and linearly independent, and hence span $HH_2^\sigma(\mathcal{A}) \cong k^2$. \square

It follows that $\ker(B_1)/\text{im}(B_0) = 0$, and $HH_2^\sigma(\mathcal{A})/\text{im}(B_1) = 0$. So:

1. $HC_0^\sigma(\mathcal{A}) = HH_0^\sigma(\mathcal{A})$
2. $HC_1^\sigma(\mathcal{A}) = HH_1^\sigma(\mathcal{A})/\text{im}(B_0) \cong k^2$
3. $HC_{2n+2}^\sigma(\mathcal{A}) = (HH_2^\sigma(\mathcal{A})/\text{im}(B_1)) \oplus \ker(B_0) = 0$
4. $HC_{2n+3}^\sigma(\mathcal{A}) = \ker(B_1)/\text{im}(B_0) = 0$

This completes the proof of Proposition 5.6. \square

Dually, we have $HC_\sigma^0(\mathcal{A}) \cong k^2$, generated by the two 0-cocycles $h_{[a^{M+1}b^{N+1}]}$, $h_{[d^{M+1}c^{N+1}]}$ defined in (48). To give the generators of $HC_\sigma^1(\mathcal{A}) \cong k^2$, define a new derivation $\partial' = (N+1)\partial_a - (M+1)\partial_b$. We have

$$\partial'(a^{M+1}b^{N+1}) = 0 = \partial'(d^{M+1}c^{N+1})$$

so by Lemma 4.4 the twisted Hochschild 1-cocycles ϕ_1, ϕ'_1 defined by

$$\phi_1(x, y) = h_{[a^{M+1}b^{N+1}]}(x\partial'(y)), \quad \phi'_1(x, y) = h_{[d^{M+1}c^{N+1}]}(x\partial'(y))$$

are also twisted cyclic 1-cocycles, and satisfy

$$\phi_1(a^M b^{N+1}, a) = N+1, \quad \phi_1(d^M c^{N+1}, d) = 0$$

$$\phi'_1(a^M b^{N+1}, a) = 0, \quad \phi'_1(d^M c^{N+1}, d) = -(N+1)$$

In fact ϕ_1, ϕ'_1 are a basis for $HC_\sigma^1(\mathcal{A}) \cong k^2$.

PROPOSITION 5.10 *In case 4, $\mu = q^{-(M+1)}$, $\lambda = q^{-(N+1)}$, $M, N \geq 0$,*

1. $HC_0^\sigma(\mathcal{A}) \cong k^2$, with generators $[d^{M+1}b^{N+1}]$, $[a^{M+1}c^{N+1}]$.
2. $HC_1^\sigma(\mathcal{A}) \cong k^2$, generated by $[(d^{M+1}b^N, b)]$, $[(a^{M+1}c^N, c)]$, equivalently $[(d^M b^{N+1}, d)]$, $[(a^M c^{N+1}, a)]$.
3. $HC_n^\sigma(\mathcal{A}) = 0$, $n \geq 2$.

The proof is completely analogous to that of Proposition 5.6. We also have $HC_\sigma^0(\mathcal{A}) \cong k^2$, generated by the two 0-cocycles $h_{[a^{M+1}c^{N+1}]}$, $h_{[d^{M+1}b^{N+1}]}$, and $HC_\sigma^1(\mathcal{A}) \cong k^2$, generated by ϕ_1, ϕ'_1 defined by

$$\phi_1(x, y) = h_{[a^{M+1}c^{N+1}]}(x\partial''(y)), \quad \phi'_1(x, y) = h_{[d^{M+1}b^{N+1}]}(x\partial''(y))$$

where $\partial'' = (N+1)\partial_a + (M+1)\partial_b$.

The remaining case is the trivial one:

PROPOSITION 5.11 *In case 5 ($\mu = q^{\pm(M+1)}$, $M \geq 0$, $\lambda \notin q^{-\mathbb{N}}$, and $\mu \notin q^{\mathbb{Z}}$, $\lambda \neq 1$), we have $HC_n^\sigma(\mathcal{A}) = 0$ for all $n \geq 0$.*

Proof. In each case $HH_n^\sigma(\mathcal{A}) = 0$ for all $n \geq 0$, so the spectral sequence stabilises at the first page, with all entries being zero. \square

6 Covariant differential calculi

In this section we identify the classes in twisted cyclic cohomology of $\mathcal{A}(SL_q(2))$ of the twisted cyclic cocycles arising from the three and four dimensional covariant differential calculi originally discovered for quantum $SU(2)$ by Woronowicz.

6.1 THREE DIMENSIONAL LEFT-COVARIANT CALCULUS

The automorphism of $\mathcal{A}(SL_q(2))$ corresponding to Woronowicz's three-dimensional left-covariant calculus over quantum $SU(2)$ is

$$\sigma(a) = q^{-2}a, \quad \sigma(b) = q^4b, \quad \sigma(c) = q^{-4}c, \quad \sigma(d) = q^2d \quad (71)$$

The twisted cyclic 3-cocycle over $\mathcal{A}(SL_q(2))$ arising from this calculus was written down in [17] section 3 (denoted by $\tau_{\omega,h}$) and [12] section 5 (corresponding to the linear functional f).

THEOREM 6.1 *For $\mathcal{A} = \mathcal{A}(SL_q(2))$, we have $HC_\sigma^3(\mathcal{A}) = 0$ for the automorphism (71), hence the twisted cyclic 3-cocycle corresponding to $\tau_{\omega,h}$ and f is a trivial element of twisted cyclic cohomology.*

Specializing Proposition 5.6 to the automorphism (71), we obtain:

PROPOSITION 6.2 *For $\sigma = \sigma_{\lambda,\mu}$, with $\lambda = q^{-2}$, $\mu = q^4$, we have*

1. $HC_0^\sigma(\mathcal{A}) = HH_0^\sigma(\mathcal{A}) \cong k^2$, with generators $[d^4c^2]$, $[a^4b^2]$,
2. $HC_1^\sigma(\mathcal{A}) \cong k^2$ generated by $[(a^4b, b)]$, $[(d^4c, c)]$, equivalently $[(d^3c^2, d)]$, $[(a^3b^2, a)]$.
3. $HC_n^\sigma(\mathcal{A}) = 0$, $n \geq 2$

By duality between twisted cyclic homology and cohomology we have

COROLLARY 6.3 $HC_\sigma^0(\mathcal{A}) \cong k^2 \cong HC_\sigma^1(\mathcal{A})$, $HC_\sigma^n(\mathcal{A}) = 0$ for $n \geq 2$.

So the twisted cyclic 3-cocycles coming from Woronowicz's three dimensional calculus that appear in [17] section 3 (denoted by $\tau_{\omega,h}$) and [12] section 5 (corresponding to the linear functional f) are necessarily trivial elements of twisted cyclic cohomology, thus proving Theorem 6.1.

6.2 FOUR DIMENSIONAL BICOVARIANT CALCULI

It is well-known (see [17] for example) that the twisted cyclic 4-cocycles on $\mathcal{A}(SL_q(2))$ coming from both Woronowicz's four-dimensional bicovariant calculi over quantum $SU(2)$ are both simply S^2h , the promotion of the twisted 0-cocycle given by the Haar functional h to a 4-cocycle via the periodicity operator S . Explicitly (up to a normalising constant),

$$(S^2h)(a_0, a_1, a_2, a_3, a_4) = h(a_0a_1a_2a_3a_4) \quad (72)$$

On basis elements, h is given by

$$h(a^{i+1}b^j c^k) = 0 = h(d^{i+1}b^j c^k)$$

$$h(b^j c^k) = \begin{cases} (-q)^{-k}(1-q^{-2})(1-q^{-2(k+1)})^{-1} & : j = k \\ 0 & : j \neq k \end{cases} \quad (73)$$

From (27) we see that h is a well-defined σ_{mod}^{-1} -twisted cyclic 0-cocycle, given by $\lambda = q^2$, $\mu = 1$, and hence corresponds to Case 1. By inspection, we see that h is exactly the twisted 0-cocycle $h_{[1]}$ defined in (40).

7 Conclusions

The original motivation for this work was the belief that calculating twisted cyclic cohomology would give new insight into existing classification results [8, 9] for covariant differential calculi over quantum $SL(2)$ and quantum $SU(2)$. However, we see from the Woronowicz four-dimensional calculi that nonisomorphic calculi can give rise to cohomologous cocycles, and as Theorem 6.1 shows, interesting differential calculi can correspond to trivial elements of twisted cyclic cohomology.

The striking result that twisting via the modular automorphism overcomes the dimension drop in Hochschild homology seems to offer the most promising direction for future work. Similar results have been obtained by the first author for Podleś quantum spheres [7], and by Sitarz for quantum hyperplanes [20]. It seems natural to ask whether the modular automorphism can overcome the dimension drop in Hochschild homology for larger classes of quantum groups, and look for applications of these results.

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