

# REAL RANK AND PROPERTY (SP) FOR DIRECT LIMITS OF RECURSIVE SUBHOMOGENEOUS ALGEBRAS

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ABSTRACT. Let  $A$  be a unital simple direct limit of recursive subhomogeneous algebras with no dimension growth. We give criteria which specify exactly when  $A$  has real rank zero, and exactly when  $A$  has the Property (SP): every nonzero hereditary subalgebra of  $A$  contains a nonzero projection. Specifically,  $A$  has real rank zero if and only if the image of  $K_0(A)$  in  $\text{Aff}(T(A))$  is dense, and  $A$  has the Property (SP) if and only if for every  $\varepsilon > 0$  there is  $\eta \in K_0(A)$  such that the corresponding affine function  $f$  on  $T(A)$  satisfies  $0 < f(\tau) < \varepsilon$  for all tracial states  $\tau$ . By comparison with results for unital simple direct limits of homogeneous  $C^*$ -algebras with no dimension growth, one might hope that weaker conditions might suffice. We give examples to show that several plausible weaker conditions do not suffice for the results above.

If  $A$  has real rank zero and at most countably many extreme tracial states, we apply results of H. Lin to show that  $A$  has tracial rank zero and is classifiable.

## 0. INTRODUCTION

Let  $A$  be a unital simple direct limit of recursive subhomogeneous algebras with no dimension growth. In [27], we proved that  $A$  must have stable rank one, and that the order on projections over  $A$  is determined by traces (essentially Blackadar's Second Fundamental Comparability Question). The first part generalizes [8], where the result is proved for finite direct sums of algebras of the form  $C(X, M_n)$  in place of recursive subhomogeneous algebras.

In this paper, we determine, in terms of  $K$ -theory and traces, when a simple direct limit as above has real rank zero. For the case that the algebras in the direct system are finite direct sums of algebras of the form  $C(X, M_n)$ , for connected finite complexes  $X$ , it is shown in [2] that  $\text{RR}(A) = 0$  if and only if the projections in  $A$  distinguish the tracial states. In our situation, this condition does not suffice. We prove that  $\text{RR}(A) = 0$  if and only if the canonical map  $K_0(A) \rightarrow \text{Aff}(T(A))$ , to the real affine continuous functions on the tracial state space, has dense range. We show by example that several conditions between ours and that of [2] also do not imply real rank zero.

We do have a three part condition for real rank zero which looks more like that of [2]: the projections in  $A$  distinguish the tracial states;  $K_0(A)$  is a Riesz group (torsion is allowed); and  $A$  has Property (SP), that is, every nonzero hereditary

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subalgebra contains a nonzero projection. Examples show that none of the three parts of this condition can be omitted.

We also prove that a unital simple direct limit  $A$  of recursive subhomogeneous algebras with no dimension growth has Property (SP) if and only if for every  $\varepsilon > 0$  there is  $\eta \in K_0(A)$  such that  $0 < \tau_*(\eta) < \varepsilon$  for all tracial states  $\tau$  on  $A$ . One might hope that it would suffice to require that  $\tau_*(K_0(A))$  be dense in  $\mathbf{R}$  for all tracial states  $\tau$ , but we show by example that this is false.

We leave open the question of when  $A$  as above is approximately divisible in the sense of [5], and when  $K_0(A)$  is a Riesz group. Both are automatic for direct limits, with no dimension growth, of finite direct sums of algebras of the form  $C(X, M_n)$ , for connected compact metric spaces  $X$ . See [12] for approximate divisibility, and see Theorem 2.7 of [16] for  $K_0(A)$  being a Riesz group (under much more general hypotheses). Our examples rule out some possible conditions for these properties, but we have no positive results. Some further discussion can be found in Section 4, and the examples are in Section 5.

We also mention the paper [14]. The building blocks there, section algebras of locally trivial bundles with fiber  $M_n$  and possibly nontrivial Dixmier-Douady class, are a special case of recursive subhomogeneous algebras. The criterion given there for real rank zero is of a very different nature, using eigenvalue lists associated with the maps of the direct system.

This paper is organized as follows. In Section 1, we recall the definitions of recursive subhomogeneous algebras and dimension growth, and some other definitions and terminology used in the paper, as well as proving several results for which we have been unable to find references. In Section 2, we analyze hereditary subalgebras of recursive subhomogeneous algebras. Section 3 contains the main technical result. In Section 4 we state and prove the main results, and give corollaries related to classification. Finally, Section 5 contains the counterexamples mentioned above.

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## 1. PRELIMINARIES

In this section, we collect three kinds of preliminary results. First, we recall for convenience the definition of a recursive subhomogeneous algebra and some useful associated terminology. Second, we record for clarity the equivalence, in our context, of several versions of Riesz decomposition and Riesz interpolation. Third, we give several results on traces and the map  $K_0(A) \rightarrow \text{Aff}(T(A))$  which we regard as folklore but for which we have been unable to find references. These results include the easy directions of our characterizations of real rank zero and Property (SP). We also establish related notation.

Definition 1.1 is from Definitions 1.1 and 1.2 of [26]. First recall that if  $A$ ,  $B$ , and  $C$  are  $C^*$ -algebras, and  $\varphi: A \rightarrow C$  and  $\rho: B \rightarrow C$  are homomorphisms, then the pullback  $A \oplus_C B$  is given by

$$A \oplus_C B = \{(a, b) \in A \oplus B : \varphi(a) = \rho(b)\}.$$

**Definition 1.1.** A recursive subhomogeneous algebra is a  $C^*$ -algebra of the form

$$R = \left[ \cdots \left[ \left[ C_0 \oplus_{C_1^{(0)}} C_1 \right] \oplus_{C_2^{(0)}} C_2 \right] \cdots \right] \oplus_{C_l^{(0)}} C_l,$$

with  $C_k = C(X_k, M_{n_k})$  for compact Hausdorff spaces  $X_k$  and positive integers  $n_k$ , with  $C_k^{(0)} = C(X_k^{(0)}, M_{n_k})$  for compact subsets  $X_k^{(0)} \subset X_k$  (possibly empty), and where the maps  $C_k \rightarrow C_k^{(0)}$  are always the restriction maps. An expression of this type will be referred to as a (*recursive subhomogeneous*) *decomposition* of  $R$  (over  $\coprod_{k=0}^l X_k$ ).

Associated with this decomposition are:

- (1) its *length*  $l$ ;
- (2) its *base spaces*  $X_0, X_1, \dots, X_l$  and *total space*  $X = \coprod_{k=0}^l X_k$ ;
- (3) its *matrix sizes*  $n_0, \dots, n_l$ , and *matrix size function*  $n: X \rightarrow \mathbf{N} \cup \{0\}$ , defined by  $n(x) = n_k$  when  $x \in X_k$  (the *matrix size of  $A$  at  $x$* );
- (4) its *minimum matrix size*  $\min_k n_k$ ;
- (5) its *topological dimension*  $\dim(X)$  (the covering dimension of  $X$ , Definition 1.6.7 of [13]; here equal to  $\max_k \dim(X_k)$ ), and *topological dimension function*  $d: X \rightarrow \mathbf{N} \cup \{0\}$ , defined by  $d(x) = \dim(X_k)$  when  $x \in X_k$  (this is called the *topological dimension of  $A$  at  $x$* );
- (6) its *standard representation*  $\sigma = \sigma_R: R \rightarrow \bigoplus_{k=0}^l C(X_k, M_{n_k})$ , defined by forgetting the restriction to a subalgebra in each of the fibered products in the decomposition;
- (7) the associated *evaluation maps*  $\text{ev}_x: R \rightarrow M_{n_k}$  for  $x \in X_k$ , defined to be the restriction of the usual evaluation map to  $R$ , identified with a subalgebra of  $\bigoplus_{k=0}^l C(X_k, M_{n_k})$  via  $\sigma$ .

**Definition 1.2.** We say that a direct system  $(A_n)_{n \in \mathbf{N}}$  of recursive subhomogeneous algebras has *no dimension growth* if there is  $d \in \mathbf{N}$  such that every  $A_n$  has a recursive subhomogeneous decomposition with topological dimension at most  $d$ . By abuse of terminology, we also say that the direct limit  $A = \lim_{\rightarrow} A_n$  has no dimension growth.

See Section 1 of [27] for more on dimension growth conditions.

Now we turn to the Riesz conditions.

**Proposition 1.3.** Let  $A$  be an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth. Then the following are equivalent (see Section 1.1 of [16] for definitions):

- (1) The ordered group  $K_0(A)$  has the Riesz interpolation property.
- (2) The ordered group  $K_0(A)$  has the Riesz decomposition property.
- (3) The projections in  $M_\infty(A)$  satisfy Riesz interpolation.
- (4) The projections in  $M_\infty(A)$  satisfy Riesz decomposition.

*Proof.* The projections in  $M_\infty(A)$  satisfy cancellation, by Theorem 2.2 of [27]. Given this, the equivalence of the four conditions is found in Section 1.1 of [16]. ■

**Definition 1.4.** A directed partially ordered abelian group  $(G, G_+)$  is a *Riesz group* if it has the Riesz decomposition property.

This definition is in 1.1 of [16]. (It differs, for example, from Section IV.6 of [9], where Riesz groups are required to be unperforated and hence torsion free.)

**Proposition 1.5.** ([32], Theorem 1.1) Let  $A$  be any infinite dimensional simple unital  $C^*$ -algebra with real rank zero. The projections in  $M_\infty(A)$  satisfy Riesz decomposition.

Finally, we consider traces.

**Notation 1.6.** Let  $A$  be a unital  $C^*$ -algebra. Then  $T(A)$  denotes the space of tracial states on  $A$ , equipped with the weak\* topology. For any compact convex set  $\Delta$ , we let  $\text{Aff}(\Delta)$  denote the space of continuous affine real valued functions on  $\Delta$ , with the supremum norm.

We further let  $\rho_A: K_0(A) \rightarrow \text{Aff}(T(A))$  (or  $\rho$  when  $A$  is understood) denote the group homomorphism given by  $\rho(\eta)(\tau) = \tau_*(\eta)$  for  $\eta \in K_0(A)$  and  $\tau \in T(A)$ .

Note that  $T(A)$  is always a Choquet simplex (Theorem 3.1.18 of [29]), and that  $\text{Aff}(\Delta)$  is always a real Banach space (Chapter 7 of [15]).

**Definition 1.7.** If  $A$  is a  $C^*$ -algebra, then we define

$$[A, A] = \text{span}(\{ab - ba : a, b \in A\});$$

we use obvious modifications for subsets of  $A$ . Note that  $[A, A]_{\text{sa}} = i[A_{\text{sa}}, A_{\text{sa}}]$ .

Following Section 2 of [10], we define the *universal trace* on  $A$  to be the quotient map  $T: A \rightarrow A/\overline{[A, A]}$ . (No confusion should arise with the notation  $T(A)$  for the tracial state space of  $A$ .)

**Remark 1.8.** The universal trace  $T$  is a (Banach space valued) trace, that is,  $T(ab) = T(ba)$  for all  $a, b \in A$ . By Lemma 1(d) of [10], it induces a group homomorphism

$$T_*: K_0(A) \rightarrow (A/\overline{[A, A]})_{\text{sa}} = A_{\text{sa}}/\overline{[A, A]_{\text{sa}}}.$$

(If

$$p = \begin{pmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{n,1} & \cdots & p_{n,n} \end{pmatrix} \in M_n(A)$$

is a projection, then  $T_*([p]) = \sum_{j=1}^n T(p_{j,j})$ .)

**Proposition 1.9.** Let  $A$  be a unital  $C^*$ -algebra. Define  $S_0: A_{\text{sa}} \rightarrow \text{Aff}(T(A))$  by  $S_0(a)(\tau) = \tau(a)$ . Then  $S_0$  induces a map  $S: A_{\text{sa}}/\overline{[A, A]_{\text{sa}}} \rightarrow \text{Aff}(T(A))$  which is an isometric isomorphism of real Banach spaces. Moreover,  $S \circ T_* = \rho$  as maps from  $K_0(A)$  to  $\text{Aff}(T(A))$ .

*Proof.* Clearly  $S_0$  is continuous. Since the tracial states are continuous and vanish on commutators, it induces a map  $S$  as described, and the relation  $S \circ T_* = \rho$  is obvious.

It remains to show that  $S$  is isometric and surjective. Let  $E$  be the real Banach space of bounded selfadjoint tracial functionals on  $A$ . By the Hahn-Banach Theorem, the obvious map from  $E$  to the dual of  $A_{\text{sa}}/\overline{[A, A]_{\text{sa}}}$  is an isometric isomorphism. Now let  $A_0, A^q = A_{\text{sa}}/A_0$ , and the quotient map  $q: A_{\text{sa}} \rightarrow A^q$  be as at the beginning of Section 2 of [7]. Proposition 2.7 of [7] states that the obvious map from  $E$  to the dual of  $A^q$  is also an isometric isomorphism. Therefore  $A_0 = \overline{[A, A]_{\text{sa}}}$ . So it suffices to prove that the map  $R: A^q \rightarrow \text{Aff}(T(A))$ , coming from the inclusion  $T(A) \subset E$  and the identification of  $E$  with the dual of  $A^q$ , is isometric and surjective.

Clearly  $\|R\| \leq 1$ . Assume therefore that  $a \in A_{\text{sa}}$  and  $\|R(q(a))\| \leq 1$ . To prove that  $\|q(a)\| \leq 1$ , it suffices to prove that  $|\sigma(a)| \leq \|\sigma\|$  for all  $\sigma \in E$ . By Propositions 2.7 and 2.8 of [7], there are nonnegative numbers  $\alpha_+$  and  $\alpha_-$ , and tracial states  $\rho_+$  and  $\rho_-$ , such that  $\sigma = \alpha_+\rho_+ - \alpha_-\rho_-$  and  $\alpha_+ + \alpha_- = \|\sigma\|$ . (In the notation of [7],  $\sigma_+ = \alpha_+\rho_+$  and  $\sigma_- = \alpha_-\rho_-$ .) Now

$$|\sigma(a)| \leq \alpha_+|\rho_+(a)| + \alpha_-|\rho_-(a)| \leq \alpha_+\|R(q(a))\| + \alpha_-\|R(q(a))\| \leq \alpha_+ + \alpha_- = \|\sigma\|,$$

as desired.

This shows that  $R$  is isometric. Therefore, in particular, its image is closed. Moreover, its image is a real vector space which separates the points of  $T(A)$  and contains the constant functions. Therefore the image is dense, by Corollary 7.4 of [15]. ■

**Proposition 1.10.** Let  $A$  be any infinite dimensional simple unital  $C^*$ -algebra with real rank zero. Then:

- (1)  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$ .
- (2) For the universal trace  $T: A \rightarrow A/\overline{[A, A]}$  (see Definition 1.7), we have  $T_*(K_0(A))$  dense in  $(A/\overline{[A, A]})_{\text{sa}}$ .

*Proof.* The two parts of the conclusion are equivalent by Proposition 1.9. We therefore prove (1).

Let  $P(A)$  be the set of projections in  $A$ . Then  $\text{span}_{\mathbf{R}}(P(A))$  is dense in  $A_{\text{sa}}$  by real rank zero, so  $\text{span}_{\mathbf{R}}(\rho_A(K_0(A)))$  is dense in  $\text{Aff}(T(A))$  by Proposition 1.9. Therefore it suffices to show that  $\rho_A(K_0(A))$  is closed under multiplication by real scalars. In fact, it is enough to show that if  $n, r \in \mathbf{N}$  and if  $p \in M_r(A)$  is a projection, then  $2^{-n}\rho_A([p]) \in \rho_A(K_0(A))$ .

Let  $\varepsilon > 0$ . Choose  $k \in \mathbf{N}$  with  $2^{-(2n+k)}r < \varepsilon$ . Since  $M_r(A)$  is simple with real rank zero, Theorem 1.1(i) of [33] gives projections  $e, f \in M_r(A)$  such that  $2^{n+k}[e] + [f] = [p]$  in  $K_0(A)$  and  $f$  is Murray-von Neumann equivalent to a subprojection of  $e$ . As functions on  $T(A)$ , we therefore have

$$2^k\rho_A([e]) \leq 2^{-n}\rho_A([p]) \leq (2^k + 2^{-n})\rho_A([e]).$$

Because  $2^{n+k}\rho_A([e]) \leq \rho_A([p]) \leq r$ , this gives

$$\|2^k\rho_A([e]) - 2^{-n}\rho_A([p])\|_{\infty} \leq 2^{-n} \left( \frac{r}{2^{n+k}} \right) < \varepsilon.$$

Since  $2^k\rho_A([e]) \in \rho_A(K_0(A))$ , we are done. ■

An earlier result, Lemma III.3.4 of [3], states that if  $A$  is a stably finite  $C^*$ -algebra with “stable (HP)” (now known to be equivalent to real rank zero; see [6]), with cancellation of projections, with no finite dimensional representations, and such that  $K_0(A)$  is weakly unperforated, then  $\rho_A(K_0(A)_+)$  is dense in  $\text{Aff}(T(A))_+$ . This result applies in particular to any infinite dimensional unital simple direct limit of recursive subhomogeneous algebras with no dimension growth.

**Proposition 1.11.** Let  $A$  be any infinite dimensional simple unital  $C^*$ -algebra with Property (SP). Then for every  $\varepsilon > 0$  there is  $\eta \in K_0(A)$  such that  $0 < \tau_*(\eta) < \varepsilon$  for all tracial states  $\tau$  on  $A$ .

*Proof.* Let  $\varepsilon > 0$ . Choose  $n \in \mathbf{N}$  with  $\frac{1}{n} < \varepsilon$ , and use Lemma 3.2 of [18] to find  $n$  nonzero mutually orthogonal projections  $q_1, q_2, \dots, q_n \in A$  whose  $K_0$ -classes are all equal. Take  $\eta$  to be this common  $K_0$ -class. ■

## 2. HEREDITARY SUBALGEBRAS OF RECURSIVE SUBHOMOGENEOUS C\*-ALGEBRAS

The purpose of this section is to prove that the unitization of a hereditary subalgebra of a recursive subhomogeneous algebra has a useful recursive subhomogeneous decomposition.

We let  $A^+$  denote the unitization of the C\*-algebra  $A$ ; a new identity is added even if  $A$  already has an identity.

**Lemma 2.1.** Let  $A$ ,  $B$ , and  $C$  be C\*-algebras, and let  $\alpha: A \rightarrow C$  and  $\beta: B \rightarrow C$  be homomorphisms. Then the obvious map  $A \oplus_{C,\alpha,\beta} B \rightarrow A^+ \oplus_{C^+,\alpha^+,\beta^+} B^+$  determines an isomorphism  $(A \oplus_{C,\alpha,\beta} B)^+ \cong A^+ \oplus_{C^+,\alpha^+,\beta^+} B^+$ .

The proof is easy.

**Lemma 2.2.** Let  $B$ ,  $C$ , and  $D$  be C\*-algebras, let  $\varphi: B \rightarrow D$  be a homomorphism, and let  $\rho: C \rightarrow D$  be a surjective homomorphism. Let  $A = B \oplus_D C$ , and let  $\kappa: A \rightarrow B$  and  $\pi: A \rightarrow C$  be the projection maps. Let  $A_0$  be a hereditary subalgebra of  $A$ , and let  $B_0$ ,  $C_0$ , and  $D_0$  be the hereditary subalgebras of  $B$ ,  $C$ , and  $D$  generated by  $\kappa(A_0)$ ,  $\pi(A_0)$ , and  $\varphi \circ \kappa(A_0) = \rho \circ \pi(A_0)$  respectively. Let  $\tilde{\varphi}: B_0 \rightarrow D_0$  and  $\tilde{\rho}: C_0 \rightarrow D_0$  be the restrictions of  $\varphi$  and  $\rho$ . Then  $\tilde{\rho}$  is surjective, and  $A_0$  is canonically isomorphic to  $B_0 \oplus_{D_0} C_0$ .

*Proof.* Since  $\rho$  is surjective, the image under  $\rho$  of a hereditary subalgebra is again a hereditary subalgebra. Therefore  $\tilde{\rho}$  is surjective.

It is obvious that the canonical image in  $A = B \oplus_D C$  of  $B_0 \oplus_{D_0} C_0$  contains  $A_0$ . It therefore suffices to prove the reverse inclusion. So let  $a = (b, c) \in B_0 \oplus_{D_0} C_0$ . Let  $(e_\lambda)_{\lambda \in \Lambda}$  be an approximate identity for  $A_0$ . Then  $e_\lambda = (\kappa(e_\lambda), \pi(e_\lambda))$ , and  $(\kappa(e_\lambda))_{\lambda \in \Lambda}$  and  $(\pi(e_\lambda))_{\lambda \in \Lambda}$  are approximate identities for  $\kappa(A_0)$  and  $\pi(A_0)$ . So they are also approximate identities for the hereditary subalgebras  $B_0$  and  $C_0$  generated by  $\kappa(A_0)$  and  $\pi(A_0)$ . Therefore  $\kappa(e_\lambda)b\kappa(e_\lambda) \rightarrow b$  and  $\pi(e_\lambda)c\pi(e_\lambda) \rightarrow c$ , whence  $e_\lambda a e_\lambda \rightarrow a$ . Also  $e_\lambda a e_\lambda \in A_0$  because  $A_0$  is hereditary. So  $a \in A_0$ , as desired. ■

**Corollary 2.3.** Let

$$A = \left[ \cdots \left[ \left[ C(X_0, M_{n_0}) \oplus_{C(X_1^{(0)}, M_{n(1)})} C(X_1, M_{n(1)}) \right] \right. \right. \\ \left. \left. \oplus_{C(X_2^{(0)}, M_{n(2)})} C(X_2, M_{n(2)}) \right] \cdots \right] \oplus_{C(X_l^{(0)}, M_{n(l)})} C(X_l, M_{n(l)}).$$

be a recursive subhomogeneous algebra, with unital maps

$$\varphi_k: C(X_{k-1}, M_{n(k-1)}) \rightarrow C(X_k^{(0)}, M_{n(k)})$$

and restriction maps

$$\rho_k: C(X_k, M_{n(k)}) \rightarrow C(X_k^{(0)}, M_{n(k)}).$$

Let  $B \subset A$  be a hereditary subalgebra. Let  $B_k$  and  $B_k^{(0)}$  be the hereditary subalgebras of  $C(X_k, M_{n(k)})$  and  $C(X_k^{(0)}, M_{n(k)})$  generated by the images of  $B$  in these algebras, and let  $\tilde{\varphi}_k: B_{k-1} \rightarrow B_k^{(0)}$  and  $\tilde{\rho}_k: B_k \rightarrow B_k^{(0)}$  be the restrictions of  $\varphi_k$  and  $\rho_k$ . Then each  $\tilde{\rho}_k$  is surjective, and  $B$  is canonically isomorphic to the iterated pullback

$$\left[ \cdots \left[ \left[ B_0 \oplus_{B_1^{(0)}} B_1 \right] \oplus_{B_2^{(0)}} B_2 \right] \cdots \oplus_{B_l^{(0)}} B_l \right]$$

with respect to the maps  $\tilde{\varphi}_k$  and  $\tilde{\rho}_k$ .

*Proof.* This follows from the previous lemma by induction. ■

**Lemma 2.4.** Let  $X$  be a compact metric space with  $\dim(X) \leq d$ . Let  $B \subset C(X, M_n)$  be a hereditary subalgebra. Then  $B^+$  has a recursive subhomogeneous decomposition with topological dimension at most  $d$ .

*Proof.* By Theorem 5.5.5 of [22], the primitive ideal space  $\text{Prim}(B)$  can be identified with the primitive ideal space of the ideal  $I$  in  $C(X, M_n)$  generated by  $B$ , which is an open subset  $U \subset X$ . For each  $m$ , let  $\text{Prim}_m(A)$  denote the subspace of  $\text{Prim}(A)$  consisting of the kernels of  $m$  dimensional representations of  $A$ . Then  $\dim(\text{Prim}_m(B)) \leq \dim(X) \leq d$  by Theorems 1.1.2 and 1.7.7 of [13].

If  $m \neq 1$ , then  $\text{Prim}_m(B^+) = \text{Prim}_m(B)$ . Also,  $\text{Prim}_1(B^+)$  is the union of  $\text{Prim}_1(B)$  and the one point set whose element is the kernel of the unitization map  $B^+ \rightarrow \mathbf{C}$ . Corollary 1.5.6 and Theorem 1.7.7 of [13] now imply that  $\dim(\text{Prim}_1(B^+)) \leq d$ .

We now apply (3)  $\implies$  (2) of Theorem 2.16 of [26] to conclude that  $B^+$  has a recursive subhomogeneous decomposition with topological dimension at most  $d$ . ■

**Proposition 2.5.** Let  $A$  be a separable recursive subhomogeneous algebra having a recursive subhomogeneous decomposition with topological dimension at most  $d$  and total space  $X$ . Let  $B \subset A$  be a hereditary subalgebra. Then  $B^+$  is a recursive subhomogeneous algebra, and has a recursive subhomogeneous decomposition with topological dimension at most  $d$ . If  $\inf_{x \in X} \text{rank}(\text{ev}_x(B)) \geq 2$ , then the recursive subhomogeneous decomposition for  $B^+$  can in addition be chosen to have base spaces  $Y_0, Y_1, \dots, Y_m$  such that  $Y_0$  consists of a single point and the matrix size on every other  $Y_k$  is at least  $\inf_{x \in X} \text{rank}(\text{ev}_x(B))$ .

*Proof.* That  $B^+$  has a recursive subhomogeneous decomposition with topological dimension at most  $d$  follows by induction from Proposition 3.2 of [26], Lemma 2.4, Corollary 2.3, and Lemma 2.1.

We prove the last statement. As in the previous proof, for any C\*-algebra  $A$  let  $\text{Prim}_m(A)$  denote the subspace of  $\text{Prim}(A)$  consisting of the kernels of  $m$  dimensional representations of  $A$ .

Recall, from the constructions in Section 2 of [26] leading up to the proof of Theorem 2.16 there, that if  $A$  is a separable recursive subhomogeneous algebra, then  $A$  is isomorphic to an iterated pullback

$$\left[ \cdots \left[ \left[ A_0 \oplus_{A_1^{(0)}} A_1 \right] \oplus_{A_2^{(0)}} A_2 \right] \cdots \oplus_{A_l^{(0)}} A_l \right],$$

in which each  $A_m$  is the section algebra of a locally trivial bundle with fiber  $M_m$  and base space  $X_m$  equal to a suitable compactification of  $\text{Prim}_m(A)$ . (The space  $\text{Prim}_m(A)$  is finite dimensional, as in the proof of Lemma 2.4. Therefore the corresponding subquotient is a the section algebra of a locally trivial continuous field of finite type, using Lemma 2.5, Lemma 2.6, and Theorem 2.12 of [26]. Now apply Lemma 2.11 of [26] and induction.) Taking  $B^+$  for  $A$ , the assumption on the ranks implies that  $\text{Prim}_1(B^+)$ , and hence also  $X_1$ , consists of a single point. Moreover, from the proof of Proposition 1.7 of [26], one sees that for any collection of finite closed covers  $\mathcal{F}_m$  of the spaces  $X_m$  over whose sets the corresponding bundles are trivial, this recursive subhomogeneous algebra has a recursive subhomogeneous decomposition whose base spaces are exactly the sets in  $\bigcup_{m=1}^l \mathcal{F}_m$ , and such that the

matrix size over such a space  $Y$  is  $m$  when  $Y \subset X_m$ . We therefore produce a recursive subhomogeneous decomposition as demanded in the last statement simply by choosing  $\mathcal{F}_1 = \{X_1\}$ . ■

### 3. INTERPOLATION BY PROJECTIONS

The main result of this section is Proposition 3.5, in which we show that if  $A = \varinjlim A_n$  is a direct limit of recursive subhomogeneous algebras which should have real rank zero, and if  $a, b, c$  are positive elements in one of the algebras of the system such that  $ba = a$  and  $cb = b$ , then there is a projection  $p$  in an algebra farther out in the system such that  $pa = a$  and  $cp = p$ . As in previous work with recursive subhomogeneous algebras and direct limits of them [26], [27], it is necessary to be able to extend standard constructions in  $C(X, M_n)$  when values on a closed subset of  $X$  are already specified.

The following lemma is a variant of Proposition 3.1 of [27].

**Lemma 3.1.** Let  $X$  be a compact Hausdorff space, and let  $a, b, c \in C(X, M_n)_{sa}$  be positive elements such that  $ba = a$  and  $cb = b$ . Then there exist open sets  $U_k$ , for  $0 \leq k \leq n$ , and continuous rank  $k$  projections  $p_k: U_k \rightarrow M_n$ , such that:

- (1)  $\bigcup_{k=0}^n U_k = X$ .
- (2) If  $k \leq l$  and  $x \in U_k \cap U_l$ , then  $p_k(x) \leq p_l(x)$ .
- (3) For all  $x \in U_k$ , we have  $p_k(x)a(x) = a(x)$  and  $c(x)p_k(x) = p_k(x)$ .

*Proof.* Without loss of generality  $\|b\| \leq 1$ . For  $x \in X$ , write the eigenvalues of  $b(x)$  as

$$\beta_1(x) \geq \beta_2(x) \geq \cdots \geq \beta_n(x)$$

(repeated according to multiplicity). It follows from Theorem 8.1 of [1] that the  $\beta_k$  are continuous functions on  $X$ . Further set  $\beta_0(x) = 1$  and  $\beta_{n+1}(x) = 0$  for all  $x$ . For  $0 \leq k \leq n$  define

$$\lambda_k(x) = \frac{1}{2}[\beta_k(x) + \beta_{k+1}(x)] \quad \text{and} \quad U_k = \{x \in X : \beta_k(x) > \lambda_k(x) > \beta_{k+1}(x)\}.$$

Then use functional calculus to define  $p_k(x) = \chi_{(\lambda_k(x), \infty)}(b(x))$  for  $x \in U_k$ .

We verify that these sets and projections satisfy the conclusion of the proposition. The  $U_k$  are open because the functions  $\beta_k$  and  $\lambda_k$  are continuous. They cover  $X$  because the relations  $\beta_0(x) = 1$  and  $\beta_{n+1}(x) = 0$  show that the  $\beta_k(x)$  are not all equal. To see that  $p_k$  is continuous, rewrite  $p_k(x) = f_x(b(x))$ , where

$$f_x(t) = \begin{cases} 1 & t \geq \beta_k(x) \\ \frac{t - \beta_{k+1}(x)}{\beta_k(x) - \beta_{k+1}(x)} & \beta_k(x) \geq t \geq \beta_{k+1}(x) \\ 0 & \beta_{k+1}(x) \geq t. \end{cases}$$

The function  $(t, x) \mapsto f_x(t)$  is jointly continuous, so  $x \mapsto f_x(a(x))$  is continuous by Proposition 2.12 of [25]. Clearly  $\text{rank}(p_k(x)) = k$  for all  $x$ . It is also obvious that if  $k \leq l$  then  $p_k(x) \leq p_l(x)$  wherever both are defined.

We verify part (3). First consider  $k = 0$ . If  $x \in U_0$ , then  $p_0(x) = 0$ , so trivially  $c(x)p_0(x) = p_0(x)$ . Moreover,  $\beta_1(x) < 1$ , so  $a(x) = 0$ , whence  $p_0(x)a(x) = a(x)$ .

Next suppose  $k = n$ . If  $x \in U_n$ , then  $p_n(x) = 1$ , so trivially  $p_n(x)a(x) = a(x)$ . Moreover,  $\beta_n(x) > 0$ , so  $c(x) = 1$ , whence  $c(x)p_n(x) = p_n(x)$ .

Finally, let  $1 \leq k \leq n - 1$  and let  $x \in U_k$ . Then  $0 < \lambda_k(x) < 1$ . By functional calculus, there is a sequence  $(g_r)_{r \in \mathbf{N}}$  of polynomials with real coefficients and no



constant term, such that  $p_k(x) = \lim_{r \rightarrow \infty} g_r(b(x))$ . From  $b(x)a(x) = a(x)$  we get  $g_r(b(x))a(x) = a(x)$  for all  $r$ , whence  $p_k(x)a(x) = a(x)$ . Similarly,  $c(x)b(x) = b(x)$  implies  $c(x)g_r(b(x)) = b(x)$  for all  $r$  and  $c(x)p_k(x) = p_k(x)$ . ■

The following lemma is an approximate relative version of Lemma C of [2]. Note that in the hypotheses we start with  $avv^* = vv^*$  on  $X^{(0)}$ , but in the conclusion we only have  $cvv^* = vv^*$  on  $X$ .

**Lemma 3.2.** Let  $X$  be a compact Hausdorff space, and let  $a, b, c \in C(X, M_n)_{\text{sa}}$  be positive elements such that  $ba = a$  and  $cb = b$ . Let  $p \in C(X, M_n)$  be a projection such that

$$\text{rank}(p(x)) \leq \text{rank}(a(x)) - \frac{1}{2}(\dim(X) - 1)$$

for all  $x \in X$ . Let  $X^{(0)} \subset X$  be closed, and let  $v^{(0)} \in C(X^{(0)}, M_n)$  be a partial isometry such that

$$(v^{(0)})^* v^{(0)} = p|_{X^{(0)}} \quad \text{and} \quad (a|_{X^{(0)}})v^{(0)}(v^{(0)})^* = v^{(0)}(v^{(0)})^*.$$

Then there is a partial isometry  $v \in C(X, M_n)$  such that

$$v|_{X^{(0)}} = v^{(0)}, \quad v^*v = p, \quad \text{and} \quad cvv^* = vv^*.$$

*Proof.* By partitioning the space  $X$ , without loss of generality  $p$  has constant rank, say  $r$ . Also set  $d = \dim(X)$  and  $q^{(0)} = v^{(0)}(v^{(0)})^*$ . We are then assuming that

$$r \leq \text{rank}(a(x)) - \frac{1}{2}(d - 1) \quad \text{and} \quad (a|_{X^{(0)}})q^{(0)} = q^{(0)}.$$

For  $0 \leq k \leq n$ , choose  $U_k$  and  $p_k$  as in Lemma 3.1. For  $x \in U_k \cap X^{(0)}$ , we then have

$$p_k(x)q^{(0)}(x) = p_k(x)a(x)q^{(0)}(x) = a(x)q^{(0)}(x) = q^{(0)}(x),$$

that is,  $q^{(0)}(x) \leq p_k(x)$ . Also,  $c(x)p_k(x) = p_k(x)$  whenever  $x \in U_k$ . Moreover, if  $x \in U_k$  then  $p_k(x)a(x) = a(x)$  implies  $\text{rank}(p_k(x)) \geq \text{rank}(a(x))$ , whence

$$\text{rank}(p_k(x)) - \text{rank}(p(x)) \geq \text{rank}(a(x)) - \text{rank}(p(x)) \geq \frac{1}{2}(d - 1).$$

Let  $f_0, f_1, \dots, f_n$  be a partition of unity on  $X$  such that  $\text{supp}(f_k) \subset U_k$ . Then the sets  $X_k = \text{supp}(f_k)$  are closed subsets of  $X$ , with  $X_k \subset U_k$ , which still cover  $X$ . Set  $Y_k = X^{(0)} \cup X_0 \cup X_1 \cup \dots \cup X_k$ . Note that  $Y_n = X$ . We construct the partial isometry  $v_k = v|_{Y_k}$ , satisfying  $v_k|_{X^{(0)}} = v^{(0)}$  and  $v_k^*v_k = p|_{Y_k}$ , as well as

$$(v_k v_k^*)|_{X_j} \leq p_j|_{X_j}$$

for  $j \leq k$ , by induction on  $k$ .

Let  $k_0$  be the least integer such that  $X_{k_0} \neq \emptyset$ . For  $k < k_0$ , we have  $Y_k = X^{(0)}$ , and we simply take  $v_k = v^{(0)}$ . Suppose now we have  $v_{k-1}$ , and that  $k \geq k_0$ ; we construct  $v_k$ . Since  $p_{k_0}(x)a(x) = a(x)$  for  $x \in X_{k_0}$  and  $r \leq \text{rank}(a(x)) - \frac{1}{2}(d - 1)$ , we have  $k - r \geq \frac{1}{2}(d - 1)$ . Apply Proposition 4.2(1) of [26] with  $X_k$  in place of  $X$ , with  $X^{(0)} \cap X_k$  in place of  $Y$ , with  $p_k|_{X_k}$  in place of  $p$ , with  $p|_{X_k}$  in place of  $q$ , and with  $v^{(0)}|_{X^{(0)} \cap X_k}$  in place of  $s_0$ . Let  $s \in C(X_k, M_n)$  be the partial isometry resulting from the application of this proposition. Thus,

$$s^*s = p|_{X_k}, \quad ss^* \leq p_k|_{X_k}, \quad \text{and} \quad s|_{X^{(0)} \cap X_k} = v^{(0)}|_{X^{(0)} \cap X_k}.$$

Noting that  $Y_k = Y_{k-1} \cup X_k$ , define  $v_k: Y_k \rightarrow M_n$  by

$$v_k(x) = \begin{cases} s(x) & x \in X_k \\ v_{k-1}(x) & x \in Y_{k-1}. \end{cases}$$

This is well defined, continuous, and clearly satisfies

$$v_k|_{X^{(0)}} = v^{(0)}, \quad v_k^*v_k = p|_{Y_k}, \quad \text{and} \quad (v_kv_k^*)|_{X_k} \leq p_k|_{X_k}.$$

The relation

$$(v_kv_k^*)|_{X_j} \leq p_j|_{X_j},$$

for  $j < k$ , follows from the assumption on  $v_{k-1}$ . This completes the induction step.

Now take  $v = v_n$ . That  $v|_{X^{(0)}} = v^{(0)}$  and  $v^*v = p$  are clear. For  $x \in X$ , choose  $k$  such that  $x \in Y_k$ . Then, because  $v(x)v(x)^* \leq p_k(x)$ , we have

$$c(x)v(x)v(x)^* = c(x)p_k(x)v(x)v(x)^* = p_k(x)v(x)v(x)^* = v(x)v(x)^*.$$

Thus,  $cvv^* = vv^*$ . ■

The following lemma is the analog for recursive subhomogeneous algebras of Lemma C of [2]. In the hypotheses, however, we must assume ahead of time the existence of a projection of the right “size”. Without such an assumption, the lemma is false, since a recursive subhomogeneous algebra need have no nontrivial projections at all.

**Lemma 3.3.** Let  $A$  have a recursive subhomogeneous decomposition with total space  $X$  and topological dimension function  $d: X \rightarrow \mathbf{N} \cup \{0\}$ . Let  $a, b, c \in A_{\text{sa}}$  be positive elements such that  $ba = a$  and  $cb = b$ . Let  $p \in A$  be a projection such that

$$\text{rank}_x(p) \leq \text{rank}(\text{ev}_x(a)) - \frac{1}{2}(d(x) - 1)$$

for all  $x \in X$ . Then there is a partial isometry  $v \in A$  such that  $v^*v = p$  and  $cvv^* = vv^*$ .

*Proof.* The proof is by induction on the length  $l$  of the recursive subhomogeneous decomposition. If  $l = 0$  then  $A = C(X_0, M_n)$ , and this is the case  $X^{(0)} = \emptyset$  of Lemma 3.2.

For the general case, write  $A = A_0 \oplus_{C(X^{(0)}, M_n)} C(X, M_n)$ , with respect to a unital homomorphism  $\varphi: A_0 \rightarrow C(X^{(0)}, M_n)$  and the restriction map  $\rho: C(X, M_n) \rightarrow C(X^{(0)}, M_n)$ . Further let  $\pi: A \rightarrow A_0$  and  $\kappa: A \rightarrow C(X, M_n)$  be the obvious projections. Assume  $A_0$  is given with a recursive subhomogeneous decomposition of length  $l - 1$ , so that the conclusion of the lemma is known to hold in  $A_0$ . We then prove it for  $A$ . Note that the total space of this recursive subhomogeneous decomposition for  $A$  is the disjoint union of  $X$  and the total space of  $A_0$ , and that topological dimension function for  $A$  is equal to  $\dim(X)$  on  $X$  and equal to the topological dimension function for  $A_0$  on the total space of  $A_0$ .

Define continuous functions  $f, g, h: [0, \infty) \rightarrow [0, 1]$  as follows:

$$f(t) = \begin{cases} 0 & 0 \leq t \leq \frac{2}{3} \\ 3t - 2 & \frac{2}{3} \leq t \leq 1 \\ 1 & 1 \leq t, \end{cases}$$

$$g(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{3} \\ 3t - 1 & \frac{1}{3} \leq t \leq \frac{2}{3} \\ 1 & \frac{2}{3} \leq t, \end{cases}$$

and

$$h(t) = \begin{cases} 3t & 0 \leq t \leq \frac{1}{3} \\ 1 & \frac{1}{3} \leq t. \end{cases}$$

Then  $f(b), g(b), h(b) \in A_{\text{sa}}$  are positive elements satisfying  $g(b)f(b) = f(b)$  and  $h(b)g(b) = g(b)$ . Moreover, approximating  $h$  by polynomials with no constant term, we see that  $ch(b) = h(b)$ , and similarly  $f(b)a = a$ .

Apply the induction assumption to  $A_0$ , with  $\pi(a), \pi(f(b))$ , and  $\pi(g(b))$  in place of  $a, b$ , and  $c$ , and with  $\pi(p)$  in place of  $p$ . This gives a partial isometry  $v_0 \in A_0$  such that  $v_0^*v_0 = \pi(p)$  and  $\pi(g(b))v_0v_0^* = v_0v_0^*$ . In particular,  $\pi(c)v_0v_0^* = v_0v_0^*$ . Now apply Lemma 3.2 with  $X$  and  $X^{(0)}$  as given, with  $\kappa(g(b)), \kappa(h(b))$ , and  $\kappa(c)$  in place of  $a, b$ , and  $c$ , with  $\kappa(p)$  in place of  $p$ , and with  $\varphi(v_0)$  in place of  $v^{(0)}$ . This gives a partial isometry in  $C(X, M_n)$ , which we call  $v_1$ , such that

$$v_1^*v_1 = \kappa(p), \quad \kappa(c)v_1v_1^* = v_1v_1^*, \quad \text{and} \quad \rho(v_1) = \varphi(v_0).$$

Set  $v = (v_0, v_1)$ , which is in  $A$  by construction. We have  $cvv^* = vv^*$  because  $\pi(c)v_0v_0^* = v_0v_0^*$  and  $\kappa(c)v_1v_1^* = v_1v_1^*$ . ■

**Lemma 3.4.** Let  $A = \varinjlim A_n$  be an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2), and such that the maps of the system are unital and injective. Let  $\varphi_{k,l}: A_k \rightarrow A_l$  and  $\varphi_k: A_k \rightarrow A$  be the maps associated with the direct limit. Let  $a \in A_k$  satisfy  $0 \leq a \leq 1$ , and let  $p \in A_k$  be a projection such that  $\tau(\varphi_k(a)) - \tau(\varphi_k(p)) > 0$  for all  $\tau \in T(A)$ . Let  $M \in \mathbf{N}$ . Then, for all sufficiently large  $n$ , the images  $\varphi_{k,n}(a)$  and  $\varphi_{k,n}(p)$  satisfy

$$\text{rank}(\text{ev}_x(\varphi_{k,n}(a))) - \text{rank}(\text{ev}_x(\varphi_{k,n}(p))) \geq M$$

for every  $x$  in the total space of  $A_n$ .

*Proof.* Without loss of generality  $k = 0$ . Suppose the lemma fails. By passing to a subsystem, we may assume that for every  $n$  there is some  $x_n$  in the total space  $X_n$  of  $A_n$  such that

$$\text{rank}(\text{ev}_{x_n}(\varphi_{0,n}(a))) - \text{rank}(\text{ev}_{x_n}(\varphi_{0,n}(p))) < M.$$

Let  $\text{tr}_n$  be the tracial state on the codomain of  $\text{ev}_{x_n}$ , and define a tracial state  $\tau_n: A_n \rightarrow C$  by  $\tau_n = \text{tr}_n \circ \text{ev}_{x_n}$ . Since  $\varphi_n$  is injective, we may regard  $\tau_n$  as a state on a subalgebra of  $A$ . Use the Hahn-Banach Theorem to extend to a state  $\omega_n$  on  $A$  such that  $\omega_n \circ \varphi_n = \tau_n$ . By Alaoglu's Theorem, the sequence  $(\omega_n)_{n \in \mathbf{N}}$  has a weak\* limit point  $\tau$ . Clearly  $\tau|_{\varphi_n(A_n)}$  is a tracial state (being the pointwise limit of tracial states), so  $\tau$  is a tracial state.

Let  $m_n$  be the minimum matrix size in the recursive subhomogeneous decomposition of  $A_n$ , and note that  $\lim_{n \rightarrow \infty} m_n = \infty$  by Lemma 1.8 of [27]. We have

$$\begin{aligned} \omega_n(\varphi_0(a)) - \omega_n(\varphi_0(p)) &= \text{tr}_n(\text{ev}_{x_n}(\varphi_{0,n}(a))) - \text{tr}_n(\text{ev}_{x_n}(\varphi_{0,n}(p))) \\ &\leq \frac{1}{m_n} [\text{rank}(\text{ev}_{x_n}(\varphi_{0,n}(a))) - \text{rank}(\text{ev}_{x_n}(\varphi_{0,n}(p)))] \leq \frac{M}{m_n}. \end{aligned}$$

Therefore  $\lim_{n \rightarrow \infty} [\omega_n(\varphi_0(a)) - \omega_n(\varphi_0(p))] \leq 0$ , whence  $\tau(\varphi_0(a)) - \tau(\varphi_0(p)) \leq 0$ . This is a contradiction. ■

Part of the proof of the following proposition follows the proof of Lemma E of [2].

**Proposition 3.5.** Let  $A = \varinjlim A_n$  be an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2), and such that the maps of the system are unital and injective. Let  $\varphi_{k,l}: A_k \rightarrow A_l$  be the maps of the direct system. Assume that  $\rho_A(K_0(A))$  is dense

in  $\text{Aff}(T(A))$ . Let  $a, b, c \in (A_k)_{\text{sa}}$  be positive elements such that  $ba = a$  and  $cb = b$ . Then there exists  $n \geq k$  and a projection  $p \in A_n$  such that  $p\varphi_{k,n}(a) = \varphi_{k,n}(a)$  and  $\varphi_{k,n}(c)p = p$ .

*Proof.* We identify all  $A_k$  with their images in  $A$ , so that  $A_0 \subset A_1 \subset \cdots \subset A$ , and we must prove that there is a projection  $p \in A_n$  such that  $pa = a$  and  $cp = p$ . Without loss of generality  $k = 0$  and  $0 \leq a \leq b \leq c \leq 1$ . We also assume  $a \neq 0$  and  $b \neq 1$ , since otherwise  $p = 0$  or  $p = 1$  will satisfy the conclusion. If  $\text{sp}(b) \neq [0, 1]$ , then choose  $\alpha \in [0, 1] \setminus \text{sp}(b)$ , and take  $n = 0$  and  $p = \chi_{[\alpha, \infty)}(b) \in A_0$ . Therefore we may assume that  $\text{sp}(b) = [0, 1]$ .

In a manner similar to the proof of Lemma 3.3 (but using more functions), we find  $b_1, b_2, \dots, b_7 \in (A_0)_{\text{sa}}$  such that

$$0 \leq a \leq b_1 \leq b_2 \leq \cdots \leq b_7 \leq c \leq 1,$$

and such that

$$b_{j+1} \neq b_j, \quad b_{j+1}b_j = b_j, \quad b_1a = a, \quad \text{and} \quad cb_7 = b_7.$$

Since  $A$  is simple, all tracial states are faithful, and compactness of  $T(A)$  provides  $\varepsilon > 0$  such that

$$\inf_{\tau \in T(A)} \tau(b_4 - b_3) > \varepsilon \quad \text{and} \quad \inf_{\tau \in T(A)} \tau(b_5 - b_4) > \varepsilon.$$

Because  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$  in the supremum norm, there is  $\eta \in K_0(A)$  such that  $|\tau_*(\eta) - \tau(b_4)| < \frac{1}{2}\varepsilon$  for all tracial states  $\tau$ . In particular,

$$0 < \tau(b_3) < \tau_*(\eta) < \tau(b_5) < 1$$

for all tracial states  $\tau$ . Because the order on  $K_0(A)$  is determined by traces (Theorem 2.3 of [27]), there exists a projection  $q \in A$  such that  $[q] = \eta$  in  $K_0(A)$ . In fact, we may assume that  $q \in A_{n_0}$  for some  $n_0$ . Note that

$$\tau(b_3) + \frac{1}{2}\varepsilon < \tau(q) < \tau(b_5) - \frac{1}{2}\varepsilon$$

for all  $\tau \in T(A)$ .

By assumption, there is an integer  $d$  such that the given recursive subhomogeneous decomposition of every  $A_n$  has topological dimension at most  $d$ . By Lemma 3.4, there is  $n_1 \geq n_0$  such that for every  $m \geq n_1$  and every  $x$  in the total space  $X_m$  of  $A_m$ , we have

$$\text{rank}_x(b_5) - \left(\frac{1}{2}d + 2\right) \geq \text{rank}_x(q) \quad \text{and} \quad \text{rank}_x(1 - b_3) - \left(\frac{1}{2}d + 2\right) \geq \text{rank}_x(1 - q).$$

We apply Lemma 3.3 in  $A_{n_1}$  twice, the first time with  $b_5, b_6,$  and  $b_7$  in place of  $a, b,$  and  $c,$  and the second time with  $1 - b_3, 1 - b_2,$  and  $1 - b_1$  in place of  $a, b,$  and  $c.$  We obtain projections  $e_0, f \in A_{n_1},$  with  $f$  Murray-von Neumann equivalent to  $q$  and  $e_0$  Murray-von Neumann equivalent to  $1 - q,$  such that

$$b_7f = f \quad \text{and} \quad (1 - b_1)e_0 = e_0.$$

Because projections in  $A$  satisfy cancellation (Theorem 2.2 of [27]), there is  $n_2 \geq n_1$  such that  $e = 1 - e_0$  is Murray-von Neumann equivalent to  $q$  in  $A_{n_2}.$  Then  $e, f \in A_{n_2}$  are both Murray-von Neumann equivalent to  $q,$  and

$$b_7f = f \quad \text{and} \quad eb_1 = b_1.$$

Choose  $v \in A_{n_2}$  such that  $v^*v = e$  and  $vv^* = f.$  Define  $r = vb_1.$  Then

$$r^*r = b_1^*v^*vb_1 \leq b_1^2 \leq b_1 \leq b_7 \quad \text{and} \quad rr^* = vb_1b_1^*v^* \leq vv^* \leq b_7.$$

Therefore  $r$  is in the hereditary subalgebra  $B \subset A_{n_2}$  generated by  $b_7$ .

We now study the subalgebra  $B^+$  of  $A_{n_2}$ . Let  $X_{n_2}$  be the total space of  $A_{n_2}$ . For  $x \in X_{n_2}$ , we note that the matrix size  $\text{rank}_x(B)$  of  $\text{ev}_x(B)$  satisfies

$$\text{rank}_x(B) = \text{rank}(\text{ev}_x(b_7)) \geq \text{rank}(\text{ev}_x(b_5)) \geq \frac{1}{2}d + 2.$$

In particular,  $\text{rank}_x(B) \geq 2$  for all  $x \in X_{n_2}$ . Lemma 2.5 implies that  $B^+$  has a recursive subhomogeneous decomposition with base spaces  $Y_0, Y_1, \dots, Y_l$  and total space  $Y$ , such that  $\dim(Y_k) \leq d$  for all  $k$ , such that  $Y_0$  is a one point space, and such that the matrix size on every  $Y_k$ , for  $k > 0$ , is at least  $\frac{1}{2}d + 2$ .

Let  $d_Y$  be the topological dimension function for this recursive subhomogeneous decomposition of  $B^+$ . We claim that

$$\text{rank}_y(B^+) - \text{rank}_y(r) \geq \frac{1}{2}d_Y(y)$$

for all  $y \in Y$ . So let  $y \in Y$ . Write  $\text{ev}_y$  as a direct sum of irreducible representations  $\bigoplus_{j=1}^k \pi_j$ . There are three cases.

First, suppose that  $y \in Y_0$ . Then  $d_Y(y) = 0$ , so the right hand side of the desired inequality is zero. The left hand side is nonnegative because  $r \in B^+$ , so the inequality holds.

Next, suppose that  $y \notin Y_0$  but that every  $\pi_j$  is equivalent to the map  $B^+ \rightarrow \mathbf{C}$  coming from the unitization. Then  $\pi(r) = 0$  since  $r \in B$ , and  $\text{rank}_y(B) \geq \frac{1}{2}d + 2$  by the above.

Finally, suppose that some  $\pi_j$  is not equivalent to the unitization map. It suffices to prove that

$$\text{rank}(\pi_j(B)) - \text{rank}(\pi_j(r)) \geq \frac{1}{2}d$$

for this representation  $\pi_j$ , since at least  $\text{rank}(\pi_i(B^+)) - \text{rank}(\pi_i(r)) \geq 0$  for all other  $i$ . Now  $\pi_j|_B$  is an irreducible representation of  $B$ . Because  $B$  is a hereditary subalgebra, there is some irreducible representation  $\sigma$  of  $A$  whose restriction to  $B$  is the direct sum of  $\pi_j|_B$  and a zero representation. By Lemma 2.1 of [26], we may assume that  $\sigma = \text{ev}_x$  for some  $x \in X_{n_2}$ . Now

$$\text{rank}(\pi_j(B)) - \text{rank}(\pi_j(r)) = \text{rank}_x(B) - \text{rank}_x(r) \geq \text{rank}_x(b_7) - \text{rank}_x(b_1) \geq \frac{1}{2}d,$$

as desired. The claim is proved.

By Proposition 3.4 of [27], for every  $\varepsilon > 0$  there is a unitary  $u \in B^+$  such that  $\|r - u(r^*r)^{1/2}\| < \varepsilon$ . Therefore  $r$  is a norm limit of invertible elements in  $B^+$ .

Let  $r = s|r| = s(r^*r)^{1/2}$  be the polar decomposition of  $r$  in the second dual  $(B^+)^{''}$ . Choose continuous functions  $h, h_0: [0, \infty) \rightarrow [0, \infty)$  such that  $h$  vanishes on a neighborhood of zero and  $h(1) = 1$ , and such that  $h_0$  vanishes on a (smaller) neighborhood of zero and  $th_0(t) = h(t)^{1/2}$  for all  $t$ . Since  $r \in \overline{\text{inv}(B^+)}$ , Corollary 8 of [24] provides a unitary  $w \in B^+$  such that  $wh_0(|r|) = sh_0(|r|)$ . Then

$$rh_0(|r|) = sh_0(|r|)|r| = wh_0(|r|)|r| = wh(|r|)^{1/2}.$$

Therefore

$$rh_0(|r|)^2 r^* = wh(|r|)w^*.$$

Using polynomial approximations to the function  $t \mapsto h_0(t^{1/2})^2$ , we get

$$wh(|r|)w^* = rh_0(|r|)^2 r^* = h_0((r^*r)^{1/2})^2 r^* = h(|r^*|).$$

Define  $p = w^*fw$ . Then  $p \in B$  because  $f \in B$  and  $w \in B^+$ . We further have  $cp = p$  because  $cb_7 = b_7$  implies  $cx = x$  for all  $x \in B$ .

We complete the proof by showing that  $pa = a$ . From  $vv^* = f$  we get  $fv = v$ , so  $r = vb_1$  implies  $fr r^* = rr^*$ . Therefore also  $fh(|r^*|) = h(|r^*|)$ . So

$$ph(|r|) = w^* fwh(|r|) = w^* fh(|r^*|)w = w^* h(|r^*|)w = h(|r|).$$

Also, using  $b_1 a = a$ ,  $v^* v = e$ , and  $eb_1 = b_1$ , we get

$$r^* r a = b_1 v^* v b_1 a = b_1 e a = a.$$

So  $(r^* r)^{1/2} a = a$ , and from  $h(1) = 1$  we now get  $h(|r|)a = a$ . Combining this with  $ph(|r|) = h(|r|)$ , we obtain  $pa = a$ . ■

#### 4. DIRECT LIMITS WITH REAL RANK ZERO

In this section, we prove the main results, namely characterizations of Property (SP) and of real rank zero for infinite dimensional unital simple direct limits of recursive subhomogeneous algebras with no dimension growth. As an application, we prove that if such an algebra has real rank zero and not too many extreme tracial states, then it has tracial rank zero in the sense of [19], and is thus classifiable.

We begin with Property (SP).

**Theorem 4.1.** Let  $A$  be an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2). Then the following are equivalent:

- (1)  $A$  has Property (SP), that is, every nonzero hereditary subalgebra of  $A$  contains a nonzero projection.
- (2) For every  $\varepsilon > 0$  there is  $\eta \in K_0(A)$  such that  $0 < \tau_*(\eta) < \varepsilon$  for all tracial states  $\tau$  on  $A$ .

*Proof.* That (1) implies (2) is Proposition 1.11. We therefore prove the converse. By Proposition 1.10 of [27], we may assume all the maps of the direct system are injective. We identify all  $A_k$  with their images in  $A$ , so that  $A_0 \subset A_1 \subset \cdots \subset A$ .

Let  $B \subset A$  be a nonzero hereditary subalgebra. Choose a positive element  $r \in B$  with  $\|r\| = 1$ . Define continuous functions  $f, g, h: [0, \infty) \rightarrow [0, 1]$  as in the proof of Lemma 3.3. Choose  $n_0$  and a positive element  $r_0 \in A_{n_0}$  with  $\|r_0\| = 1$  and with  $\|r_0 - r\|$  so small that  $\|h(r_0) - h(r)\| < \frac{1}{4}$ .

We construct a nonzero projection  $q \in A$  such that  $h(r_0)q = q$ . If  $\text{sp}(r_0) \neq [0, 1]$ , then functional calculus immediately produces such a projection. Otherwise, set  $a = f(r_0)$ ,  $b = g(r_0)$ , and  $c = h(r_0)$ . These elements are nonzero, and  $ab = a$  and  $cb = b$ . All traces on  $A$  are faithful, and  $T(A)$  is compact, so  $\varepsilon = \inf_{\tau \in T(A)} \tau(a) \in (0, 1)$ . Apply the hypothesis (2) with this  $\varepsilon$ , and let  $\eta$  be the resulting element of  $K_0(A)$ . Because the order on  $K_0(A)$  is determined by traces (Theorem 2.3 of [27]), there exists a projection  $q_0 \in A$  such that  $[q] = \eta$  in  $K_0(A)$ . In fact, we may assume that  $q_0 \in A_{n_1}$  for some  $n_1$ . Let  $d$  be a finite upper bound for the topological dimensions of the  $A_k$ . Using Lemma 3.4, there is  $n \geq \max(n_0, n_1)$  such that, regarding  $q_0$  and  $a$  as elements of  $A_n$ , we have

$$\text{rank}_x(p) \leq \text{rank}(\text{ev}_x(a)) - \frac{1}{2}(d - 1)$$

for every  $x$  in the total space of  $A_n$ . Lemma 3.3 now provides a partial isometry  $v \in A$  such that  $v^*v = q_0$  and  $cvv^* = vv^*$ . Then  $q = vv^*$  is the required projection.

We have  $h(r_0)qh(r_0) = q$  and  $\|h(r_0) - h(r)\| < \frac{1}{4}$ . Since  $\|h(r)\| \leq 1$  and  $\|h(r_0)\| \leq 1$ , it follows that  $\|h(r)qh(r) - q\| < \frac{1}{2}$ . Therefore  $h(r)qh(r)$  is an element of  $B$  whose

spectrum does not contain  $\frac{1}{2}$ , and functional calculus produces a projection  $p \in B$  which is Murray-von Neumann equivalent to  $q$ . Since  $q$  is nonzero, so is  $p$ . ■

We can now give several characterizations of real rank zero. In Condition (5) of the next theorem, none of the three parts can be omitted. For the Property (SP), see the version of Example 5.4 in which  $K_0(A)$  is a Riesz group. For the Riesz group condition, see Example 5.6. For the requirement that the projections in  $A$  distinguish the tracial states, use the algebra  $A_3$  of Example 1.6 of [4]. It has Property (SP) by Corollary 1.10 of [4], and  $K_0(A_3)$  is a Riesz group by Theorem 2.7 of [16].

**Theorem 4.2.** Let  $A$  be an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2). Then the following are equivalent:

- (1)  $A$  has real rank zero.
- (2)  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$ .
- (3) For the universal trace  $T: A \rightarrow A/\overline{[A, A]}$  (see Definition 1.7), we have  $T_*(K_0(A))$  dense in  $(A/\overline{[A, A]})_{\text{sa}}$ .
- (4) The projections in  $A$  distinguish the tracial states,  $K_0(A)$  is a Riesz group, and for every  $\varepsilon > 0$  there is  $\eta \in K_0(A)$  such that  $0 < \tau_*(\eta) < \varepsilon$  for all tracial states  $\tau$  on  $A$ .
- (5) The projections in  $A$  distinguish the tracial states,  $K_0(A)$  is a Riesz group, and  $A$  has Property (SP).

*Proof.* We prove (1)  $\implies$  (5)  $\implies$  (4)  $\implies$  (2)  $\implies$  (1) and (2)  $\iff$  (3).

(1)  $\implies$  (5): The only nontrivial part is that  $K_0(A)$  is a Riesz group, which follows from Proposition 1.5 and Proposition 1.3.

(5)  $\implies$  (4): This is Proposition 1.11.

(4)  $\implies$  (2): Let  $S(K_0(A))$  be the state space (Chapter 6 of [15]) of the scaled ordered group  $K_0(A)$ , and let  $\rho_0: K_0(A) \rightarrow \text{Aff}(S(K_0(A)))$  be the canonical homomorphism. Theorem 3.5 of [23] implies that  $\rho_0(K_0(A))$  is dense in  $\text{Aff}(S(K_0(A)))$ . (The notion of ‘‘asymptotic refinement group’’ appearing there is defined after Proposition 2.1 of [23], and includes all Riesz groups, even with torsion. A discrete state is one whose range is discrete; there are none, by the last part of (4).)

Every tracial state on  $A$  defines a state on  $K_0(A)$ , yielding a continuous affine function  $\Phi: T(A) \rightarrow S(K_0(A))$ , and hence a contractive linear map of Banach spaces  $\Phi^*: \text{Aff}(S(K_0(A))) \rightarrow \text{Aff}(T(A))$ . Also, the map  $\rho: K_0(A) \rightarrow \text{Aff}(T(A))$  factors through the canonical map  $\rho_0: K_0(A) \rightarrow \text{Aff}(S(K_0(A)))$  as  $\rho = \Phi^* \circ \rho_0$ . Since projections distinguish traces,  $\Phi$  is injective. By Theorem 6.1 of [28], every state on  $K_0(A)$  comes in this way from a normalized quasitrace on  $A$ . By Theorem II.4.9 of [3] every 2-quasitrace on a direct limit of type 1  $C^*$ -algebras, in particular on  $A$ , is a trace. (The terminology in these two papers differs. In 4.2 of [28], a quasitrace is required to extend, with the same properties, to  $M_n(A)$  for all  $n$ . In Definition II.1.1 of [3], a quasitrace is defined only on  $A$ , and a 2-quasitrace is required to extend to  $M_2(A)$ . Proposition II.4.1 of [3] shows that every 2-quasitrace in this sense automatically extends to  $M_n(A)$  for all  $n$ .)

It follows that  $\Phi$  is surjective. So  $\Phi^*$  is an isometric isomorphism of Banach spaces, and density of  $\rho_0(K_0(A))$  in  $\text{Aff}(S(K_0(A)))$  implies density of  $\rho(K_0(A))$  in  $\text{Aff}(T(A))$ .

(2)  $\implies$  (1): If all the maps of the direct system are injective, we combine Proposition 3.5 with Lemma A of [2]. The general case can be reduced to this case by Proposition 1.10 of [27].

(2)  $\iff$  (3): This is immediate from Proposition 1.9.  $\blacksquare$

In this proof, we didn't actually use Proposition 1.10. Note, though, that it gives (1)  $\implies$  (2) without using quasitraces and [23].

The condition in the following proposition is probably also equivalent to real rank zero, but we don't know how to prove that real rank zero implies approximate divisibility.

**Proposition 4.3.** Let  $A$  be an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2). If  $A$  is approximately divisible in the sense of [5], and if the projections in  $A$  distinguish the tracial states, then  $A$  has real rank zero.

*Proof.* Theorem II.4.9 of [3] implies that every quasitrace (in the sense used in [5], defined before Proposition 3.3 there) is a trace. Therefore Proposition 3.14(b) of [5] implies that  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$ . (See the discussion before Proposition 3.13 of [5] for the definition of the space  $V_0$  appearing in this result.)  $\blacksquare$

It also remains to decide when an infinite dimensional separable unital simple direct limit  $A$  of recursive subhomogeneous algebras, with no dimension growth, is approximately divisible, and when  $K_0(A)$  is a Riesz group. It follows from Corollary 3.15 of [5] that if  $A$  is such an algebra, if  $A$  is approximately divisible, and if the state space of  $K_0(A)$  is a simplex, then  $K_0(A)$  is a Riesz group. However, the discussion after that result points out that approximate divisibility by itself does not imply that  $K_0(A)$  is a Riesz group, and in Example 5.7 we give an infinite dimensional separable unital simple direct limit  $A$  of recursive subhomogeneous algebras, with no dimension growth, which is approximately divisible but such that  $K_0(A)$  is not a Riesz group. This algebra is also not an AH algebra. The version of Example 5.4 in which  $K_0(A)$  a Riesz group shows that this property by itself does not imply approximate divisibility. We don't know what happens if one also requires Property (SP).

Applying results of H. Lin, we obtain the following consequences of Theorem 4.2.

**Theorem 4.4.** Let  $A$  be an infinite dimensional separable unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2). Assume that  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$  (or any of the other equivalent conditions of Theorem 4.2). If in addition  $A$  has at most countably many extreme tracial states, then  $A$  is tracially AF in the sense of Definition 2.1 of [18].

*Proof.* We verify the conditions of Theorem 4.15 of [21]. That  $A$  has stable rank one is Theorem 3.6 of [27]. That  $A$  has real rank zero is Theorem 4.2. That  $K_0(A)$  is weakly unperforated (unperforated for the strict order) is Theorem 2.4 of [27], using Proposition 1.10 of [27] to reduce to the case of injective maps in the system. To see that every tracial state on  $A$  is approximately AC in the sense of [21], we apply Proposition 5.4 of [21], keeping in mind Definitions 2.8 (both parts) and 5.1 of [21]. Thus, the result follows from Theorem 4.15 of [21].  $\blacksquare$

**Theorem 4.5.** Let  $A$  and  $B$  be infinite dimensional separable unital simple direct limits of recursive subhomogeneous algebras, with no dimension growth (Definition 1.2). Assume that  $\rho_A(K_0(A))$  is dense in  $\text{Aff}(T(A))$  (or any of the other



equivalent conditions of Theorem 4.2), and similarly for  $B$ . Assume moreover that  $A$  and  $B$  have at most countably many extreme tracial states. If there is an order isomorphism

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B)),$$

then  $A \cong B$ .

*Proof.* By Theorem 4.4, we may apply Theorem 3.10 of [20]. (The class  $\mathcal{BD}$  appearing there is defined in Definition 3.1 of [20].) ■

The limitation on the number of extreme tracial states in Theorem 4.4 should not be necessary.

**Conjecture 4.6.** Let  $A$  be an infinite dimensional separable unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth. If  $A$  has real rank zero, then  $A$  is tracially AF.

## 5. EXAMPLES

In this section, we give examples showing that various weakenings of the conditions in Theorems 4.1 and 4.2 do not suffice.

We first show (using Villadsen's example [31]) that the restriction to no dimension growth in Theorem 4.2 can't be dropped. The counterexample does not, however, have stable rank 1. It remains an open question whether a simple unital C\*-algebra  $A$ , with stable rank 1 and such that  $\rho(K_0(A))$  is dense in  $\text{Aff}(T(A))$ , must have real rank zero.

**Example 5.1.** There is a simple separable unital nuclear C\*-algebra  $A$ , in fact a direct limit of homogeneous C\*-algebras (although not with slow dimension growth) such that  $\rho(K_0(A))$  is dense in  $\text{Aff}(T(A))$  but such that  $A$  does not have real rank zero.

The C\*-algebra  $A$  is taken from [31]. Fix  $n \geq 1$ . Let  $A$  be the C\*-algebra with stable rank  $n + 1$  constructed there. Theorem 10 of [31] implies that  $\text{RR}(A) \geq n$ , and in particular  $\text{RR}(A) \neq 0$ . On the other hand,  $A$  has a unique tracial state  $\tau$ , by the remark at the end of Section 6 of [31]. So  $\text{Aff}(T(A)) \cong \mathbf{R}$  and this isomorphism identifies  $\rho$  with  $\tau_*$ .

Examining the construction in Section 3 of [31], we see that  $A = \varinjlim A_k$  for C\*-algebras  $A_k \cong p_k(C(X_k) \otimes K)p_k$  with suitable connected compact metric spaces  $X_k$  and projections  $p_k \in C(X_k) \otimes K$ . Moreover,  $\text{rank}(p_k) \rightarrow \infty$  as  $k \rightarrow \infty$ . However (see the end of Section 3 of [31]) there is a trivial rank one projection  $q_k \in A_k$ . In the direct limit, we must have

$$\tau(q_k) = \frac{\text{rank}(q_k)}{\text{rank}(p_k)} = \frac{1}{\text{rank}(p_k)},$$

from which it easily follows that the range of  $\tau_*$  is dense.

We note, however, that if a simple C\*-algebra has finite tracial topological rank in the sense of Lin (Definition 3.1 of [19]), if the image of  $K_0(A)$  in  $\text{Aff}(T(A))$  is dense, and if  $A$  has only countably many extreme tracial states, then  $A$  does have real rank zero (in fact, tracial topological rank zero). See Remark 7.8 of [19].

The remaining examples rule out various weakenings of the conditions on tracial states and projections, and several conjectures one might make involving approximate divisibility. Most of them will be constructed using a theorem of Thomsen [30],

or a generalization due to Elliott [11], so we start by setting up the machinery. The following is stated without proof in the introduction to [30]. The proof given here simplifies our earlier version considerably, and was provided by Ken Goodearl. See Page 4 of [15] for the definition of an order unit.

**Lemma 5.2.** Let  $\Delta$  be a metrizable Choquet simplex. Let  $G$  be a countable abelian group, and let  $\psi: G \rightarrow \text{Aff}(\Delta)$  be a homomorphism whose image contains the constant function 1. Make  $G$  a scaled partially ordered group by setting

$$G_+ = \{g \in G: \psi(g)(x) > 0 \text{ for all } x \in \Delta\} \cup \{0\}$$

and taking the order unit to be any element  $g_0 \in G$  such that  $\psi(g_0) = 1$ . Then every state on  $G$  has the form  $\text{ev}_x \circ \psi$  for some point  $x \in \Delta$ .

*Proof.* Let  $\omega: G \rightarrow \mathbf{R}$  be a state on  $G$ . First, observe, as in the proof of Theorem 14.17(a) of [15], that  $\omega$  vanishes on  $\text{Ker}(\psi)$ . (If  $g \in \text{Ker}(\psi)$ , then  $-g_0 \leq ng \leq g_0$  for all  $n \in \mathbf{Z}$ ; since  $\omega(g_0) = 1$ , this forces  $\omega(g) = 0$ .) So  $\omega$  defines a homomorphism  $\bar{\omega}: \psi(G) \rightarrow \mathbf{R}$ , clearly a state. (We give  $\psi(G) \subset \text{Aff}(\Delta)$ , and  $\text{Aff}(\Delta)$  itself, the order and order unit defined by the obvious analog of the formula for  $G_+$ .) By Corollary 4.3 of [15], there is a state on  $\text{Aff}(\Delta)$  whose restriction to  $\psi(G)$  is  $\bar{\omega}$ . By Corollary 7.2 of [15], this state is given by evaluation at some  $x \in \Delta$ . Clearly  $\omega(g) = \psi(g)(x)$  for all  $g \in G$ . ■

**Theorem 5.3.** Let  $G$ ,  $\Delta$ , and  $\psi: G \rightarrow \text{Aff}(\Delta)$  be as in Lemma 5.2, with order and scale on  $G$  as there. Assume in addition that  $G$  is torsion free. Then there exists a simple separable unital C\*-algebra  $A$ , which is a direct limit of recursive subhomogeneous algebras of topological dimension at most 1, such that

$$(K_0(A), K_0(A)_+, [1], T(A), \rho_A) \cong (G, G_+, g_0, \Delta, \psi).$$

(The isomorphism means that there is an isomorphism  $\varphi: K_0(A) \rightarrow G$  of partially ordered scaled groups, and an affine homeomorphism  $R: \Delta \rightarrow T(A)$ , such that for every  $\eta \in K_0(A)$ , the functions  $\rho_A(\eta) \circ R$  and  $\psi(\varphi(\eta))$  are equal in  $\text{Aff}(\Delta)$ .)

*Proof.* We use Theorem A of [30], or Theorem 5.2.3.2 of [11]. First,  $G$  is simple. Indeed, by Lemma 14.1 of [15], it suffices to show that every nonzero element of  $G_+$  is a order unit. This follows directly from the fact that continuous functions on the compact space  $\Delta$  have maximum and minimum values. The group  $G$  is unperforated because it is torsion free. (Any perforation must lie in  $\text{Ker}(\psi)$ , and 0 is the only element of  $G_+ \cap \text{Ker}(\psi)$ .)

Next, Lemma 5.2 shows that the obvious map from  $\Delta$  to  $S(G)$ , the state space of  $G$ , is surjective, and it is trivially continuous and affine. So Theorem A of [30] produces a C\*-algebra  $A$  as above, except that it is a direct limit of the “building blocks” of [30]. By inspection, these building blocks are recursive subhomogeneous algebras with topological dimension 1 (and length 1). Alternately, use Theorem 5.2.3.2 of [11], and use Theorem 2.16 of [26] to see that the building blocks, from Section 5.1.2 of [11], are recursive subhomogeneous algebras with topological dimension at most 2. ■

The proof using Theorem 5.2.3.2 of [11] has the advantage that one can also specify  $K_1(A)$ , which we don’t need here. The proof using Theorem A of [30] has the advantage that the building blocks are simpler.

**Example 5.4.** There is an infinite dimensional unital simple direct limit  $A$  of recursive subhomogeneous algebras, with no dimension growth, which has a unique tracial state and no nontrivial projections, and such that  $K_0(A)$  is a Riesz group. There is also an algebra  $A$  with all the other properties listed but such that  $K_0(A)$  is not a Riesz group. In these algebras, the projections distinguish the tracial states for trivial reasons, but the algebra doesn't even have Property (SP), let alone real rank zero.

To get such an algebra with  $K_0(A)$  a Riesz group, apply Theorem 5.3 with  $\Delta$  a one point space and  $G = \mathbf{Z} \cdot 1$ . An algebra with these properties also appears in [17].

To get such an algebra with  $K_0(A)$  not a Riesz group, apply Theorem 5.3 with  $\Delta$  consisting of one point,  $\text{Aff}(\Delta) = \mathbf{R}$ ,  $G = \mathbf{Z}^2$ , and  $\psi(m, n) = m$ . An algebra with these properties also appears in Example 4.8 of [27].

**Example 5.5.** There is an infinite dimensional unital simple direct limit  $A$  of recursive subhomogeneous algebras, with no dimension growth, such that  $\tau_*(K_0(A))$  is dense in  $\mathbf{R}$  for every tracial state  $\tau$ , and in which the projections distinguish the tracial states, but such that  $A$  does not have Property (SP). In particular,  $A$  does not have real rank zero.

Let  $\mathbf{N} = \{1, 2, \dots\}$ , and let  $\mathbf{N}^+$  be its one point compactification. Let  $\Delta$  be the simplex consisting of the Borel probability measures on  $\mathbf{N}^+$ . (This is a Choquet simplex because it is  $T(C(\mathbf{N}^+))$ . See Theorem 3.1.18 of [29].) Let  $R: C(\mathbf{N}^+) \rightarrow \text{Aff}(\Delta)$  be the obvious linear map. Define functions  $f_0, f_1, f_2, \dots \in C(\mathbf{N}^+)$  as follows. Take  $f_0$  to be the constant function 1. For  $n \geq 1$ , set

$$f_n(k) = \begin{cases} 1 & k = n \\ \frac{1}{2^n} & k \in \mathbf{N}^+ \setminus \{n\}. \end{cases}$$

Let  $G_0$  be the subgroup of  $C(\mathbf{N}^+)$  generated by  $f_0, f_1, f_2, \dots$ , and let  $G = R(G_0) \subset \text{Aff}(\Delta)$ .

Apply Theorem 5.3 with this  $G$  and  $\Delta$ , and with  $\psi$  being the inclusion, obtaining a  $C^*$ -algebra  $B$ . Take  $A = M_2(B)$ .

First, we show that  $\tau_*(K_0(A))$  is dense in  $\mathbf{R}$  for every tracial state  $\tau$ . By construction, this is equivalent to showing that for every Borel probability measure  $\mu$  on  $\mathbf{N}^+$ , the set

$$H = \left\{ \int_{\mathbf{N}^+} f d\mu : f \in G_0 \right\}$$

is dense in  $\mathbf{R}$ . In fact,  $\int_{\mathbf{N}^+} f_n d\mu > 0$  for all  $n$ , because  $f_n$  is strictly positive. Moreover,  $f_n \rightarrow 0$  pointwise and  $0 \leq f_n \leq 1$ , so  $\int_{\mathbf{N}^+} f_n d\mu \rightarrow 0$  as  $n \rightarrow \infty$  by the Dominated Convergence Theorem. Thus  $H$  is a subgroup of  $\mathbf{R}$  which contains a sequence of strictly positive numbers converging to 0, so is dense.

Next, we show that the projections distinguish the tracial states. Since  $0 \leq f_n \leq 2f_0$  in the order on  $G_0$  determined by that on  $G$ , it follows that all  $f_n$  correspond to projections in  $A = M_2(B)$ . If two tracial states  $\sigma$  and  $\tau$  are not distinguished by the projections in  $A$ , then the corresponding Borel probability measures  $\mu$  and  $\nu$  on  $\mathbf{N}^+$  must satisfy

$$\int_{\mathbf{N}^+} f_n d\mu = \int_{\mathbf{N}^+} f_n d\nu$$

for all  $n$ . So  $\mu - \nu$  is a signed measure on  $\mathbf{N}^+$  such that  $\int_{\mathbf{N}^+} f d(\mu - \nu) = 0$  for all  $f \in \overline{\text{span}}_{\mathbf{C}}(f_0, f_1, f_2, \dots) \subset C(\mathbf{N}^+)$ . Now  $\text{span}_{\mathbf{C}}(f_0, f_1, f_2, \dots)$  trivially contains

the constant function 1, and is easily seen to contain for all  $n$  the function

$$k \mapsto \begin{cases} 1 & k = n \\ 0 & k \in \mathbf{N}^+ \setminus \{n\}. \end{cases}$$

Therefore  $\text{span}_{\mathbf{C}}(f_0, f_1, f_2, \dots)$  is dense in  $C(\mathbf{N}^+)$ , whence  $\mu - \nu = 0$ . So  $\sigma = \tau$ , and projections distinguish tracial states.

Finally, we show that  $K_0(A)$  contains no element  $\eta$  such that  $0 < \tau_*(\eta) < \frac{1}{8}$  for all tracial states  $\tau$ . By Theorem 4.1, this will imply that  $A$  does not have Property (SP). It suffices to show that  $G_0$  contains no function  $f$  such that  $0 < f(n) < \frac{1}{4}$  for all  $n \in \mathbf{N}^+$ . Suppose  $f = k_0 f_0 + k_1 f_1 + \dots + k_n f_n$  is such a function. Let  $1 \leq r \leq n$ . Observe that  $f_j(r) = f_j(\infty)$  for  $j \in \mathbf{N} \cup \{0\}$  with  $j \neq r$ . Therefore

$$\frac{1}{2} > |f(r) - f(\infty)| = |k_r| \cdot |f_r(r) - f_r(\infty)| = \left(1 - \frac{1}{2^r}\right) |k_r|.$$

Since  $k_r \in \mathbf{Z}$  and  $1 - \frac{1}{2^r} \geq \frac{1}{2}$ , it follows that  $k_r = 0$ . This is true for  $1 \leq r \leq n$ , so  $f = k_0 f_0$ . But no function  $f = k_0 f_0$  satisfies  $0 < f(n) < \frac{1}{4}$  for all  $n \in \mathbf{N}^+$ . So  $f$  does not exist.

**Example 5.6.** There is an infinite dimensional unital simple direct limit  $A$  of recursive subhomogeneous algebras, with no dimension growth, such that  $\rho_A(K_0(A))$  contains arbitrarily small strictly positive constant functions, and in which the projections distinguish the tracial states, but such that  $A$  does not have real rank zero and is not approximately divisible in the sense of [5], and  $K_0(A)$  is not a Riesz group. However, the state space of  $K_0(A)$  is a simplex. The algebra  $A$  has Property (SP) by Theorem 4.1. Therefore having projections distinguish the tracial states, even combined with Property (SP), does not imply real rank zero, and Property (SP) does not imply that  $K_0(A)$  is a Riesz group. Moreover, both implications remain false even if one adds the assumption that the state space of  $K_0(A)$  is a simplex, despite Corollary 3.15 of [5].

Let  $\Delta$  be the Choquet simplex  $[0, 1]$ . Define  $f, g \in \text{Aff}(\Delta)$  by  $f(t) = 1$  and  $g(t) = \frac{1}{3} + \frac{1}{3}t$  for all  $t$ . Let  $\mathbf{Z}[\frac{1}{2}]$  be the subset of  $\mathbf{Q}$  consisting of those rationals whose denominators are powers of 2, and define  $G = \mathbf{Z}[\frac{1}{2}]f + \mathbf{Z}g \subset \text{Aff}(\Delta)$ . Apply Theorem 5.3 with this  $G$  and  $\Delta$ , and with  $\psi$  being the inclusion, obtaining a  $\mathbf{C}^*$ -algebra  $A$ .

That  $\rho_A(K_0(A))$  contains the constant functions with values in  $\mathbf{Z}[\frac{1}{2}]$  is clear. To see that projections distinguish the tracial states, we note that  $g$  is a positive element of  $G$  which distinguishes the points of  $[0, 1]$ , and  $g \leq 1$  in the order of  $G$ , so there is a projection in  $A$  whose class is  $g$ , and this projection necessarily distinguishes the tracial states.

The group  $G$  is not dense in  $\text{Aff}(\Delta)$  because the functional  $g \mapsto g(1) - g(0)$  has range  $\frac{1}{3}\mathbf{Z}$ , which is not dense in  $\mathbf{R}$ . Therefore  $A$  does not have real rank zero, by Theorem 4.2. Also,  $S(G) = \Delta$  by Lemma 5.2, and both extreme points are states whose range includes  $\mathbf{Z}[\frac{1}{2}]$  and so is dense. Since  $G$  is not dense in  $\text{Aff}(\Delta)$ , the implication (4)  $\implies$  (2) of Theorem 4.2 implies that  $G$  is not a Riesz group. Since the projections distinguish the tracial states, and since every quasitrace is a trace (Theorem II.4.9 of [3]), Corollary 3.15 of [5] implies that  $A$  is not approximately divisible.

**Example 5.7.** There is an infinite dimensional unital simple direct limit  $A$  of recursive subhomogeneous algebras, with no dimension growth, such that  $A$  is approximately divisible and has Property (SP), but such that  $K_0(A)$  is not a Riesz group, the state space of  $K_0(A)$  is not a simplex, and  $A$  is not an AH algebra with slow dimension growth. The ordered  $K_0$ -group even satisfies a stronger condition than that of Theorem 4.1, namely that for every  $\varepsilon > 0$ , as a group  $K_0(A)$  is generated by elements  $\eta$  such that  $0 < \tau_*(\eta) < \varepsilon$  for every  $\tau \in T(A)$ .

This example is essentially the same as the simple example mentioned after Corollary 3.15 of [5], despite the rather different construction. We will use Theorem 5.3. Set

$$G = \left\{ \eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \left(\mathbf{Z} \left[\frac{1}{2}\right]\right)^4 : \eta_1 + \eta_2 = \eta_3 + \eta_4 \right\} \cong \left(\mathbf{Z} \left[\frac{1}{2}\right]\right)^3.$$

Let  $\Delta$  be the standard Choquet simplex in  $\mathbf{R}^4$ ,

$$\Delta = \{x \in \mathbf{R}^4 : x_k \geq 0 \text{ for } 1 \leq k \leq 4 \text{ and } x_1 + x_2 + x_3 + x_4 = 1\}.$$

Set  $g_0 = (1, 1, 1, 1)$ . Define  $\psi: G \rightarrow \text{Aff}(\Delta)$  by  $\psi(\eta)(x) = \sum_{k=1}^4 \eta_k x_k$ . The hypotheses of Theorem 5.3 are clearly satisfied, and give

$$G_+ = \{0\} \cup \{\eta \in G : \eta_k > 0 \text{ for } 1 \leq k \leq 4\},$$

and order unit  $g_0$ .

Let  $A_0$  be the  $C^*$ -algebra obtained from Theorem 5.3, let  $B$  be the  $2^\infty$  UHF algebra, and set  $A = A_0 \otimes B$ . Set  $A_n = M_{2^n} \otimes A_0$ , and write  $A = \varinjlim A_n$ , with maps  $a \mapsto \text{diag}(a, a)$  at each stage. Because multiplication by 2 is an order isomorphism from  $G$  to itself, these maps induce isomorphisms

$$\begin{aligned} & (K_0(A_n), K_0(A_n)_+, [1_{A_n}], T(A_n), \rho_{A_n}) \rightarrow \\ & (K_0(A_{n+1}), K_0(A_{n+1})_+, [1_{A_{n+1}}], T(A_{n+1}), \rho_{A_{n+1}}). \end{aligned}$$

It follows that, apart from the  $K_1$ -groups,  $A$  and  $A_0$  have the same Elliott invariants, so

$$(K_0(A), K_0(A)_+, [1], T(A), \rho_A) \cong (G, G_+, g_0, \Delta, \psi).$$

Moreover,  $A$  is again an infinite dimensional unital simple direct limit of recursive subhomogeneous algebras, with no dimension growth.

For every  $n$ , the range of  $\psi$  contains the constant function with value  $\frac{1}{2^n}$ . Therefore Theorem 4.1 implies that  $A$  has Property (SP). Furthermore,  $A$  is approximately divisible because its tensor factor  $B$  is.

We now compute the state space  $S(K_0(A))$  of  $K_0(A)$ . Applying Proposition 6.9 of [15] to  $G$ , we see that it is equivalent to compute the state space of  $G$  with the order unit  $g_0 = (1, 1, 1, 1)$  and positive cone

$$G_0 = \{\eta \in G : \eta_k \geq 0 \text{ for } 1 \leq k \leq 4\}.$$

(The group  $(G, G_0, g_0)$  is the scaled ordered  $K_0$ -group of the nonsimple example after Corollary 3.15 of [5].) For  $(x, y) \in [0, 1]^2$  define a homomorphism  $s_{x,y}: G \rightarrow \mathbf{R}$  by

$$s_{x,y} = \frac{1}{2} [(x + y - 1)\eta_1 + (1 - x - y)\eta_2 + (1 + x - y)\eta_3 + (1 - x + y)\eta_4].$$

We claim that  $(x, y) \mapsto s_{x,y}$  is an affine homeomorphism from  $[0, 1]^2$  to  $S(G, G_0, g_0)$ .

We first show that  $s_{x,y}$  is a state. That  $s_{x,y}(1, 1, 1, 1) = 1$  is immediate. Also,

$$s_{x,y}(1, 0, 1, 0) = x \geq 0 \quad \text{and} \quad s_{x,y}(1, 0, 0, 1) = y \geq 0,$$

and

$$s_{x,y}(0, 1, 1, 0) = 1 - y \geq 0 \quad \text{and} \quad s_{x,y}(0, 1, 0, 1) = 1 - x \geq 0.$$

Now let  $\eta \in G_0$  be arbitrary. If  $\eta_1 \leq \eta_2, \eta_3, \eta_4$ , then we can use  $\eta_1 + \eta_2 = \eta_3 + \eta_4$  to write

$$\eta = \eta_1(1, 1, 1, 1) + (\eta_3 - \eta_1)(0, 1, 1, 0) + (\eta_4 - \eta_1)(0, 1, 0, 1),$$

giving

$$s_{x,y}(\eta) = \eta_1 + (\eta_3 - \eta_1)(1 - y) + (\eta_4 - \eta_1)(1 - x) \geq 0.$$

Similar calculations show that  $s_{x,y}(\eta) \geq 0$  when  $\min(\eta_1, \eta_2, \eta_3, \eta_4)$  is  $\eta_2, \eta_3$ , or  $\eta_4$ . So  $s_{x,y}$  is a state. Clearly  $(x, y) \mapsto s_{x,y}$  is injective.

Next, given any state  $s$  on  $(G, G_0, g_0)$ , set

$$x = s(1, 0, 1, 0) \quad \text{and} \quad y = s(1, 0, 0, 1).$$

These are nonnegative by definition, and also

$$1 - x = s(g_0) - s(1, 0, 1, 0) = s(0, 1, 0, 1) \geq 0;$$

similarly,  $1 - y = s(0, 1, 1, 0) \geq 0$ . Therefore  $(x, y) \in [0, 1]^2$ . Since  $(1, 1, 1, 1)$ ,  $(1, 0, 1, 0)$ , and  $(1, 0, 0, 1)$  generate  $G \cap \mathbf{Z}^4$  as a group and  $G = \mathbf{Z} \left[ \frac{1}{2} \right] \cdot (G \cap \mathbf{Z}^4)$ , it is easy to check that  $s$  is determined by its values on these three elements. So  $s = s_{x,y}$ . We have shown that  $(x, y) \mapsto s_{x,y}$  is bijective. That this map is an affine homeomorphism is now easy, and the claim is proved.

As in the discussion after Corollary 10.8 of [15], it follows that  $S(K_0(A))$  is not a simplex. Corollary 10.6 of [15] now shows that  $K_0(A)$  is not a Riesz group. So Theorem 2.7 of [16] implies that  $A$  is not an AH algebra with slow dimension growth.

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