

THE BAUM-CONNES CONJECTURE, NONCOMMUTATIVE POINCARÉ DUALITY AND THE BOUNDARY OF THE FREE GROUP

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ABSTRACT. For every hyperbolic group Γ with Gromov boundary $\partial\Gamma$, one can form the cross product C^* -algebra $C(\partial\Gamma) \rtimes \Gamma$. For each such algebra we construct a canonical K -homology class, which induces a Poincaré duality map $K_*(C(\partial\Gamma) \rtimes \Gamma) \rightarrow K^{*+1}(C(\partial\Gamma) \rtimes \Gamma)$. We show that this map is an isomorphism in the case of $\Gamma = \mathbb{F}_2$ the free group on two generators. We point out a direct connection between our constructions and the Baum-Connes Conjecture and eventually use the latter to deduce our result.

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1. INTRODUCTION

The aim of this note is to point out a connection between the Baum-Connes conjecture with coefficients for the free group \mathbb{F}_2 on two generators, and a Poincaré duality result for the ‘noncommutative space’ $\partial\mathbb{F}_2/\mathbb{F}_2$, where $\partial\mathbb{F}_2$ is the Gromov boundary of \mathbb{F}_2 , acted upon minimally by \mathbb{F}_2 through homeomorphisms.

In order to formulate what Poincaré duality should mean for a noncommutative space such as $\partial\mathbb{F}_2/\mathbb{F}_2$, one passes to the C^* -algebra cross product $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$ and to K -theory and K -homology for C^* -algebras. Poincaré duality for $\partial\mathbb{F}_2/\mathbb{F}_2$ then means an isomorphism between the K -theory and K -homology of $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$, induced by cap product with a fixed K -homology class.

More generally one can speak of C^* -algebras having Poincaré duality, or, as we call them in this paper, Poincaré duality algebras. It seems that such algebras are in some sense noncommutative analogs of spin^c manifolds. For the commutative examples of such C^* -algebras are given precisely by the C^* -algebras $C(M)$, where M is a compact spin^c manifold. Such a manifold has, corresponding to the spin^c -structure, a canonical elliptic operator on it - the Dirac operator - and thus (see e.g. [9]) a canonical K -homology class. Cap product with this class induces the Poincaré duality isomorphism.

Various noncommutative examples of Poincaré duality C^* -algebras have been produced by A. Connes, the first of which was the irrational rotation algebra A_θ . Several other examples now exist, but all have the same character insofar as they are in some sense deformations of actual spin^c -manifolds. Our example is somewhat different. The geometric data underlying $\partial\mathbb{F}_2/\mathbb{F}_2$ is highly singular: the space $\partial\mathbb{F}_2$ is not a homology manifold, and the group \mathbb{F}_2 is not a Poincaré duality group. It turns out to be true, however, that in factoring the space by the action of the group, i.e. by forming the cross product C^* -algebra $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$, the resulting noncommutative space satisfies Poincaré duality.

Part of our goal is thus to point out this example and also to place it in its proper context: that of hyperbolic groups acting on their Gromov boundaries. The second part is

to show as mentioned above, a connection between our constructions and the Baum-Connes conjecture for \mathbb{F}_2 .

We begin by constructing - in the full generality of hyperbolic groups - the K -homology class cap product with which will induce our Poincaré duality isomorphism. It turns out that with Gromov hyperbolic groups Γ in general there is a certain duality between functions continuous on the Gromov boundary $\partial\Gamma$ of Γ , and right translation operators on $l^2\Gamma$. Using this duality, we produce an algebra homomorphism $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$, where $\mathcal{Q}(l^2\Gamma) = \mathcal{B}(l^2\Gamma)/\mathcal{K}(l^2\Gamma)$ denotes the Calkin algebra of $l^2\Gamma$, and where Γ is an arbitrary hyperbolic group. Since $C(\partial\Gamma) \rtimes \Gamma$ is nuclear ([3]), such an algebra homomorphism yields via the Stinespring construction a class $\Delta \in KK^1(C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma, \mathbb{C})$, i.e. a class Δ in the K -homology of $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma$. Kasparov product with Δ gives the required ‘cap-product’ map $\cap\Delta : K_*(C(\partial\Gamma) \rtimes \Gamma) \rightarrow K^{*+1}(C(\partial\Gamma) \rtimes \Gamma)$.

We next wish to prove that cap product with Δ as above gives an isomorphism in the case of $\Gamma = \mathbb{F}_2$, the general case of hyperbolic groups being beyond the scope of this paper. To this end we observe that a sort of geodesic flow on the Cayley graph of \mathbb{F}_2 may be used to construct a dual element to Δ , this time in the K -theory of $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$, playing the same role in this context as does the Thom class of the normal bundle of M in $M \times M$ in the commutative setting. We obtain a putative inverse map $K^*(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2) \rightarrow K_{*+1}(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$.

We then set about calculating the composition of these two maps. The connection with the Baum-Connes conjecture appears in that the composition $K_*(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2) \rightarrow K_*(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$ turns out to be multiplication by the γ -element constructed by Julg and Vallette.

As mentioned, the construction of our fundamental class Δ makes sense for a general hyperbolic group acting on its boundary, and in fact several of our other constructions have their counterparts for arbitrary hyperbolic groups; thus for instance it is possible by means of work of Gromov to make sense of ‘geodesic flow’ for an arbitrary hyperbolic group. Furthermore, although the statement ‘ $\gamma = 1$ ’ for general hyperbolic groups is false due to the possible presence of Property T, it is nevertheless true by work of Tu ([12]) that $\gamma_{\partial\Gamma \rtimes \Gamma} = 1_{C(\partial\Gamma)}$, where $\gamma_{\partial\Gamma \rtimes \Gamma}$ is the γ -element for the amenable *groupoid* $\partial\Gamma \rtimes \Gamma$, which weaker statement is all we need. Nevertheless, the arguments for the general case, being substantially more involved, will be dealt with in a later paper. We have chosen to emphasise the free group case for two reasons: one, that the relationship to the Baum-Connes conjecture is extremely explicit, and two, that the geometry of our constructions is particularly visible.

Finally, we note that our arguments tend to suggest that the phenomenon of Poincaré duality for amenable groupoid algebras constructed from boundary actions of discrete groups is relatively common. Specifically, the author expects similar results for uniform lattices in semisimple lie groups acting on their Furstenberg boundaries, and for discrete, cocompact isometry groups of affine buildings acting on the boundaries of these buildings. Along these lines, we draw the attention of the reader to the work of Kaminker and Putnam on Cuntz-Krieger algebras (see [8]); indeed, our result (in the case of the free group of two generators) can be deduced from theirs. In fact, our work was partly motivated by the idea of finding a geometric explanation for theirs.

2. GEOMETRIC PRELIMINARIES

In this section we work in the generality of a Gromov hyperbolic group Γ (see [5] or [4]). So let Γ be such. Thus, we have fixed a generating set S for Γ and the corresponding metric $d(\gamma_1, \gamma_2) = |\gamma_1^{-1}\gamma_2|$, where $|\cdot|$ denotes the word length of a group element with respect to S , and with this metric Γ is hyperbolic in the sense of Gromov as a metric space. Note that the metric is clearly invariant under left translation.

Recall that with the hypothesis of hyperbolicity, the group Γ viewed as a metric space can be compactified by addition of a boundary: thus there exists a compact metrizable space $\bar{\Gamma} = \Gamma \cup \partial\Gamma$ such that Γ sits densely in $\bar{\Gamma}$, and $\bar{\Gamma}$ is compact. The compactification is Γ -equivariant in the sense that the left translation action of Γ extends to an action by homeomorphisms on $\bar{\Gamma}$.

There turns out to be an interesting duality between functions on Γ which extend continuously to the Gromov compactification $\bar{\Gamma}$, and a certain class of operators on $l^2\Gamma$, as follows. First we recall a definition. For what follows, let e_x, e_y , etc, denote the standard basis vectors in $l^2\Gamma$ corresponding to points $x, y \in \Gamma$. Also, if \tilde{f} is a function on Γ , we shall denote by $M_{\tilde{f}}$ the corresponding multiplication operator on $l^2\Gamma$.

Definition 1. An operator $T \in \mathcal{B}(l^2\Gamma)$ is finite propagation if there exists $R > 0$ such that $\langle T(e_x), e_y \rangle = 0$ whenever $d(x, y) \geq R$.

The duality we have alluded to is stated in the following:

Lemma 2. *If \tilde{f} is a function on Γ which extends continuously to $\bar{\Gamma}$, then $[M_{\tilde{f}}, T]$ is a compact operator for all finite propagation operators T on $l^2\Gamma$.*

For the proof, we shall need to use the following fact about the Gromov compactification of a hyperbolic group (see [5]).

Note that here and elsewhere in this paper, $B_r(x)$, for $r > 0$ and $x \in \Gamma$, denotes the ball of word-metric radius r centered at x .

Lemma 3. *If \tilde{f} is a continuous function on $\bar{\Gamma}$, then for every $R > 0$, we have*

$$\lim_{x \rightarrow \infty} \sup\{|f(x) - f(y)| \mid y \in B_R(x)\} = 0.$$

Proof. (of Lemma 2)

Let T be a finite propagation operator with propagation R , and \tilde{f} a bounded function on Γ which extends continuously to $\bar{\Gamma}$. Then $[M_{\tilde{f}}, T](e_x) = \sum_{y \in B_R(x)} (\tilde{f}(x) - \tilde{f}(y)) T_{xy} e_y$ where T_{xy} denotes as usual $\langle T(e_x), e_y \rangle$. Therefore $\langle [M_{\tilde{f}}, T](e_x), e_y \rangle = 0$ if $d(x, y) \geq R$, and equals $(\tilde{f}(x) - \tilde{f}(y)) T_{xy}$ else. The result follows immediately from Lemma 3. □

Let $\gamma \in \Gamma$, and $\rho(\gamma)$ denote the unitary $l^2\Gamma \rightarrow l^2\Gamma$ induced from right translation by γ , $\rho(\gamma)e_x = e_{x\gamma^{-1}}$. The relevance of the above remarks to us lies in the following observation:

Lemma 4. *$\rho(\gamma)$ is a finite propagation operator on $l^2\Gamma$ for all $\gamma \in \Gamma$.*

Proof. One has $d(x, x\gamma^{-1}) \leq |\gamma|$, from which the result follows with $R = |\gamma|$. □

Corollary 5. *If $\gamma \in \Gamma$ and \tilde{f} is a function on Γ which extends continuously to $\bar{\Gamma}$, then $[\rho(\gamma), M_{\tilde{f}}]$ is a compact operator.*

Now, consider the unitary $I : l^2\Gamma \rightarrow l^2\Gamma$, induced from inversion $\iota : \Gamma \rightarrow \Gamma$. Then $I\rho(\gamma)I = \lambda(\gamma)$, where $\lambda(\gamma)$ denotes left translation by γ ; and $IM_{\tilde{f}}I = M_{\tilde{f}\circ\iota}$. The following follows from conjugating the equation appearing in Corollary 5 by the unitary I :

Corollary 6. *The commutator $[\lambda(\gamma), M_{\tilde{f}\circ\iota}]$ is a compact operator, for every $\gamma \in \Gamma$ and \tilde{f} a function on Γ extending continuously to $\bar{\Gamma}$.*

In Section 3 we will show how the above constructions can be organized to produce a K -homology class inducing a Poincaré duality isomorphism.

3. KK-THEORETIC PRELIMINARIES

In this section we recall some basic facts from KK -theory. For further details we refer the reader to [1], or to [9].

KK. KK can be understood categorically ([6]): there is a category **KK** whose objects are separable, nuclear C^* -algebras and whose morphisms $A \rightarrow B$ are the elements of $KK(A, B)$. There is a functor from the category of C^* -algebras to the category **KK**. If $\phi : A \rightarrow B$ is an algebra homomorphism $A \rightarrow B$, we denote its image under this functor as $[\phi]$. There is a composition, or intersection product operation $KK(A, D) \times KK(D, B) \rightarrow KK(A, B)$ which we denote by $(\alpha, \beta) \mapsto \alpha \otimes_D \beta$. If $\phi : A \rightarrow B$ is an algebra homomorphism, and D is any C^* -algebra, we thus have a map $\phi_* : KK(D, A) \rightarrow KK(D, B)$, given by $\alpha \mapsto \alpha \otimes_A [\phi]$. Similarly we have a map $\phi^* : KK(B, D) \rightarrow KK(A, D)$ given by $\beta \mapsto [\phi] \otimes_B \beta$.

We will sometimes use the notations $\phi_*([\beta])$ and $[\phi] \otimes_B \beta$ interchangeably, as is warranted by clarity of notation. Similarly with ϕ_* .

If D is a C^* -algebra, there is a natural map $KK(A, B) \rightarrow KK(A \otimes D, B \otimes D)$, $\alpha \mapsto \alpha \otimes 1_D$, and similarly a map $KK(A, B) \rightarrow KK(D \otimes A, D \otimes B)$.

Graded Commutativity. There are higher KK groups $KK^i(A, B)$ for all $i \in \mathbb{Z}$, defined by $KK^i(A, B) = KK(A, B \otimes C_i)$ where C_i is the i th complex Clifford algebra, and one of the features of the theory is that the intersection product is graded commutative. If A_1, \dots, A_n are C^* -algebras, let σ_{ij} denote the map

$$A_1 \otimes \dots \otimes A_i \otimes \dots \otimes A_j \otimes \dots \otimes A_n \rightarrow A_1 \otimes \dots \otimes A_j \otimes \dots \otimes A_i \otimes \dots \otimes A_n$$

obtained by flipping the two factors. Then by graded commutativity we mean the following: if $\alpha \in KK^i(A_1, B_1)$ and $\beta \in KK^j(A_2, B_2)$, then

$$(\alpha \otimes 1_{A_2}) \otimes_{B_1 \otimes A_2} (1_{B_1} \otimes \beta) = (-1)^{ij} (\sigma_{12})_* \sigma_{12}^* ((\beta \otimes 1_{A_1}) \otimes (1_{B_2} \otimes \alpha)) \in KK(A_1 \otimes A_2, B_1 \otimes B_2).$$

K-theory and K-homology. For any C^* -algebra A , $KK^i(\mathbb{C}, A) = K_i(A)$ is the topological K -theory of A , and $KK^i(A, \mathbb{C}) = K^i(A)$ is the K -homology of A by definition.

Description of Even Cycles. We let $\mathcal{B}(\mathcal{E})$ denote bounded operators on a Hilbert module \mathcal{E} , $\mathcal{K}(\mathcal{E})$ compact operators, and $\mathcal{Q}(\mathcal{E})$ the Calkin algebra $\mathcal{B}(\mathcal{E})/\mathcal{K}(\mathcal{E})$. The quotient map $\mathcal{B}(\mathcal{E}) \rightarrow \mathcal{Q}(\mathcal{E})$ will always be denoted by π .

Following Kasparov ([9]), if \mathcal{E} is a Hilbert B -module and A acts on \mathcal{E} by a homomorphism $A \rightarrow \mathcal{B}(\mathcal{E})$, we will refer to \mathcal{E} as a Hilbert (A, B) -bimodule.

Because all the algebras in this paper are ungraded – or alternatively, have trivial grading – we can make certain simplifications in the definitions of the KK groups (see [1]). With such ungraded A and B , cycles for $KK(A, B)$ are given simply by pairs (\mathcal{E}, F) where \mathcal{E} is an (A, B) -bimodule, F commutes modulo compact operators with the action of A , and $a(F^*F - 1)$ and $a(FF^* - 1)$ are compact for every $a \in A$.

Description of Odd Cycles. Cycles for $KK^1(A, B)$ are given by pairs (\mathcal{E}, P) for which P is an operator on the (A, B) -bimodule \mathcal{E} satisfying the three conditions $[a, P]$, $a(P^2 - P)$, and $a(P - P^*)$ are compact for all $a \in A$.

Let (\mathcal{E}, P) be an odd cycle. Then we obtain a homomorphism $A \rightarrow \mathcal{Q}(\mathcal{E})$ by the formula $a \mapsto \pi(PaP)$.

Conversely, let $\tau : A \rightarrow \mathcal{Q}(\mathcal{E})$ be a homomorphism. Under the assumption of nuclearity of all algebras concerned, there exists a potentially larger Hilbert B -module $\tilde{\mathcal{E}}$, a representation of A on $\tilde{\mathcal{E}}$, an isometry $U : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$, and an operator P on $\tilde{\mathcal{E}}$ such that $a(P^2 - P)$, $[a, P]$, and $a(P - P^*)$ are compact for all $a \in A$, and $\pi(U^*PaPU) = \tau(a)$ for all $a \in A$ (see [1]). The data $(\tilde{\mathcal{E}}, P)$ makes up an odd cycle. The process of constructing a $\tilde{\mathcal{E}}$, U , and P , from an extension, we shall refer to as the Stinespring construction.

As a consequence, for A and B nuclear, we may regard $KK^1(A, B)$ as given by classes of maps $\tau : A \rightarrow \mathcal{Q}(\mathcal{E})$, where \mathcal{E} is a right Hilbert B -module. This description of KK^1 -classes will be particularly appropriate to our purposes.

Bott Periodicity. Recall that $KK^{-1}(\mathbb{C}, C_0(\mathbb{R})) \cong \mathbb{Z}$ and is generated by the class $[\hat{d}_{\mathbb{R}}]$ of the multiplier $f(x) = \frac{x}{\sqrt{1+x^2}}$ of $C_0(\mathbb{R})$, suitably interpreted in terms of the Clifford gradings. The class $[\hat{d}_{\mathbb{R}}]$ allows us to identify, for any C^* -algebras A and B , the groups $KK^1(C_0(\mathbb{R}) \otimes A, B)$, and $KK(A, B)$, by the map $KK^1(C_0(\mathbb{R}) \otimes A, B) \rightarrow KK(A, B)$, $x \mapsto [\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} x$. We shall need to compute this map at the level of cycles in several simple cases.

Let ψ be the function $\psi(t) = \frac{-2i}{t+i}$ in $C_0(\mathbb{R})$. It has the property that $\psi + 1$ is unitary in $C_0(\mathbb{R})^+$. We begin by stating the simplest version of what we shall need.

Lemma 7. *Let τ be a homomorphism $C_0(\mathbb{R}) \rightarrow \mathcal{Q}(H)$ to the Calkin algebra of some Hilbert space H . Let $[\tau]$ denote the class in $KK^1(C_0(\mathbb{R}), \mathbb{C})$ corresponding to τ . Then the class $[\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [\tau] \in KK(\mathbb{C}, \mathbb{C})$ is represented by the cycle $(H, U + 1)$, where U is any operator on H such that $\pi(U) = \tau(\psi)$.*

The significance of this simple lemma is that in the given setting it is not necessary to *explicitly* represent $[\tau]$ as a KK -cycle (that is, perform the Stinespring construction) in order to calculate the Kasparov product of $[\hat{d}_{\mathbb{R}}]$ and $[\tau]$. This is true also of the situation in the following lemma, which will be of direct use to us.

Lemma 8. *Let A_1, A_2 be C^* -algebras and \mathcal{E} be a right Hilbert A_2 -module. Let h be a homomorphism $C_0(\mathbb{R}) \otimes A_1 \rightarrow \mathcal{Q}(\mathcal{E})$ and $[h]$ its class, regarded as an element of $KK^1(C_0(\mathbb{R}) \otimes A_1, A_2)$. Assume that h has the form $x \otimes a_1 \mapsto h'(x)h''(a_1)$, where h' and h'' are homomorphisms. Suppose that the homomorphism h'' lifts to a homomorphism $\tilde{h}'' : A_1 \rightarrow \mathcal{B}(\mathcal{E})$. Then the class $[\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [h] \in KK(A_1, A_2)$ is represented by the following cycle. The module is \mathcal{E} with its original right A_2 -module structure and the left A_1 -module structure given by the homomorphism \tilde{h}'' . The operator is given by $U + 1$ where U is any operator on \mathcal{E} such that $\pi(U) = h'(\psi)$.*

The proof of both lemmas involves an application of the axioms for the intersection product, and is omitted (see [9]).

Equivariant KK. If Γ is a group acting on C^* -algebras A and B , we have in addition to the group $KK(A, B)$, an equivariant group $KK_{\Gamma}(A, B)$. We shall discuss this group briefly in connection with the γ -element and the work of Julg and Valette. Suffice it to say

that the cycles for $KK_\Gamma(A, B)$ consist of the same cycles as for $KK(A, B)$, but with the following extra conditions. (1) Γ acts as linear isometric maps on the Hilbert (A, B) -module \mathcal{E} , in such a way that $\gamma(a\xi b) = \gamma(a)\gamma(\xi)\gamma(b)$ for $a \in A$, $b \in B$ and $\xi \in \mathcal{E}$; (2) the operator F satisfies: $\gamma(F) - F$ is compact, for all $\gamma \in \Gamma$.

Regarding \mathbf{KK}_Γ as a category in its own right, with morphisms $A \rightarrow B$ the elements of $KK_\Gamma(A, B)$, and objects Γ - C^* -algebras, there is a functor $\lambda : KK_\Gamma(A, B) \rightarrow KK(A \rtimes \Gamma, B \rtimes \Gamma)$, called *descent*. The map $\lambda : KK_\Gamma(A, B) \rightarrow KK(A \rtimes \Gamma, B \rtimes \Gamma)$ can be explicitly calculated on cycles; the formulas are given in [9]. Since λ is a functor, it takes the unit $1_A \in KK_\Gamma(A, A)$ to the unit $1_{A \rtimes \Gamma} \in KK(A \rtimes \Gamma, A \rtimes \Gamma)$, which fact we will make use of.

4. CONSTRUCTION OF THE FUNDAMENTAL CLASS

For this section, we shall return to the generality of a hyperbolic group Γ . Since Γ acts by homeomorphisms on $\partial\Gamma$, we can consider the cross product C^* -algebra $C(\partial\Gamma) \rtimes \Gamma$, which is our main object of interest in this paper. Note the cross product we are referring to is the *reduced* cross product; however, by the proof of the following lemma (whose proof can be found in [3]), the reduced and max cross products are in fact the same.

Lemma 9. *The algebra $C(\partial\Gamma) \rtimes \Gamma$ is nuclear and separable.*

Our goal is to construct an element of the K -homology of the algebra $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma$, specifically, an element of $KK^1(C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma, \mathbb{C})$. This element will be presented as an extension; that is, as a map $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(H)$ for some Hilbert space H . By our remarks in the previous section and Lemma 9, such a map does produce a canonical class in $KK^1(C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma, \mathbb{C})$.

We construct two commuting maps $\lambda, \rho : C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$. Let $f \in C(\partial\Gamma)$ and let \tilde{f} denote any extension of f to a continuous function on $\bar{\Gamma}$. Let $M_{\tilde{f}}$ denote as above the multiplication operator on $l^2\Gamma$ corresponding to \tilde{f} , and let $\lambda(f)$ be the image in $\mathcal{Q}(l^2\Gamma)$ of the operator $M_{\tilde{f}}$. Let $\lambda(\gamma)$ be the image in $\mathcal{Q}(l^2\Gamma)$ of the unitary u_γ corresponding to left translation by γ : $u_\gamma(e_x) = e_{\gamma x}$, $x \in \Gamma$. It is easy to check that the assignments $f \rightarrow \lambda(f)$, $\gamma \rightarrow \lambda(\gamma)$, define a covariant pair for the C^* -dynamical system $(C(\partial\Gamma), \Gamma)$, and so a homomorphism

$$\lambda : C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma).$$

Next, define

$$\rho : C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$$

by $\rho(a) = I\lambda(a)I$, where I is at the end of Section 2. Thus $\rho(f)$ is the image in $\mathcal{Q}(l^2\Gamma)$ of the multiplication operator $M_{\tilde{f} \circ l}$, and $\rho(\gamma)$ is the image in $\mathcal{Q}(l^2\Gamma)$ of right translation by γ , $e_x \mapsto e_{x\gamma^{-1}}$. The following follows from Corollaries 5 and 6.

Theorem 10. *The homomorphisms, $\lambda, \rho : C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$ commute, and so define a homomorphism $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$ by $a \otimes b \rightarrow \lambda(a)\rho(b)$.*

Definition 11. Let $\Delta \in KK^1(C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma, \mathbb{C})$ denote the class corresponding to the above homomorphism $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$.

We shall refer to the class Δ as the *fundamental class* of the algebra $C(\partial\Gamma) \rtimes \Gamma$.

Before proceeding, let us note the following. Let $\sigma_{12} : C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma \rightarrow C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma$ the homomorphism which interchanges factors, and $\sigma_{12}^* : KK^1(C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma, \mathbb{C}) \rightarrow KK^1(C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma, \mathbb{C})$ the corresponding homomorphism of KK

groups. The following rather simple observation reflects a common property of ‘fundamental classes,’ i.e. those classes implementing by cap product Poincaré duality isomorphisms; the author knows of no case, either commutative or not, where the fundamental class does not have it.

Lemma 12. *We have: $\sigma_{12}^*(\Delta) = \Delta$.*

Proof. For $\sigma_{12}^*(\Delta)$ is the class corresponding to the map $C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma \rightarrow \mathcal{Q}(l^2\Gamma)$, $a \otimes b \mapsto \rho(a)\lambda(b)$. But this is unitarily conjugate to the map $a \otimes b \mapsto \lambda(a)\rho(b)$ via the symmetry I . □

We can now define the ‘cap-product map’ interchanging the K -theory and K -homology of $C(\partial\Gamma) \rtimes \Gamma$, which we are going to show is an isomorphism when $\Gamma = \mathbb{F}_2$. Specifically, define:

$$\cap\Delta : K_*(C(\partial\Gamma) \rtimes \Gamma) \rightarrow K^{*+1}(C(\partial\Gamma) \rtimes \Gamma)$$

by the formula

$$x \mapsto (x \otimes 1_{C(\partial\Gamma) \rtimes \Gamma}) \otimes_{C(\partial\Gamma) \rtimes \Gamma \otimes C(\partial\Gamma) \rtimes \Gamma} \Delta.$$

Our main theorem is the following:

Theorem 13. *For $\Gamma = \mathbb{F}_2$ and Δ as in Definition 11, the map $\cap\Delta$ is an isomorphism.*

5. CONNES’ NOTION OF POINCARÉ DUALITY

In order to prove that the map $\cap\Delta$ of the previous section is an isomorphism, we shall use some ideas due to Connes.

Theorem 14. *Let A be a separable, nuclear C^* -algebra, and Δ a class in $KK^i(A \otimes A, \mathbb{C})$.*

Suppose we can find a class $\hat{\Delta} \in KK^{-i}(\mathbb{C}, A \otimes A)$ such that the following equations hold:

$$(\hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta) = 1_A, \tag{1}$$

and

$$((\sigma_{12})_* \hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \Delta) = (-1)^i 1_A. \tag{2}$$

Then the map

$$\cap\Delta : K_j(A) \rightarrow K^{j+i}(A)$$

defined previously, is an isomorphism with inverse (up to sign) the map $K^j(A) \rightarrow K_{j-i}(A)$,

$$y \mapsto \hat{\Delta} \otimes_{A \otimes A} (1_A \otimes y).$$

If A is as above, with classes Δ and $\hat{\Delta}$ satisfying Equations (1) and (2), we will call A a Poincaré duality algebra.

Proof. The hypotheses imply the two equations:

$$(\hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^*(\Delta)) = 1_A, \tag{3}$$

and

$$((\sigma_{12})_* \hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \Delta) = (-1)^i 1_A. \tag{4}$$

We show that as a consequence of these two equations,

$$\hat{\Delta} \otimes_{A \otimes A} (1_A \otimes (y \cap \Delta)) = (-1)^{ij} y, \quad y \in KK^j(\mathbb{C}, A) \tag{5}$$

Expanding the product involved in (5), we obtain:

$$\hat{\Delta} \otimes_{A \otimes A} (1_A \otimes y \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \Delta).$$

Consider the term $(1_A \otimes y \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \Delta)$. It is easy to check this is the same as $(1_A \otimes 1_A \otimes y) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^*(\Delta))$. Returning to the original product (5), we see the latter can be written

$$(\hat{\Delta} \otimes_{A \otimes A} (1_{A \otimes A} \otimes y)) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^*(\Delta)).$$

Now, by skew-commutativity of the external tensor product,

$$\hat{\Delta} \otimes_{A \otimes A} (1_{A \otimes A} \otimes y) = (-1)^{ij} (\sigma_{23})_*(\sigma_{12})_*(y \otimes_A (1_A \otimes \hat{\Delta})) = (-1)^{ij} y \otimes_A (\sigma_{23})_*(\sigma_{12})_*(1_A \otimes \hat{\Delta}).$$

Furthermore, $(\sigma_{23})_*(\sigma_{12})_*(1_A \otimes \hat{\Delta}) = \hat{\Delta} \otimes 1_A$. Hence, putting back into the main product, we see that (5) can be written

$$(-1)^{ij} y \otimes_A ((\hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^*(\Delta))) = (-1)^{ij} y,$$

where the last equality follows from equation (1). □

Remark 15. We note that if we happen to have Δ and $\hat{\Delta}$ as above, and

$$\sigma_{12}^*(\Delta) = \Delta$$

and

$$(\sigma_{12})_*(\hat{\Delta}) = (-1)^i \hat{\Delta},$$

then the two equations (1) and (2) above would be the same, and it would suffice to show that one of them holds. This is the case in the commutative setting of a compact spin^c-manifold, and will be the case for us, also, part of which we have already proven (Lemma 12).

We now set about proving Theorem 13 in the case of $\Gamma = \mathbb{F}_2$ by verifying the equations (1) and (2) of Theorem 14 above, with, i.e. $A = C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$ and Δ the fundamental class of Definition 11. We need first produce an element $\hat{\Delta} \in KK^{-1}(\mathbb{C}, C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$ playing the role of the dual element in Theorem 14. We will then verify equation (1), the other being rendered superfluous as a consequence of Remark 15, which is applicable in this case.

It will turn out, rather surprisingly, that equation (1) can be shown to be equivalent to the equation

$$\gamma_{\partial\mathbb{F}_2 \rtimes \mathbb{F}_2} = 1_{C(\partial\mathbb{F}_2)},$$

where $\gamma_{\partial\mathbb{F}_2 \rtimes \mathbb{F}_2}$ is the γ -element for the groupoid $\partial\mathbb{F}_2 \rtimes \mathbb{F}_2$. Since this latter equation has been established by Julg and Valette, and also by J.L. Tu, we will by this device, i.e. by means of the Baum-Connes Conjecture, be done.

6. CONSTRUCTION OF A DUAL ELEMENT

In this section as for the rest of this note we specialize to the free group \mathbb{F}_2 on two generators. We are going to define an element $\hat{\Delta} \in KK^{-1}(\mathbb{C}, C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$ serving as an ‘inverse’ to Δ .

$\hat{\Delta}$ shall be constructed by use of the fact that any two points of $\partial\mathbb{F}_2$ may be connected by a unique geodesic.

By “geodesic” we shall mean an isometric map $r : \mathbb{Z} \rightarrow \mathbb{F}_2$. Topologize the collection of such r by means of the metric

$$d_{G\mathbb{F}_2}(r_1, r_2) = \sum_{n \in \mathbb{Z}} 2^{-|n|} d(r_1(n), r_2(n))$$

and denote the resulting metric space by $G\mathbb{F}_2$ (we follow [4]). Both \mathbb{F}_2 and \mathbb{Z} act freely and properly on $G\mathbb{F}_2$, the former by translation $(\gamma r)(n) = \gamma r(n)$, and the latter by flow $(g^n r)(k) = r(k - n)$. These actions commute. Note that $G\mathbb{F}_2/\mathbb{F}_2$ is compact, whereas $G\mathbb{F}_2/\mathbb{Z}$ may be identified with the \mathbb{F}_2 -space

$$\partial^2 \mathbb{F}_2 = \{(a, b) \in \partial \mathbb{F}_2 \times \partial \mathbb{F}_2 \mid a \neq b\}.$$

All these observations are easy to check. As a consequence of them, the C^* -algebras $C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z}$ and $C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2$ are strongly Morita equivalent (see [10]). Let $[E]$ denote the class of the strong Morita equivalence bimodule. It is an element of $KK(C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z}, C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2)$.

On the other hand, if u is the generator of $\mathbb{Z} \subset C^*(\mathbb{Z}) \subset C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z}$, we obtain a natural homomorphism $C_0(\mathbb{R}) \rightarrow C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z}$ by the formula $\psi \mapsto u - 1$ where, recall, ψ is a specified generator of $C_0(\mathbb{R})$ satisfying $\psi + 1 \in C_0(\mathbb{R})^+$ is unitary.

We denote the class in $KK(C_0(\mathbb{R}), C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z})$ of this homomorphism by $[u - 1]$.

It will be convenient for our later computations to define an auxilliary class $[D]$, which will lie in $KK^{-1}(C, C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2)$, as follows.

Definition 16. The class $[D] \in KK^{-1}(C, C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2)$ shall be defined by

$$[D] = [\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [u - 1] \otimes_{C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z}} [E].$$

Next, note that the cross product $C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2$ may be regarded as a subalgebra of $C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2$, via the composition of inclusions:

$$C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2 \rightarrow C(\partial \mathbb{F}_2 \times \partial \mathbb{F}_2) \rtimes \mathbb{F}_2 \cong (C(\partial \mathbb{F}_2) \otimes C(\partial \mathbb{F}_2)) \rtimes \mathbb{F}_2 \rightarrow C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2.$$

Let i denote this composition.

Our class $\hat{\Delta}$ will be defined by:

Definition 17. Let $\hat{\Delta} = [D] \otimes_{C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2} [i] \in KK^{-1}(C, C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial \mathbb{F}_2) \rtimes \mathbb{F}_2)$, where $[\hat{d}_{\mathbb{R}}]$ is as in Section 3, and $[u - 1]$ and $[E]$ are as above.

It will be convenient to calculate more explicitly the cycle corresponding to the class $[u - 1] \otimes_{C(G\mathbb{F}_2/\mathbb{F}_2) \rtimes \mathbb{Z}} [E] \in KK(C_0(\mathbb{R}), C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2)$. We will express it as a homomorphism $C_0(\mathbb{R}) \rightarrow C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2$; that is, as an element $w \in C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2$ such that $w + 1$ is unitary in $(C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2)^+$.

We will first describe an element $v \in C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2$ satisfying $v^*v = vv^* = \chi$, where χ is a projection. We will then set $w = v - \chi$. Then, of course, $w + 1 = v + 1 - \chi$ will be unitary in $(C_0(\partial^2 \mathbb{F}_2) \rtimes \mathbb{F}_2)^+$.

As the method of discovering such an explicit description (that is, of transferring K -classes under strong Morita equivalences) is well known (see [2] in which a similar calculation is carried out in the context of A_θ) we give the outcome without further discussion.

As a function on $\partial^2 \mathbb{F}_2 \times \mathbb{F}_2$, $v(a, b, \gamma) = 1$ if and only if there exists a geodesic $r_{a,b}$ such that $r_{a,b}(-\infty) = a$, $r_{a,b}(+\infty) = b$, $r_{a,b}(0) = e$, and $r_{a,b}(-1) = \gamma$. And $v(a, b, \gamma) = 0$ else.

Note that $\chi = v^*v = vv^*$ is the locally constant function on $\partial^2\mathbb{F}_2$ given by $\chi(a, b) = 1$ if some (therefore any) geodesic from a to b passes through e , and equals 0 else.

We can describe v in group-algebra notation as follows. Fix γ a generator. Then $v(\cdot, \cdot, \gamma)$ is a function on $\partial^2\mathbb{F}_2$, and in particular is a function on $\partial\mathbb{F}_2 \times \partial\mathbb{F}_2$, whose representation as a tensor product of two functions on $\partial\mathbb{F}_2$ is:

$$v(\cdot, \cdot, \gamma) = \chi_\gamma \otimes (1 - \chi_\gamma),$$

where

$$\chi_\gamma(a) = \begin{cases} 1 & \gamma \in [e, a] \\ 0 & \text{else} \end{cases}.$$

We can therefore represent v as

$$v = \sum_{|\gamma|=1} \chi_\gamma \otimes (1 - \chi_\gamma) \in C_0(\partial^2\mathbb{F}_2) \rtimes \mathbb{F}_2 \subset C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2.$$

Similarly we represent the function χ by $\chi = \sum \chi_\gamma \otimes (1 - \chi_\gamma)$, and it is easy to check that $v^*v = vv^* = \chi$, as claimed.

Finally, we note the following:

Lemma 18. *The class $\hat{\Delta}$ satisfies $(\sigma_{12})_*(\hat{\Delta}) = -\hat{\Delta}$.*

Proof. We have $\hat{\Delta} = i_*([D])$, and so $(\sigma_{12})_*(\hat{\Delta}) = (\sigma_{12})_*i_*([D]) = (\sigma_{12} \circ i)_*([D]) = (\bar{\sigma}_{12})_*([D])$, where $\bar{\sigma}_{12} : C_0(\partial^2\mathbb{F}_2) \rtimes \mathbb{F}_2 \rightarrow C_0(\partial^2\mathbb{F}_2) \rtimes \mathbb{F}_2$ is the algebra homomorphism induced by the \mathbb{F}_2 -equivariant map $\partial^2\mathbb{F}_2 \rightarrow \partial^2\mathbb{F}_2$, $(a, b) \mapsto (b, a)$. Now $[D] = [\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [v - \chi]$ and hence $(\bar{\sigma}_{12})_*([D]) = [\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} (\bar{\sigma}_{12})_*([v - \chi]) = [\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [v^* - \chi] = -[\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [v^* - \chi]$ by a direct calculation and we are done. \square

In the following sections we will show that in an appropriate sense $\hat{\Delta}$ provides an ‘inverse’ to the extension Δ . More precisely, we will show that the conditions of Theorem 14 are met by Δ the fundamental class, and the element $\hat{\Delta}$ above.

7. THE γ -ELEMENT

Before proceeding to verify the equations of Theorem 14, we will need to recall the work of Julg and Vallette ([7]).

Up to now we have adopted the convention of writing even KK -cycles in the form (\mathcal{E}, F) , where F is an operator on the module \mathcal{E} . A different definition is possible, in which *two* modules are involved, and F is an operator between them. This was the set-up in the paper of [7]. We will retain their notation temporarily. In a moment we will describe a means of *geometrically* describing their class in a way consistent with our conventions.

Consider the Cayley graph Σ for \mathbb{F}_2 , which is a tree with edges Σ^1 and vertices Σ^0 . Note that we work with *geometric* edges, i.e. set theoretic pairs of vertices $\{x, x'\}$. If x is a vertex, let x' be the vertex one unit closer to e , the origin, and let $s(x)$ be the edge $\{x, x'\}$. Define an operator

$$b : l^2\Sigma^0 \rightarrow l^2\Sigma^1$$

by

$$b(e_x) = \begin{cases} e_{s(x)} & x \neq e \\ 0 & x = e \end{cases}.$$

Then it is clear that b is an isometry, is Fredholm, and has index 1. Next, note that \mathbb{F}_2 acts unitarily on $l^2(\Sigma^0)$ and $l^2(\Sigma^1)$, and that, furthermore, $\gamma b \gamma^{-1} - b$ is a compact (in fact finite rank) operator, for all $\gamma \in \mathbb{F}_2$.

It follows that the pair $(l^2\Sigma^0 \oplus l^2\Sigma^1, \begin{pmatrix} 0 & b^* \\ b & 0 \end{pmatrix})$ defines a cycle for $KK_{\mathbb{F}_2}(\mathbb{C}, \mathbb{C})$.

Let γ denote its class. That $\gamma = 1$ in this group implies the Baum-Connes conjecture for \mathbb{F}_2 . This fact (that $\gamma = 1$) was proved by Julg and Valette in [7].

We can produce a cycle for $KK_{\mathbb{F}_2}(C(\partial\mathbb{F}_2), C(\partial\mathbb{F}_2))$, whose class we will denote by $\gamma_{\partial\mathbb{F}_2}$, by tensoring all the above data with $C(\partial\mathbb{F}_2)$. Thus, let $\mathcal{E}^0 = C(\partial\mathbb{F}_2; l^2(\Sigma^0))$, and $\mathcal{E}^1 = C(\partial\mathbb{F}_2; l^2(\Sigma^1))$. Let $B : \mathcal{E}^0 \rightarrow \mathcal{E}^1$ be defined by $(B\xi)(a) = b(\xi(a))$. The Hilbert $C(\partial\mathbb{F}_2)$ -modules \mathcal{E}^i carry obvious actions of \mathbb{F}_2 . Let $\gamma_{\partial\mathbb{F}_2}$ be the class of the cycle $(\mathcal{E}^0 \oplus \mathcal{E}^1, \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix})$.

It is easy to check that the process of tensoring with $C(\partial\mathbb{F}_2)$ in this way preserves units; that is:

$$\gamma = 1 \Rightarrow \gamma_{\partial\mathbb{F}_2} = 1_{C(\partial\mathbb{F}_2)}$$

in the ring $KK_{\mathbb{F}_2}(C(\partial\mathbb{F}_2), C(\partial\mathbb{F}_2))$. Hence, we have:

Lemma 19. *The cycle $(\mathcal{E}^0 \oplus \mathcal{E}^1, \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix})$ is equivalent to the cycle corresponding to $1_{C(\partial\mathbb{F}_2)}$ in the group $KK_{\mathbb{F}_2}(C(\partial\mathbb{F}_2), C(\partial\mathbb{F}_2))$.*

We now set about describing a cycle equivalent to the above but which is in some sense simpler. To do this it will be notationally and conceptually simpler to work with fields. Thus, we note that \mathcal{E}^0 and \mathcal{E}^1 may be viewed as sections of the constant fields of Hilbert spaces $\{H_a^0 \mid a \in \partial\mathbb{F}_2\}$, respectively $\{H_a^1 \mid a \in \partial\mathbb{F}_2\}$, with $H_a^0 = l^2(\Sigma^0)$ and $H_a^1 = l^2(\Sigma^1)$ for all $a \in \partial\mathbb{F}_2$, and that the operator B may be regarded as the constant family of operators $\{b_a \mid a \in \partial\mathbb{F}_2\}$ with $b_a = b$ for all $a \in \partial\mathbb{F}_2$. What we are going to do is eliminate edges from the cycle at the expense of changing the constant field of operators to a nonconstant field.

To this end consider the field of unitary maps $\{U_a : H_a^1 \rightarrow H_a^0 \mid a \in \partial\mathbb{F}_2\}$ given by $U_a(e_s) = e_x$, where x is the vertex of s farthest from a . Note that the assignment $a \mapsto U_a$, though not constant, is strongly continuous. For if a and b are two boundary points, then $U_a = U_b$ except for edges lying on the geodesic (a, b) . Consequently, if s is a fixed edge, and a and b are close enough, then $U_a(e_s) = U_b(e_s)$, since if a and b are sufficiently close, s does not lie on (a, b) .

Now, consider the composition

$$l^2\mathbb{F}_2 = H_a^0 \xrightarrow{b_a} H_a^1 \xrightarrow{U_a} H_a^0 = l^2\mathbb{F}_2,$$

which we denote by W_a . We see that for $x = e$, $W_a(e_x) = 0$, and for $x \neq e$ we have:

$$W_a(e_x) = \begin{cases} e_{x'} & x \in [e, a) \\ e_x & \text{else} \end{cases},$$

where x' is the vertex one unit closer to e than x .

Since the assignment $a \rightarrow W_a$ is continuous, we obtain a Hilbert $C(\partial\mathbb{F}_2)$ -module map $\mathcal{E}^0 \rightarrow \mathcal{E}^0$ by defining for $\xi \in C(\partial\mathbb{F}_2; l^2\mathbb{F}_2)$, $(W\xi)(a) = W_a(\xi(a))$. Then, by unitary invariance of KK and the work of Julg and Valette, we see:

Lemma 20. *The cycle $(\mathcal{E}^0 \oplus \mathcal{E}^0, \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix})$ is equivalent to the cycle corresponding to $1_{\partial\mathbb{F}_2}$ in the group $KK_{\mathbb{F}_2}(C(\partial\mathbb{F}_2), C(\partial\mathbb{F}_2))$.*

Since we have now altered the cycle of Julg and Valette up to equivalence so that only one Hilbert module is involved (it is now otherwise known as an ‘evenly graded’ Fredholm module), we may now return as promised to our conventions and write it simply

$$(C(\partial\mathbb{F}_2; l^2\mathbb{F}_2), W),$$

consistent with the way we have been writing (even) KK -cycles up to now.

To summarize, we have:

$$[(C(\partial\mathbb{F}_2; l^2\mathbb{F}_2), W)] = [1_{C(\partial\mathbb{F}_2)}] \in KK_{\mathbb{F}_2}(C(\partial\mathbb{F}_2), C(\partial\mathbb{F}_2)).$$

We shall next apply the descent map

$$\lambda : KK_{\mathbb{F}_2}(C(\partial\mathbb{F}_2), C(\partial\mathbb{F}_2)) \rightarrow KK(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2, C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$$

to the cycle described above, thus producing a cycle for $KK(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2, C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$ which by functoriality of descent will be equivalent to the cycle corresponding to $1_{C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2}$. A direct appeal to the definition of λ (see [9]) produces the cycle $(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes l^2\mathbb{F}_2, \bar{W})$, where, regarding $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes l^2\mathbb{F}_2$ as given by functions $\mathbb{F}_2 \rightarrow C(\partial\mathbb{F}_2) \otimes l^2\mathbb{F}_2$, the action of \bar{W} on these functions is given by the formula $(\bar{W}\xi)(\gamma) = W(\xi(\gamma))$. We have:

Lemma 21. *The cycle $(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes l^2\mathbb{F}_2, \bar{W})$ is equivalent to the cycle corresponding to $1_{C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2}$ in $KK(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2, C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2)$.*

This concludes our preparatory work. We will now show that the class of the cycle given in the above lemma is the same as the class of the Kasparov product of the elements $\hat{\Delta}$ and Δ , concluding thus as a consequence of the work of Julg and Valette that equation (1) holds.

8. UNTWISTING

We are interested in calculating the cycle corresponding to the Kasparov product

$$(\hat{\Delta} \otimes 1_{C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2}) \otimes_{C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2} (1_{C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2} \otimes \sigma_{12}^* \Delta).$$

In this section we will do something we call - following an analogous procedure in [8] - ‘untwisting.’ A simple but fundamental property of hyperbolic groups - and in particular of the free group - will be used: specifically, *if two points a and b on $\partial\mathbb{F}_2$ are sufficiently far apart then any geodesic connecting them passes quite close to the identity e of the group.* This follows immediately from the definition of the topology on the compactified space $\overline{\mathbb{F}_2}$. More precisely:

Lemma 22. *Let \tilde{N} be a neighbourhood of the diagonal $\{(a, a) \mid a \in \partial\mathbb{F}_2\}$ in $\partial\mathbb{F}_2 \times \overline{\mathbb{F}_2}$. Then there exists $R > 0$ such that if $(a, b) \in (\partial\mathbb{F}_2 \times \overline{\mathbb{F}_2}) \setminus \tilde{N}$, then the (unique) geodesic from a to b passes through $B_R(e)$.*

Note 23. To simplify notation in this section, we shall denote by A the cross product $C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2$, and by B the algebra $C_0(\partial^2\mathbb{F}_2) \rtimes \mathbb{F}_2$.

Consider then the product $(\hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta)$ involved on the left hand side of equation (1).

Since $\hat{\Delta} = i_*([D]) = [D] \otimes_B [i]$, we have

$$(\hat{\Delta} \otimes 1_A) \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta) = ([D] \otimes 1_A) \otimes_{B \otimes A} [i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta).$$

We will begin by examining the term $[i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta) \in KK^1(B \otimes A, A)$. It is easy to describe the corresponding cycle explicitly. For since $\sigma_{12}^* \Delta$ is represented by a map $A \otimes A \rightarrow \mathcal{Q}(l^2 \mathbb{F}_2)$, so also $1_A \otimes \sigma_{12}^* \Delta$ is represented by a map $A \otimes A \otimes A \rightarrow \mathcal{Q}(A \otimes l^2 \mathbb{F}_2)$, and $[i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta)$ is represented by a map $B \otimes A \rightarrow \mathcal{Q}(A \otimes l^2 \mathbb{F}_2)$. By construction, this map is given on the set of elementary tensors by the formula

$$a_1 \otimes a_2 \otimes a_3 \mapsto a_1 \otimes \rho(a_2) \lambda(a_3), \quad (6)$$

where we have suppressed the inclusion $i : B \rightarrow A \otimes A$, so that $a_1 \otimes a_2$ in the above expression is understood as an element of B .

We shall first show that the above map up to unitary equivalence can be rewritten in a much more tractable way.

Before proceeding, let \tilde{G} be a function on $\partial \mathbb{F}_2 \times \mathbb{F}_2$ not necessarily continuous in the second variable, but continuous in the first. Then \tilde{G} can be made to act on the right A -module $A \otimes l^2 \mathbb{F}_2$ by the formula

$$\tilde{G} \cdot (a \otimes e_y) = \tilde{G}(\cdot, y) a \otimes e_y,$$

noting that for each $y \in \mathbb{F}_2$, $\tilde{G}(\cdot, y) \in C(\partial \mathbb{F}_2) \subset A$.

Now let F be a continuous, compactly supported function on $\partial^2 \mathbb{F}_2$. Thus F is a continuous function on $\partial \mathbb{F}_2 \times \partial \mathbb{F}_2$ vanishing in a neighbourhood of the diagonal. So we can extend it to a continuous function \tilde{F} on $\partial \mathbb{F}_2 \times \overline{\mathbb{F}_2}$ by the Tietze Extension Theorem and restrict the result to $\partial \mathbb{F}_2 \times \mathbb{F}_2$. Let \tilde{F}' denote the function on $\partial \mathbb{F}_2 \times \mathbb{F}_2$ given by $(a, x) \mapsto \tilde{F}(x^{-1}a, x^{-1})$. Note that \tilde{F}' is continuous in the first variable but not in the second. Hence \tilde{F}' may be made to act on the Hilbert A -module $A \otimes l^2 \mathbb{F}_2$ by the remark in the previous paragraph. We can thus regard \tilde{F}' as an element of $\mathcal{B}(A \otimes l^2 \mathbb{F}_2)$. Let $\tau(F)$ denote the image of the operator \tilde{F}' in $\mathcal{Q}(A \otimes l^2 \mathbb{F}_2)$.

Remark that $F \mapsto \tau(F)$ is a well-defined homomorphism $C_0(\partial^2 \mathbb{F}_2) \rightarrow \mathcal{Q}(A \otimes l^2 \mathbb{F}_2)$. For any two extensions of F to functions on $\partial \mathbb{F}_2 \times \overline{\mathbb{F}_2}$ differ by a function - say \tilde{H} - vanishing on $\partial \mathbb{F}_2 \times \partial \mathbb{F}_2$. Then \tilde{H}' also vanishes on $\partial \mathbb{F}_2 \times \partial \mathbb{F}_2$, and so defines an operator lying in $\mathcal{K}(A \otimes l^2 \mathbb{F}_2)$.

Next, for $\gamma \in \mathbb{F}_2$, set $\tau(\gamma) = 1 \otimes \rho(\gamma) \in \mathcal{Q}(A \otimes l^2 \mathbb{F}_2)$. It is a routine computation to check that the assignments

$$F \mapsto \tau(F)$$

and

$$\gamma \mapsto \rho(\gamma)$$

make up a covariant pair for the dynamical system $(C_0(\partial^2 \mathbb{F}_2), \mathbb{F}_2)$, and hence a homomorphism

$$\tau : B \rightarrow \mathcal{Q}(A \otimes l^2 \mathbb{F}_2).$$

Next, define a covariant pair for the dynamical system $(C(\partial \mathbb{F}_2), \mathbb{F}_2)$ by $\varphi(f) = f \otimes 1 \in \mathcal{B}(A \otimes l^2 \mathbb{F}_2)$, and $\varphi(\gamma) = \gamma \otimes u_\gamma \in \mathcal{B}(A \otimes l^2 \mathbb{F}_2)$. It is similarly easy to check this makes up a

covariant pair and so a homomorphism

$$\varphi : A \rightarrow \mathcal{B}(A \otimes l^2\mathbb{F}_2).$$

The following proposition is key to the untwisting argument.

Proposition 24. *The class $[i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta) \in KK^1(B \otimes A, A)$ is represented by the homomorphism $\iota : B \otimes A \rightarrow \mathcal{Q}(A \otimes l^2\mathbb{F}_2)$,*

$$\iota(b \otimes a) = \tau(b)\pi(\varphi(a)), \quad b \in B, a \in A$$

where φ, τ are as above.

We note that the homomorphisms τ and $\pi \circ \varphi$ commute, and so ι actually is a homomorphism. That ι is a homomorphism also follows, however, from the proof of Lemma 24 below, which shows that ι is unitarily conjugate to the map in Equation 6.

We will require the following:

Lemma 25. *Let $k \in C_c(\partial^2\mathbb{F}_2 \times \partial\mathbb{F}_2)$, and \tilde{k} an extension of k to a continuous function on $\partial\mathbb{F}_2 \times \overline{\mathbb{F}}_2 \times \overline{\mathbb{F}}_2$. Then the two functions on $\partial\mathbb{F}_2 \times \mathbb{F}_2$*

$$(a, x) \mapsto \tilde{k}(x^{-1}(a), x^{-1}, x)$$

and

$$(a, x) \mapsto \tilde{k}(x^{-1}(a), x^{-1}, a)$$

are the same modulo $C_0(\partial\mathbb{F}_2 \times \mathbb{F}_2)$.

Proof. Let k be as in the statement of the lemma. Then for some neighbourhood N of the diagonal in $\partial\mathbb{F}_2 \times \partial\mathbb{F}_2$, k is supported on $(\partial\mathbb{F}_2 \times \partial\mathbb{F}_2 \times \partial\mathbb{F}_2) \setminus (N \times \partial\mathbb{F}_2)$. It follows that we can choose an extension \tilde{k} of k to a function on $\partial\mathbb{F}_2 \times \overline{\mathbb{F}}_2 \times \overline{\mathbb{F}}_2$ such that there is a neighbourhood \tilde{N} of the diagonal in $\partial\mathbb{F}_2 \times \overline{\mathbb{F}}_2$ such that \tilde{k} is supported in $(\partial\mathbb{F}_2 \times \overline{\mathbb{F}}_2 \times \overline{\mathbb{F}}_2) \setminus (\tilde{N} \times \overline{\mathbb{F}}_2)$.

Now by routine compactness arguments, it suffices to show that for $a \in \partial\mathbb{F}_2$ fixed and x_n a sequence in \mathbb{F}_2 converging to a boundary point $b \in \partial\mathbb{F}_2$, the sequence

$$\tilde{k}(x_n^{-1}(a), x_n^{-1}, x_n) - \tilde{k}(x_n^{-1}(a), x_n^{-1}, a)$$

converges to 0 as $n \rightarrow \infty$. We may clearly also assume without loss of generality that for all n the point $(x_n^{-1}(a), x_n^{-1})$ lies in the complement of \tilde{N} , else both terms are 0. By Lemma 22 there exists $R > 0$ such that any two points $(c, z) \in \partial\mathbb{F}_2 \times \mathbb{F}_2$ not in \tilde{N} have the property that the geodesic $[z, c]$ passes through $B_R(e)$. Thus, for all n large enough, $d(e, [x_n^{-1}, x_n^{-1}(a)]) \leq R$. But then $d(x_n, [e, a]) \leq R$ for all n . If a sequence in a hyperbolic space remains at fixed, bounded distance from a geodesic ray, it must converge to the endpoint of the ray. Hence $x_n \rightarrow a$, and we are done by continuity of \tilde{k} in the third variable. \square

Proof. (Of Proposition 24). Consider the class $[i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta)$, which is represented by the map $B \otimes A \rightarrow \mathcal{Q}(A \otimes l^2\mathbb{F}_2)$ in Equation 6.

Define a unitary map of Hilbert modules $U : A \otimes l^2\mathbb{F}_2 \rightarrow A \otimes l^2\mathbb{F}_2$ by the formula $U(a \otimes e_x) = x \cdot a \otimes e_x$. Let Ad_U denote the inner automorphism of $\mathcal{Q}(A \otimes l^2\mathbb{F}_2)$ given by $\pi(T) \mapsto \pi(UTU^*)$ and let ι' denote the homomorphism $B \otimes A \rightarrow \mathcal{Q}(A \otimes l^2\mathbb{F}_2)$

$$\iota'(a_1 \otimes a_2 \otimes a_3) = \text{Ad}_U(a_1 \otimes \rho(a_2)\lambda(a_3)).$$

We claim that $\iota' = \iota$. It is a simple matter to check that $\iota|_{B \otimes C_r^*(\mathbb{F}_2)} = \iota'|_{B \otimes C_r^*(\mathbb{F}_2)}$, where $B \otimes C_r^*(\mathbb{F}_2)$ is viewed as a sub-algebra of $B \otimes A$, and that for $b \in B$ and $f \in C(\partial\mathbb{F}_2)$,

we have $\iota(b \otimes f) = \tau(b)\pi(f \otimes 1)$ whereas $\iota'(b \otimes f) = \tau(b)(1 \otimes \lambda(f))$. Thus it remains to prove that $\tau(b)\pi(1 \otimes M_{\tilde{f}} - f \otimes 1) = 0$ in the Calkin algebra $\mathcal{Q}(A \otimes l^2\mathbb{F}_2)$ whenever $b \in B$, $f \in C(\partial\mathbb{F}_2)$ and \tilde{f} is an extension of f to $\overline{\mathbb{F}_2}$. The collection of b of the form $\sum \gamma F_\gamma$ with each $F_\gamma \in C_c(\partial^2\mathbb{F}_2)$ is dense in B , and hence it suffices to prove the result for b having this form. Hence it is sufficient to prove the result for $b = F \in C_c(\partial^2\mathbb{F}_2)$. We are now done by Lemma 25 with $k(a, b, c) = F(a, b)f(c)$. □

9. CONCLUSION OF THE PROOF

Now consider the class $[i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta)$, which we have shown has the form $[\iota]$, where ι is as in Proposition 24. We are interested in calculating the Kasparov product of the class of this extension, and the class $[D] \otimes 1_A \in KK^{-1}(A, B \otimes A)$.

Recall that

$$[D] = [\hat{d}_{\mathbb{R}}] \otimes_{C_0(\mathbb{R})} [v - \chi]$$

where $[v - \chi]$ is the class of the homomorphism $C_0(\mathbb{R}) \rightarrow B$ induced by mapping ψ to $v - \chi$.

Hence $[D] \otimes 1_A = ([\hat{d}_{\mathbb{R}}] \otimes 1_A) \otimes_{C_0(\mathbb{R}) \otimes A} ([v - \chi] \otimes 1_A)$, where $[v - \chi] \otimes 1_A$ is represented by the homomorphism $C_0(\mathbb{R}) \otimes A \rightarrow B \otimes A$ induced by mapping $\psi \otimes a \mapsto (v - \chi) \otimes a$.

The Kasparov product

$$([D] \otimes 1_A) \otimes_{B \otimes A} [i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* (\Delta))$$

therefore has the form

$$([\hat{d}_{\mathbb{R}}] \otimes 1_A) \otimes_{C_0(\mathbb{R}) \otimes A} (([v - \chi] \otimes 1_A) \otimes_{B \otimes A} [\iota])$$

and $([v - \chi] \otimes 1_A) \otimes_{B \otimes A} [\iota]$ is represented by the homomorphism $C_0(\mathbb{R}) \otimes A \rightarrow \mathcal{Q}(A \otimes l^2\mathbb{F}_2)$ induced by mapping

$$\psi \otimes a \mapsto \tau(v - \chi)\pi(\varphi(a)).$$

But this homomorphism has the form stated in the hypothesis of Lemma 8. By that lemma,

$$([\hat{d}_{\mathbb{R}}] \otimes 1_A) \otimes_{C_0(\mathbb{R}) \otimes A} ([v - \chi] \otimes 1_A) \otimes_{B \otimes A} [\iota]$$

is represented by the $KK(A, A)$ cycle $(A \otimes l^2\mathbb{F}_2, \bar{U} + 1)$, where \bar{U} is any lift to $\mathcal{B}(A \otimes l^2\mathbb{F}_2)$ of the element $\tau(v - \chi) \in \mathcal{Q}(A \otimes l^2\mathbb{F}_2)$; where the Hilbert (A, A) -bimodule $A \otimes l^2\mathbb{F}_2$ has its standard right A -module structure; and where it has the left A -module structure given by the homomorphism $\varphi : A \rightarrow \mathcal{B}(A \otimes l^2\mathbb{F}_2)$.

In particular, the *bimodule* is in fact the same as the bimodule appearing in the Julg and Vallette cycle appearing in Lemma 21.

It remains to calculate a lift \bar{U} of $\tau(v - \chi)$ and show that in fact such a lift can be chosen which agrees with the operator \bar{W} of Lemma 21.

We first construct a lift of $\tau(v)$. Recall that $v = \sum_{\gamma \in S} \chi_\gamma \gamma \otimes (1 - \chi_\gamma) \gamma$, where S is a basis for \mathbb{F}_2 . Each γ is mapped under τ to the image in the Calkin algebra of the right translation operators $1 \otimes \rho(\gamma) : A \otimes l^2\mathbb{F}_2 \rightarrow A \otimes l^2\mathbb{F}_2$. Consider each term $F_\gamma = \chi_\gamma \otimes (1 - \chi_\gamma) \in C_c(\partial^2\mathbb{F}_2)$. Let $\tilde{\chi}_\gamma$ denote the function on \mathbb{F}_2 given by

$$\tilde{\chi}_\gamma(g) = \begin{cases} 1 & \gamma \in [e, g] \\ 0 & \text{else} \end{cases}.$$

Then $\tilde{\chi}_\gamma$ extends continuously to $\overline{\mathbb{F}_2}$ and the restriction of $\tilde{\chi}_\gamma$ to $\partial\mathbb{F}_2$ is χ_γ . Let then

$$\tilde{F}_\gamma = \chi_\gamma \otimes (1 - \tilde{\chi}_\gamma),$$

which is an extension to $\partial\mathbb{F}_2 \times \overline{\mathbb{F}_2}$ of F_γ . Forming \tilde{F}'_γ as per the recipe described in the definition of τ , we obtain the function

$$\tilde{F}'_\gamma(a, g) = \tilde{F}_\gamma(g^{-1}a, g^{-1}) = \begin{cases} 1 & \gamma \in [e, g^{-1}a) \text{ and } \gamma \notin [e, g^{-1}] \\ 0 & \text{else} \end{cases}.$$

We remind the reader that the statement: “ $x \in [e, y]$,” for $x, y \in \mathbb{F}_2$ may be equivalently read: “the reduced expression of y contains x as an initial subword,” or more shortly, “ y begins with x .”

With this in mind, consider the first case above. If $g^{-1}a$ begins with γ but g^{-1} does not, it follows there is cancellation between g^{-1} and a ; more precisely, a must begin with g , followed by γ . (Since g^{-1} does not begin with γ , g does not end in γ^{-1} , and hence $g\gamma$ is in fact reduced.) We have:

$$\tilde{F}'_\gamma(a, g) = \begin{cases} 1 & g\gamma \in [e, a), \text{ and } g \text{ does not end in } \gamma^{-1} \\ 0 & \text{else} \end{cases}.$$

Now consider the operator $\tilde{F}'_\gamma(1 \otimes v_\gamma) \in A \otimes \mathcal{B}(l^2\mathbb{F}_2) \subset \mathcal{B}(A \otimes l^2\mathbb{F}_2)$. This operates by

$$\left(\sum f_h h\right) \otimes e_g \mapsto \left(\sum \tilde{F}'_\gamma(\cdot, g\gamma^{-1}) f_h h\right) \otimes e_{g\gamma^{-1}}$$

for $\sum f_h h$ an arbitrary element of the cross product A . From our above work, we see that $\tilde{F}'_\gamma(\cdot, g\gamma^{-1}) = 0$ unless g ends in γ . On the other hand, if g does end in γ , $g\gamma^{-1}$ does not end in γ^{-1} . Hence we see that the above operator sends

$$\left(\sum f_h h\right) \otimes e_g \mapsto \begin{cases} \left(\sum \chi_g f_h h\right) \otimes e_{g\gamma^{-1}} & g \text{ ends in } \gamma \\ 0 & \text{else} \end{cases}.$$

We see finally, that $\bar{V} = \sum_{\gamma \in S} \tilde{F}'_\gamma \cdot (1 \otimes v_\gamma)$, which is a lift of $\tau(v)$, acts on $A \otimes l^2\mathbb{F}_2$ by

$$\left(\sum f_h h\right) \otimes e_g \mapsto \left(\sum \chi_g f_h h\right) \otimes e_{g'},$$

where the prime notation is as in the discussion just prior to Lemma 20.

In particular, \bar{V} as an operator on $A \otimes l^2\mathbb{F}_2$, where the latter is regarded as functions $\mathbb{F}_2 \rightarrow C(\partial\mathbb{F}_2) \otimes l^2\mathbb{F}_2$, has the form

$$(\bar{V}\xi)(g) = V(\xi(g)),$$

where V is the operator $C(\partial\mathbb{F}_2) \otimes l^2\mathbb{F}_2 \rightarrow C(\partial\mathbb{F}_2) \otimes l^2\mathbb{F}_2$,

$$V(f \otimes e_g) = \chi_g f \otimes e_{g'}.$$

Otherwise expressed, let ξ be an element of $C(\partial\mathbb{F}_2; l^2\mathbb{F}_2)$ of the form $\xi(a) = \sum \xi_g(a) e_g$, where each ξ_g is a scalar-valued function on $\partial\mathbb{F}_2$. Then

$$(V\xi)(a) = \sum_{g \in [e, a)} \xi_g(a) \otimes e_{g'}.$$

Now apply the same calculations to the element $\tau(\chi)$. We obtain the operator (projection) \bar{P} on $A \otimes l^2\mathbb{F}_2$ given by $\bar{P} = \sum \tilde{F}'_\gamma \in A \otimes \mathcal{B}(l^2\mathbb{F}_2) \subset \mathcal{B}(A \otimes l^2\mathbb{F}_2)$.

We have: $\bar{V} - \bar{P}$ is an operator whose projection to the Calkin algebra is $\tau(v - \chi)$ as required. Let it be denoted by \bar{U} . Form $\bar{F} = \bar{U} + 1$.

Our calculations show that \bar{F} is an operator having the form

$$(\bar{F}\xi)(g) = F(\xi(g)),$$

where $F : C(\partial\mathbb{F}_2) \otimes l^2\mathbb{F}_2 \rightarrow C(\partial\mathbb{F}_2) \otimes l^2\mathbb{F}_2$ is the operator

$$(F\xi)(a) = \sum_{g \in [e, a]} \xi_g(a) \otimes e_{g'} + \sum_{g \notin [e, a]} \chi_g(a) \otimes e_g,$$

which is precisely the operator W of Lemma 20. That is, $F = W$ and therefore $\bar{F} = \bar{W} \in \mathcal{B}(A \otimes l^2\mathbb{F}_2)$.

We are now done, having shown by direct computation that

$$([D] \otimes 1_A) \otimes_{B \otimes A} [i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta) = [(C(\partial\mathbb{F}_2) \rtimes \mathbb{F}_2 \otimes l^2\mathbb{F}_2, \bar{W})] = \lambda(\gamma_{\partial\mathbb{F}_2 \rtimes \mathbb{F}_2})$$

and therefore that

$$([D] \otimes 1_A) \otimes_{B \otimes A} [i \otimes 1_A] \otimes_{A \otimes A \otimes A} (1_A \otimes \sigma_{12}^* \Delta) = 1_A.$$

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