

Swarming Behavior of Multi-Agent Systems *

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Abstract: In this paper we consider a continuous-time anisotropic swarm model in n -dimensional space with an attraction/repulsion function and study its aggregation properties. It is shown that the swarm members will aggregate and eventually form a cohesive cluster of finite size around the swarm center. Moreover, the numerical simulations show that all agents will eventually enter into and remain in a bounded region around the swarm center. The model is more general than isotropic swarms and our results provide further insight into the effect of the interaction pattern on individual motion in a swarm system.

Keywords: Biological systems, multiagent systems, swarms.

1 Introduction

In nature swarming can be found in many organisms ranging from simple bacteria to more advanced mammals. Examples of swarms include flocks of birds, schools of fish, herds of animals, and colonies of bacteria. Such collective behavior has certain advantages such as avoiding predators and increasing the chance of finding food. Recently, there has been a growing interest in biomimicry of foraging and swarming for using in engineering applications such as optimization, robotics, military applications and autonomous air vehicle [1]–[7]. Modeling and exploring the collective dynamics has become an important issue and many papers have appeared [8]–[12]. However, results on the anisotropic swarms are relatively few. The study of anisotropic swarm is very difficult though the anisotropic swarm is a ubiquitous phenomenon, including natural phenomena and social phenomena.

Gazi and Passino [2] proposed an isotropic swarm model and studied its aggregation, cohesion and stability properties. Subsequently, Chu and Wang [4] generalized their model, considering an anisotropic swarm model, and obtained the properties of aggregation, cohesion and completely stability. The coupling matrix W considered in [4] is symmetric, that is, the interactions between two individuals are reciprocal. In this paper, we try to study the behavior of anisotropic swarms when the coupling matrix is completely nonsymmetric. The model and the results given

here extend the work on isotropic swarms [2] and anisotropic swarms [4] to more general cases and further illustrate the effect of the interaction pattern on individual motion in swarm system.

In the next section we specify an “individual-based” continuous-time anisotropic swarm model in an n -dimensional Euclidean space which includes the isotropic model of [2] as a special case. Then, under some assumptions, we show that the swarm can exhibit aggregation in Section 3. In Section 4, we extend the results in Section 3, considering a more general attraction/repulsion function. In Section 5, under some assumptions, we provide some numerical simulations of the agent motion. We briefly summarize the results of the paper in Section 6.

2 Anisotropic Swarms

We consider a swarm of N individuals (members) in an n -dimensional Euclidean space. We model the individuals as points and ignore their dimensions. We consider the equation of motion of individual i described by

$$\dot{x}^i = \sum_{j=1}^N w_{ij} f(x^i - x^j), \quad i = 1, \dots, N, \quad (1)$$

where $x^i \in R^n$ represents the position of individual i ; $W = [w_{ij}] \in R^{N \times N}$ with $w_{ij} \geq 0$ for all $i, j = 1, \dots, N$ is the coupling matrix; $f(\cdot)$ represents the function of attraction and repulsion

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between the members. In other words, the direction and magnitude of motion of each member is determined as a weighted sum of the attraction and repulsion of all the other members on this member. The attraction/repulsion function that we consider is

$$f(y) = -y \left(a - b \exp \left(- \frac{\|y\|^2}{c} \right) \right), \quad (2)$$

where a, b , and c are positive constants such that $b > a$ and $\|y\|$ is the Euclidean norm given by $\|y\| = \sqrt{y^T y}$.

In the following discussion we always assume $w_{ii} = 0, i = 1, \dots, N$ in model (1). Moreover, we assume that there are no isolated clusters in the swarm, that is, $W + W^T$ is irreducible.

Note that the function $f(\cdot)$ is the social potential function that governs the interindividual interactions and is attractive for large distances and repulsive for small distances. By equating $f(y) = 0$, one can find that $f(\cdot)$ switches sign at the set of points defined as $\mathcal{Y} = \{y = 0 \text{ or } \|y\| = \delta = \sqrt{c \ln(b/a)}\}$. The distance δ is the distance at which the attraction and repulsion balance. Such a distance in biological swarms exists [3]. Note that it is natural as well as reasonable to require that any two different swarm members could not occupy the same position at the same time.

Remark 1: The anisotropic swarm model given here includes the isotropic model of [2] as a special case. Obviously, the present model (1) is more close to actuality and more meaningful.

3 Main Results

In this section, the main results concerning aggregation and cohesiveness of the swarm (1) are presented. In fact, it is interesting to investigate collective behavior of the system rather than to ascertain detailed behavior of each individual. And due to complex interactions among the multi-agents, in general, it is very difficult or even impossible to study the specific behavior of each agent.

Define the center of the swarm members as $\bar{x} = \frac{1}{N} \sum_{i=1}^N x^i$, then we have $\dot{\bar{x}} = \frac{1}{N} \sum_{i=1}^N \dot{x}^i$. If the coupling matrix W is symmetric, by the symmetry of $f(\cdot)$ with respect to the origin, the center \bar{x} is stationary for all t [4] and the swarm described by Eqs. (1) and (2) is not drifting on average. Note, however, that the swarm members may still have relative motions with respect to the center while the center itself stays stationary and the members will move toward the swarm center and form a cohesive cluster around it. However, if

the coupling matrix W is nonsymmetric, the center \bar{x} may not be stationary. An interesting issue is whether the members will form a cohesive cluster and which point they will move around. We will deal with this issue in the following theorem.

Theorem 1: Consider the swarm described by the model in (1) with an attraction/repulsion function $f(\cdot)$ as given in (2). Assume for any agent i , we have $\sum_{j=1}^N w_{ij} = \sum_{j=1}^N w_{ji}$. Then, all agents will eventually enter into and remain in the bounded region

$$\Omega = \left\{ x : \sum_{i=1}^N \|x^i - \bar{x}\|^2 \leq \rho^2 \right\}, \quad (3)$$

where

$$\rho = \frac{2bM\sqrt{2c}\exp(-\frac{1}{2})}{a\lambda_2};$$

and λ_2 denotes the second smallest real eigenvalue of the matrix $L + L^T$; and $M = \sum_{i,j=1}^N w_{ij}$; $L = [l_{ij}]$

with

$$l_{ij} = \begin{cases} -w_{ij}, & i \neq j, \\ \sum_{k=1, k \neq i}^N w_{ik}, & i = j; \end{cases} \quad (4)$$

Ω provides a bound on the maximum ultimate swarm size.

Proof. Let $e^i = x^i - \bar{x}$. By the definition of the center \bar{x} of the swarm and the assumption of $\sum_{j=1}^N w_{ij} = \sum_{j=1}^N w_{ji}$, we have

$$\dot{\bar{x}} = \frac{b}{N} \sum_{i=1}^N \left[\sum_{j=1}^N w_{ij} (x^i - x^j) \exp \left(- \frac{\|x^i - x^j\|^2}{c} \right) \right].$$

To estimate e^i , we let $V = \sum_{i=1}^N V_i$ be the Lyapunov function for the swarm, where $V_i = \frac{1}{2} e^{iT} e^i$. Evaluating its time derivative along solution of the system (1), we have

$$\begin{aligned} \dot{V} &= -a \sum_{i=1}^N \sum_{j=1}^N w_{ij} e^{iT} (e^i - e^j) \\ &\quad + b \sum_{i=1}^N e^{iT} \left\{ \sum_{j=1}^N w_{ij} \beta_{ij} (x^i - x^j) \right. \\ &\quad \left. - \frac{1}{N} \sum_{i=1}^N \left[\sum_{j=1}^N w_{ij} \beta_{ij} (x^i - x^j) \right] \right\} \\ &\leq -a e^T (L \otimes I) e \\ &\quad + b \sum_{i=1}^N \sum_{j=1}^N w_{ij} \beta_{ij} \|x^i - x^j\| \|e^i\| \\ &\quad + \frac{b}{N} \sum_{i=1}^N \left[\sum_{i=1}^N \sum_{j=1}^N w_{ij} \beta_{ij} \|x^i - x^j\| \|e^i\| \right], \end{aligned}$$

where $e = (e^{1T}, \dots, e^{NT})^T$ and $\beta_{ij} = \exp \left(- \frac{\|x^i - x^j\|^2}{c} \right)$, $L \otimes I$ is the Kronecker product of L

and I with L as defined in Eq. (4) and I the identity matrix of order n .

Note that each of the functions $\exp\left(-\frac{\|x^i - x^j\|^2}{c}\right)\|x^i - x^j\|$ is a bounded function whose maximum occurs at $\|x^i - x^j\| = \sqrt{c/2}$ and is given by $\sqrt{c/2}\exp(-1/2)$. Substituting this in the above inequality and using the fact that $\|e^i\| \leq \sqrt{2V}$, we obtain

$$\dot{V} \leq -ae^T(L \otimes I)e + 2bM\sqrt{c}\exp\left(-\frac{1}{2}\right)V^{\frac{1}{2}}. \quad (5)$$

To get further estimate of \dot{V} , we only need to estimate the term $e^T(L \otimes I)e$. Since

$$e^T(L \otimes I)e = \frac{1}{2}e^T((L + L^T) \otimes I)e,$$

we should analyze $e^T((L + L^T) \otimes I)e$. First considering the matrix $L + L^T$ and L as defined in Eq. (4), we have $L + L^T = [\tilde{l}_{ij}]$, where

$$\tilde{l}_{ij} = \begin{cases} -w_{ij} - w_{ji}, & i \neq j, \\ 2\sum_{k=1, k \neq i}^N w_{ik}, & i = j. \end{cases} \quad (6)$$

Using the conditions $\sum_{j=1}^N w_{ij} = \sum_{j=1}^N w_{ji}$, we can conclude that $\lambda = 0$ is an eigenvalue of $L + L^T$ and $u = (l, \dots, l)^T$ with $l \neq 0$ is the associated eigenvector. Moreover, since $L + L^T$ is symmetric and $W + W^T(L + L^T)$ is irreducible, it follows from matrix theory [5] that $\lambda = 0$ is a simple eigenvalue and all the rest eigenvalues of $L + L^T$ are real and positive. Therefore, we can order the eigenvalues of $L + L^T$ as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$. Also it is known that the identity matrix I has an n multiple eigenvalues $\mu = 1$ and n independent eigenvectors

$$u^1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad u^2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad u^n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

By matrix theory [5], the eigenvalues of $(L + L^T) \otimes I$ are $\lambda_i \mu = \lambda_i$ (n multiple for each i). Next, we consider the matrix $(L + L^T) \otimes I$. $\lambda = 0$ is an n multiple eigenvalues and the associated eigenvectors are

$$v^1 = [u^{1T}, \dots, u^{1T}]^T, \dots, v^n = [u^{nT}, \dots, u^{nT}]^T.$$

Therefore, $e^T((L + L^T) \otimes I)e = 0$ implies that e must lie in the eigenspace of $(L + L^T) \otimes I$ spanned by eigenvectors v^1, \dots, v^n corresponding to the zero eigenvalue, that is, $e^1 = e^2 = \dots = e^N$. This occurs only when $e^1 = e^2 = \dots = e^N = 0$, but this is impossible for the swarm system under consideration, because it implies that the N individuals

occupy the same position at the same time. Hence, for any solution x of system (1), e must be in the subspace spanned by eigenvectors of $(L + L^T) \otimes I$ corresponding to the nonzero eigenvalues. Then, $e^T((L + L^T) \otimes I)e \geq \lambda_2 \|e\|^2 = 2\lambda_2 V$. From (5), we have

$$\begin{aligned} \dot{V} &\leq -a\lambda_2 V + 2bM\sqrt{c}\exp\left(-\frac{1}{2}\right)V^{\frac{1}{2}} \\ &= -\left[a\lambda_2 V^{1/2} - 2bM\sqrt{c}\exp\left(-\frac{1}{2}\right)\right]V^{\frac{1}{2}} \\ &< 0 \end{aligned}$$

whenever

$$V(x) > \left(\frac{2bM\sqrt{c}\exp(-1/2)}{a\lambda_2}\right)^2.$$

Therefore, any solution of system (1) will eventually enter into and remain in Ω . \square

Theorem 1 shows that the swarm members will aggregate and form a bounded cluster around the swarm center.

Remark 2: The above discussions explicitly show the effect of the coupling matrix W on aggregation and cohesion of the swarm.

Remark 3: The conditions given in the above theorem $\sum_{j=1}^N w_{ij} = \sum_{j=1}^N w_{ji}$ include the case as a special case when the coupling matrix W is a symmetric matrix.

Remark 4: Theorem 1 provides a bound on the size of the swarm, but the bound is conservative. This is because we enlarged \dot{V} , and we used $e^{iT}(x^i - x^j) \leq \|x^i - x^j\| \|e^i\|$ and also assumed that the functions $\exp\left(-\frac{\|x^i - x^j\|^2}{c}\right)\|x^i - x^j\|$ were at their maximum value for all i and j . Therefore, the actual size of the swarm is, in general, much smaller than Ω .

Remark 5: Under the assumption of $\sum_{j=1}^N w_{ij} = \sum_{j=1}^N w_{ji}$, we obtain that the motion of the swarm center only depends on the repulsion between the swarm members.

4 Extensions

In Sections 2 and 3 we consider a specific function $f(y)$ as defined in (2). In this section, we will consider a more general function $f(y)$ that satisfies some assumptions. $f(y)$ is still the social potential function that governs the interindividual interactions and is assumed to have a long rang attraction and short range repulsion nature. Following [10], we make the assumptions on the social potential function:

Assumption 1. The attraction/repulsion function $f(\cdot)$ is of the form

$$f(y) = -y[f_a(\|y\|) - f_r(\|y\|)], y \in R^n, \quad (7)$$

where $f_a : R_+ \rightarrow R_+$ represents (the magnitude of) attraction term and has a long range, whereas $f_r : R_+ \rightarrow R_+$ represents (the magnitude of) repulsion term and has a short range, and R_+ stands for the set of nonnegative real numbers, $\|y\| = \sqrt{y^T y}$ is the Euclidean norm.

Assumption 2. There are positive constants a, b such that for any $y \in R^n$,

$$f_a(\|y\|) = a, \quad f_r(\|y\|) \leq \frac{b}{\|y\|}. \quad (8)$$

That is, we assume a fixed linear attraction function and a bounded repulsion function.

Theorem 2: Consider the swarm described by the model in (1) with an attraction/repulsion function $f(\cdot)$ as given in (7) satisfied (8). Then, all agents will eventually enter into and remain in the bounded region

$$\Omega^* = \left\{ x : \sum_{i=1}^N \|x^i - \bar{x}\|^2 \leq \rho^2 \right\}, \quad (9)$$

where $\rho = \frac{4bM}{a\lambda_2}$; and λ_2 and M are defined as in Theorem 1; Ω^* provides a bound on the maximum ultimate swarm size.

Following the proof of Theorem 1, Theorem 2 can be proved analogously.

5 Simulations

In this section we will present some numerical simulations for the nonreciprocal swarm described by Eqs. (1) and (2) in order to illustrate the theory obtained in the previous section.

In these simulations we used the $f(\cdot)$ function which is taken in the form of Eq. (2) with $a = 1, b = 20$, and $c = 0.2$. The coupling matrix W is generated randomly and satisfies the before conditions and assumptions.

Figs. 1-2 and Figs. 5-6 separately show the trajectories of the swarm members and the swarm center in which there are $N = 10$ individuals, and the four simulations run for 30s. In order to more clearly describe the motion of the swarm members and the swarm center, we also present the simulations of a five-agent swarm, that is, Figs. 3-4, and the two simulations run for 100s. It can be seen from Figs. 1-6 that at the beginning phase of the simulations of the swarm member trajectories, all of the members gradually aggregate and form a cohesive cluster. Then, they continuously move in the same direction as a group, and eventually evolve into an expanding spiral motion as time increases.

6 Conclusions

In this paper, we have considered an anisotropic swarm model and analyzed its aggregation. The model given here is a generalization of the model in [2] and [4]. And the model is more applicable to the reality and is more meaningful.

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