

# An elliptic generalization of Schur's Pfaffian identity

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## Abstract

We present a Pfaffian identity involving elliptic functions, whose rational limit gives a generalization of Schur's Pfaffian identity for  $\text{Pf}((x_j - x_i)/(x_j + x_i))$ . This identity is regarded as a Pfaffian counterpart of Frobenius identity, which is an elliptic generalization of Cauchy's determinant identity for  $\det(1/(x_i + x_j))$ .

## 1 Introduction

Let  $[x]$  denote a nonzero holomorphic function on the complex plane  $\mathbb{C}$  in the variable  $x$  satisfying the following two conditions:

- (i)  $[x]$  is an odd function, i.e.,  $[-x] = -[x]$ .
- (ii)  $[x]$  satisfies the Riemann relation:

$$[x + y][x - y][u + v][u - v] - [x + u][x - u][y + v][y - v] + [x + v][x - v][y + u][y - u] = 0.$$

It is known that such a function  $[x]$  is obtained from one of the following functions by the transformation  $[x] \rightarrow e^{ax^2+b} [cx]$  (see [10, Chap. XX, Misc. Ex. 38]):

- (a) (elliptic case)  $[x] = \sigma(x)$  (the Weierstrass sigma function).
- (b) (trigonometric case)  $[x] = e^x - e^{-x}$ .
- (c) (rational case)  $[x] = x$ .

G. Frobenius [2] gave the following determinant identity:

$$\det \left( \frac{[z + x_i + y_j]}{[z][x_i + y_j]} \right)_{1 \leq i, j \leq n} = \frac{\prod_{1 \leq i < j \leq n} [x_j - x_i][y_j - y_i]}{\prod_{1 \leq i, j \leq n} [x_i + y_j]} \cdot \frac{[z + \sum_{i=1}^n x_i + \sum_{j=1}^n y_j]}{[z]}. \quad (1)$$

If we take the limit  $z \rightarrow \infty$  in the rational case of this identity (1), we obtain the Cauchy's determinant identity [1]:

$$\det \left( \frac{1}{x_i + y_j} \right)_{1 \leq i, j \leq n} = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)}. \quad (2)$$

Hence the identity (1) can be regarded as an elliptic generalization of (2).

Among identities for Pfaffians, the Schur's Pfaffian identity [9, p. 225]

$$\text{Pf} \left( \frac{x_i - x_i}{x_i + x_i} \right)_{1 \leq i, j \leq 2n} = \prod_{1 \leq i < j \leq 2n} \frac{x_i - x_i}{x_j + x_i} \quad (3)$$

plays the same role in the theory of Schur's  $Q$ -functions as the Cauchy's determinant identity in the theory of Schur functions. Our main result is the following elliptic generalization of the Schur's Pfaffian identity.

**Theorem 1.1.** For complex variables  $x_1, \dots, x_{2n}$ ,  $z$  and  $w$ , we have

$$\begin{aligned} \text{Pf} \left( \frac{[x_j - x_i]}{[x_j + x_i]} \cdot \frac{[z + x_i + x_j]}{[z]} \cdot \frac{[w + x_i + x_j]}{[w]} \right)_{1 \leq i, j \leq 2n} \\ = \prod_{1 \leq i < j \leq 2n} \frac{[x_j - x_i]}{[x_j + x_i]} \cdot \frac{[z + \sum_{i=1}^{2n} x_i]}{[z]} \cdot \frac{[w + \sum_{i=1}^{2n} x_i]}{[w]}. \end{aligned} \quad (4)$$

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If we take the limit  $z \rightarrow \infty$ ,  $w \rightarrow \infty$  in the rational case of (4), then we recover the Schur's Pfaffian identity (3).

Recently, the Frobenius' identity (1) plays a key role in proving transformations of elliptic hypergeometric series. See [5], [8]. We expect that the identity (4) provides another tool in the theory of special functions.

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.1. In Section 3, we present a generalization of the rational case of (1) and (4), in terms of Schur functions corresponding to the staircase partitions. And we give another proof of the trigonometric case of (1) and (4) in Section 4.

## 2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. The key is to use the following Pfaffian version of the Desnanot–Jacobi formula. See [6, (1.1)] and [4, Theorem 2.6] for a proof and related formulae.

**Lemma 2.1.** Let  $A$  be the skew-symmetric matrix and denote by  $A^{i_1, \dots, i_k}$  the skew-symmetric matrix obtained by removing  $i_1$ th,  $\dots$ ,  $i_k$ th rows and columns. Then we have

$$\text{Pf } A^{12} \cdot \text{Pf } A^{34} - \text{Pf } A^{13} \cdot \text{Pf } A^{24} + \text{Pf } A^{14} \cdot \text{Pf } A^{23} = \text{Pf } A \cdot \text{Pf } A^{1234}.$$

**Proof of Theorem 1.1.** We proceed by induction on  $n$ . By applying Lemma 2.1 to the skew-symmetric matrix on the left hand side of (4), and using the induction hypothesis, we see that it is enough to show

$$\begin{aligned} & [b-a][d-c][c+a][d+a][c+b][d+b] \\ & \quad \times [z+a+b+s][z+c+d+s][w+a+b+s][w+c+d+s] \\ & - [c-a][d-b][b+a][d+a][c+b][d+c] \\ & \quad \times [z+a+c+s][z+b+d+s][w+a+c+s][w+b+d+s] \\ & + [d-a][c-b][b+a][c+a][d+b][d+c] \\ & \quad \times [z+a+d+s][z+b+c+s][w+a+d+s][w+b+c+s] \\ & = [b-a][c-a][d-a][c-b][d-b][d-c] \\ & \quad \times [z+a+b+c+d+s][z+s][w+a+b+c+d+s][w+s], \end{aligned} \tag{5}$$

where  $a = x_1$ ,  $b = x_2$ ,  $c = x_3$ ,  $d = x_4$  and  $s = \sum_{j=5}^{2n} x_j$ . Note that the case of  $s = 0$  is exactly the identity (4) with  $n = 2$ .

By replacing  $z+s$  and  $w+s$  by  $z$  and  $w$  respectively, we may assume  $s = 0$ . By applying the Riemann relation with  $(x, y, u, v) = (z/2 + a, z/2 + b, z/2 + c, z/2 + d)$ , we have

$$[b-a][z+a+b][d-c][z+c+d] = [c-a][z+a+c][d-b][z+b+d] - [d-a][z+a+d][c-b][z+b+c].$$

Hence the left hand side of (5) is equal to

$$\begin{aligned} & [c-a][z+a+c][d-b][z+b+d][d+a][c+b] \\ & \quad \times ((c+a)[d+b][w+a+b][w+c+d] - [b+a][d+c][w+a+c][w+b+d]) \\ & - [d-a][z+a+d][c-b][z+b+c][c+a][d+b] \\ & \quad \times ((d+a)[c+b][w+a+b][w+c+d] - [b+a][d+c][w+a+d][w+b+c]). \end{aligned} \tag{6}$$

Again, by using the Riemann relation with

$$x = \frac{w+2a+b+c}{2}, \quad y = \frac{w+b-c}{2}, \quad u = \frac{w+b+c+2d}{2}, \quad v = \frac{w-b+c}{2},$$

we have

$$\begin{aligned} & [c+a][d+b][w+a+b][w+c+d] - [b+a][d+c][w+a+c][w+b+d] \\ & \quad = [d-a][c-b][w][w+a+b+c+d]. \end{aligned}$$

By exchanging  $c$  and  $d$ , we have

$$\begin{aligned} [d+a][c+b][w+a+b][w+c+d] - [b+a][d+c][w+a+d][w+b+c] \\ = [c-a][d-b][w][w+a+b+c+d]. \end{aligned}$$

Hence we see that (6) is equal to

$$\begin{aligned} [c-a][d-a][c-b][d-b][w][w+a+b+c+d] \\ \times ([d+a][c+b][z+a+c][z+b+d] - [c+a][d+b][z+a+d][z+b+c]). \end{aligned} \quad (7)$$

Finally we use the Riemann relation with

$$x = \frac{z+2a+c+d}{2}, \quad y = \frac{z+c-d}{2}, \quad u = \frac{z+2b+c+d}{2}, \quad v = \frac{z-c+d}{2}.$$

Then we have

$$\begin{aligned} [d+a][c+b][z+a+c][z+b+d] - [c+a][d+b][z+a+d][z+b+c] \\ = [b-a][d-c][z][z+a+b+c+d], \end{aligned}$$

and we see that (7) is equal to the right hand side of (5). This completes the proof of Theorem 1.1.  $\square$

### 3 Rational case

In this section, we generalize the rational case of (1) and (4) to the identities involving Schur functions.

If we put  $[x] = x$ , then the identities (1) and (4) reduce to

$$\begin{aligned} \det \left( \frac{1}{x_i + y_j} \cdot \frac{z + x_i + y_j}{z} \right)_{1 \leq i, j \leq n} \\ = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \cdot \frac{z + \sum_{i=1}^n x_i + \sum_{j=1}^n y_j}{z}, \end{aligned} \quad (8)$$

$$\begin{aligned} \text{Pf} \left( \frac{x_j - x_i}{x_j + x_i} \cdot \frac{z + x_i + x_j}{z} \cdot \frac{w + x_i + x_j}{w} \right)_{1 \leq i, j \leq 2n} \\ = \prod_{1 \leq i < j \leq 2n} \frac{x_j - x_i}{x_j + x_i} \cdot \frac{z + \sum_{i=1}^{2n} x_i}{z} \cdot \frac{w + \sum_{i=1}^{2n} x_i}{w}, \end{aligned} \quad (9)$$

respectively. These identities can be generalized as follows:

**Theorem 3.1.** Let  $s_\lambda$  be the Schur function corresponding to a partition  $\lambda$ . For a nonnegative integer  $r$ , let  $\delta(r) = (r, r-1, \dots, 2, 1)$  denote the staircase partition.

(1) Let  $k \leq m$  be integers. For vectors of variables  $\mathbf{x} = (x_1, \dots, x_n)$ ,  $\mathbf{y} = (y_1, \dots, y_n)$  and  $\mathbf{z} = (z_1, \dots, z_m)$ , we have

$$\det \left( \frac{1}{x_i + y_j} \cdot \frac{s_{\delta(k)}(x_i, y_j, \mathbf{z})}{s_{\delta(k)}(\mathbf{z})} \right)_{1 \leq i, j \leq n} = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{1 \leq i, j \leq n} (x_i + y_j)} \cdot \frac{s_{\delta(k)}(\mathbf{x}, \mathbf{y}, \mathbf{z})}{s_{\delta(k)}(\mathbf{z})}. \quad (10)$$

(2) Let  $k \leq m$  and  $l \leq m'$  be integers. For vectors of variables  $\mathbf{x} = (x_1, \dots, x_{2n})$ ,  $\mathbf{z} = (z_1, \dots, z_m)$  and  $\mathbf{w} = (w_1, \dots, w_{m'})$ , we have

$$\begin{aligned} \text{Pf} \left( \frac{x_j - x_i}{x_j + x_i} \cdot \frac{s_{\delta(k)}(x_i, x_j, \mathbf{z})}{s_{\delta(k)}(\mathbf{z})} \cdot \frac{s_{\delta(l)}(x_i, x_j, \mathbf{w})}{s_{\delta(l)}(\mathbf{w})} \right)_{1 \leq i, j \leq 2n} \\ = \prod_{1 \leq i < j \leq 2n} \frac{x_j - x_i}{x_j + x_i} \cdot \frac{s_{\delta(k)}(\mathbf{x}, \mathbf{z})}{s_{\delta(k)}(\mathbf{z})} \cdot \frac{s_{\delta(l)}(\mathbf{x}, \mathbf{w})}{s_{\delta(l)}(\mathbf{w})}. \end{aligned} \quad (11)$$

The identity (10) reduces to (2) and (8) in the case of  $k = 0$  and  $k = 1$  respectively, while (11) gives (3) and (9) in the case of  $k = l = 0$  and  $k = l = 1$  respectively.

We derive Theorem 3.1 from much more general identities involving generalized Vandermonde determinants, which were conjectured by the author and proven in [3]. Let  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{a} = (a_1, \dots, a_n)$  be two vectors of variables of length  $n$ . For nonnegative integers  $p$  and  $q$  with  $p + q = n$ , we define a generalized Vandermonde matrix  $V^{p,q}(\mathbf{x}; \mathbf{a})$  to be the  $n \times n$  matrix with  $i$ th row

$$(1, x_i, \dots, x_i^{p-1}, a_i, a_i x_i, \dots, a_i x_i^{q-1}).$$

Then we have

**Theorem 3.2.** (Ishikawa–Okada–Tagawa–Zeng [3])

(a) Let  $n$  be a positive integer and let  $p$  and  $q$  be nonnegative integers. For six vectors of variables

$$\begin{aligned} \mathbf{x} &= (x_1, \dots, x_n), \quad \mathbf{y} = (y_1, \dots, y_n), \quad \mathbf{a} = (a_1, \dots, a_n), \quad \mathbf{b} = (b_1, \dots, b_n), \\ \mathbf{z} &= (z_1, \dots, z_{p+q}), \quad \mathbf{c} = (c_1, \dots, c_{p+q}), \end{aligned}$$

we have

$$\begin{aligned} \det \left( \frac{\det V^{p+1,q+1}(x_i, y_j, \mathbf{z}; a_i, b_j, \mathbf{c})}{y_j - x_i} \right)_{1 \leq i, j \leq n} \\ = \frac{(-1)^{n(n-1)/2}}{\prod_{i,j=1}^n (y_j - x_i)} \det V^{p,q}(\mathbf{z}; \mathbf{c})^{n-1} \det V^{n+p,n+q}(\mathbf{x}, \mathbf{y}, \mathbf{z}; \mathbf{a}, \mathbf{b}, \mathbf{c}). \end{aligned} \quad (12)$$

(b) Let  $n$  be a positive integer and let  $p, q, r, s$  be nonnegative integers. For seven vectors of variables

$$\begin{aligned} \mathbf{x} &= (x_1, \dots, x_{2n}), \quad \mathbf{a} = (a_1, \dots, a_{2n}), \quad \mathbf{b} = (b_1, \dots, b_{2n}), \\ \mathbf{z} &= (z_1, \dots, z_{p+q}), \quad \mathbf{c} = (c_1, \dots, c_{p+q}), \\ \mathbf{w} &= (w_1, \dots, w_{r+s}), \quad \mathbf{d} = (d_1, \dots, d_{r+s}), \end{aligned}$$

we have

$$\begin{aligned} \text{Pf} \left( \frac{\det V^{p+1,q+1}(x_i, x_j, \mathbf{z}; a_i, a_j, \mathbf{c}) \det V^{r+1,s+1}(x_i, x_j, \mathbf{w}; b_i, b_j, \mathbf{d})}{x_j - x_i} \right)_{1 \leq i, j \leq 2n} \\ = \frac{1}{\prod_{1 \leq i < j \leq 2n} (x_j - x_i)} \det V^{p,q}(\mathbf{z}; \mathbf{c})^{n-1} \det V^{r,s}(\mathbf{w}; \mathbf{d})^{n-1} \\ \times \det V^{n+p,n+q}(\mathbf{x}, \mathbf{z}; \mathbf{a}, \mathbf{c}) \det V^{n+r,n+s}(\mathbf{x}, \mathbf{w}; \mathbf{b}, \mathbf{d}). \end{aligned} \quad (13)$$

**Proof of Theorem 3.1.** In (12) (resp. (13)), we specialize

$$\begin{aligned} x_i &\rightarrow x_i^2, \quad y_i \rightarrow y_i^2, \quad z_i \rightarrow z_i^2, \quad a_i \rightarrow x_i, \quad b_i \rightarrow y_i, \quad c_i \rightarrow z_i, \\ (\text{resp. } x_i &\rightarrow x_i^2, \quad z_i \rightarrow z_i^2, \quad w_i \rightarrow w_i^2, \quad a_i \rightarrow x_i, \quad b_i \rightarrow x_i, \quad c_i \rightarrow z_i, \quad d_i \rightarrow w_i). \end{aligned} \quad (14)$$

It follows from the bi-determinant definition of Schur functions that

$$\det V^{p,q}(x_1^2, \dots, x_{p+q}^2; x_1, \dots, x_{p+q}) = \varepsilon_{p,q} s_{\delta_{p,q}}(x_1, \dots, x_{p+q}) \prod_{1 \leq i < j \leq p+q} (x_j - x_i),$$

where the partition  $\delta_{p,q}$  and the signature  $\varepsilon_{p,q}$  are defined by

$$\delta_{p,q} = \begin{cases} \delta(p-q-1) & \text{if } p > q, \\ \delta(q-p) & \text{if } p \leq q, \end{cases} \quad \varepsilon_{p,q} = \begin{cases} (-1)^{q(2p-q-1)/2} & \text{if } p > q, \\ (-1)^{p(p-1)/2} & \text{if } p \leq q. \end{cases}$$

Hence, under the specialization (14), the identities (12) and (13) in Theorem 3.2 give

$$\begin{aligned}
& \varepsilon_{p+1,q+1}^n \det \left( \frac{s_{\delta_{p+1,q+1}}(x_i, y_j, \mathbf{z})}{x_i + y_j} \right)_{1 \leq i, j \leq n} \\
&= \frac{(-1)^{n(n-1)/2} \varepsilon_{p,q}^{n-1} \varepsilon_{n+p,n+q}}{\prod_{i,j=1}^n (x_i + y_j)} \cdot s_{\delta_{p,q}}(\mathbf{z})^{n-1} \cdot s_{\delta_{n+p,n+q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \\
& \varepsilon_{p+1,q+1}^n \varepsilon_{r+1,s+1}^n \text{Pf} \left( \frac{x_j - x_i}{x_j + x_i} \cdot s_{\delta_{p,q}}(x_i, x_j, \mathbf{z}) \cdot s_{\delta_{r,s}}(x_i, x_j, \mathbf{w}) \right)_{1 \leq i, j \leq 2n} \\
&= \varepsilon_{p,q}^{n-1} \varepsilon_{r,s}^{n-1} \varepsilon_{p+n,q+n} \varepsilon_{r+n,s+n} \prod_{1 \leq i < j \leq 2n} \frac{x_j - x_i}{x_j + x_i} \\
& \quad \times s_{\delta_{p,q}}(\mathbf{z})^{n-1} \cdot s_{\delta_{r,s}}(\mathbf{w})^{n-1} \cdot s_{\delta_{n+p,n+q}}(\mathbf{x}, \mathbf{z}) \cdot s_{\delta_{n+r,n+s}}(\mathbf{x}, \mathbf{w}).
\end{aligned}$$

Here a direct computation shows

$$(-1)^{n(n-1)/2} \varepsilon_{p,q}^{n-1} \varepsilon_{n+p,n+q} = \varepsilon_{p+1,q+1}^n.$$

Now we can complete the proof by choosing  $(p, q)$  and  $(p, q, r, s)$  as follows. In the determinant case, given integers  $m$  and  $k$ , we define  $p$  and  $q$  by

$$(p, q) = \begin{cases} \left( \frac{m-k}{2}, \frac{m+k}{2} \right) & \text{if } m \text{ and } k \text{ have the same parity,} \\ \left( \frac{m+k+1}{2}, \frac{m-k-1}{2} \right) & \text{otherwise.} \end{cases}$$

In the Pfaffian case, for integers  $m, m', k$  and  $l$ , we put

$$\begin{aligned}
(p, q) &= \begin{cases} \left( \frac{m-k}{2}, \frac{m+k}{2} \right) & \text{if } m \text{ and } k \text{ have the same parity,} \\ \left( \frac{m+k+1}{2}, \frac{m-k-1}{2} \right) & \text{otherwise,} \end{cases} \\
(r, s) &= \begin{cases} \left( \frac{m'-l}{2}, \frac{m'+l}{2} \right) & \text{if } m' \text{ and } l \text{ have the same parity,} \\ \left( \frac{m'+l+1}{2}, \frac{m'-l-1}{2} \right) & \text{otherwise.} \end{cases}
\end{aligned}$$

□

## 4 Trigonometric case

In this section, we give another proof of the trigonometric case of (1) and (4) by using determinant and Pfaffian identities similar to (12) and (13).

By replacing  $[x]$  by  $x^{1/2} - x^{-1/2}$ , we can see that the trigonometric cases of (1) and (4) are equivalent to

$$\begin{aligned}
& \det \left( \frac{1 - zx_i y_j}{1 - x_i y_j} \right)_{1 \leq i, j \leq n} \\
&= (1-z)^{n-1} \left( 1 - z \prod_{i=1}^n x_i \prod_{j=1}^n y_j \right) \cdot \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)}, \tag{15}
\end{aligned}$$

$$\begin{aligned}
& \text{Pf} \left( \frac{(x_j - x_i)(1 - zx_i x_j)(1 - wx_i x_j)}{1 - x_i x_j} \right)_{1 \leq i, j \leq 2n} \\
&= (1-z)^{n-1} (1-w)^{n-1} \left( 1 - z \prod_{i=1}^{2n} x_i \right) \left( 1 - w \prod_{i=1}^{2n} x_i \right) \cdot \prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{1 - x_i x_j}, \tag{16}
\end{aligned}$$

respectively. Here we derive these identities from the following Proposition.

**Proposition 4.1.** (Okada [7])

(1) For vectors of variables  $\mathbf{x}$ ,  $\mathbf{y}$ ,  $\mathbf{a}$  and  $\mathbf{b}$  of length  $n$ , we have

$$\det \left( \frac{1 - a_i b_j}{1 - x_i y_j} \right)_{1 \leq i, j \leq n} = \frac{(-1)^{n(n+1)/2}}{\prod_{1 \leq i, j \leq n} (1 - x_i y_j)} \det \begin{pmatrix} a_1 x_1^{n-1} & a_1 x_1^{n-2} & \cdots & a_1 & x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_n x_n^{n-1} & a_n x_n^{n-2} & \cdots & a_n & x_n^{n-1} & x_n^{n-2} & \cdots & 1 \\ 1 & y_1 & \cdots & y_1^{n-1} & b_1 & b_1 y_1 & \cdots & b_1 y_1^{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n & \cdots & y_n^{n-1} & b_n & b_n y_n & \cdots & b_n y_n^{n-1} \end{pmatrix}. \quad (17)$$

(2) For vectors of variables  $\mathbf{x}$ ,  $\mathbf{a}$  and  $\mathbf{b}$  of length  $2n$ , we have

$$\text{Pf} \left( \frac{\det W^2(x_i, x_j; a_i, a_j) \det W^2(x_i, x_j; b_i, b_j)}{(x_j - x_i)(1 - x_i x_j)} \right)_{1 \leq i, j \leq 2n} = \frac{1}{\prod_{1 \leq i < j \leq 2n} (x_j - x_i)(1 - x_i x_j)} \det W^{2n}(\mathbf{x}; \mathbf{a}) \det W^{2n}(\mathbf{x}; \mathbf{b}), \quad (18)$$

where  $W^n(\mathbf{x}; \mathbf{a})$  is the  $n \times n$  matrix with  $i$ th row

$$(1 + a_i x_i^{n-1}, x_i + a_i x_i^{n-2}, \dots, x_i^{n-1} + a_i).$$

**Proof.** The first identity (12) is obtained by replacing  $a_i$  and  $x_i$  by  $1/a_i$  and  $1/x_i$  respectively in the identity (12) with  $p = q = 0$ . (This case was first given in [7, Theorem 4.2].) The second identity (13) is given in [7, Theorem 4.4].  $\square$

Now the trigonometric case (15) (resp. (16)) easily follows from (17) (resp. (13)) by substituting  $a_i = z^{1/2} x_i$  and  $b_i = z^{1/2} y_i$  (resp.  $a_i = z x_i$  and  $b_i = w x_i$ ), and by using the determinant evaluation in Lemma 4.2 (resp. 4.3).

**Lemma 4.2.**

$$\det \begin{pmatrix} t x_1^n & t x_1^{n-1} & \cdots & t x_1 & x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t x_n^n & t x_n^{n-1} & \cdots & t x_n & x_n^{n-1} & x_n^{n-2} & \cdots & 1 \\ 1 & y_1 & \cdots & y_1^{n-1} & t y_1 & t y_1^2 & \cdots & t y_1^n \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & y_n & \cdots & y_n^{n-1} & t y_n & t y_n^2 & \cdots & t y_n^n \end{pmatrix} = (-1)^{n(n+1)/2} (1 - t^2)^{n-1} \left( 1 - t^2 \prod_{i=1}^n x_i \prod_{i=1}^n y_i \right) \prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i). \quad (19)$$

**Proof.** Let  $D(\mathbf{x}, \mathbf{y}; t)$  denote the determinant of the left hand side. Since  $D(\mathbf{x}, \mathbf{y}; t)$  is divisible by  $\Delta(\mathbf{x})\Delta(\mathbf{y}) = \prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)$  and it has degree  $n$  with respect to  $x_1$ , we can write

$$D(\mathbf{x}, \mathbf{y}; t) = (c_0 + c_1 x_1) \prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i), \quad (20)$$

where  $c_0$  and  $c_1$  are polynomials in  $x_2, \dots, x_n, y_1, \dots, y_n, t$ .

By performing elementary transformations and by using the Vandermonde determinant, we see that the constant term and the coefficient of  $x_1^n$  in  $D(\mathbf{x}, \mathbf{y}; t)$  are given by

$$(1 - t^2)^{n-1} (-1)^{(n-1)(n-2)/2} \prod_{2 \leq i < j \leq n} (x_j - x_i) \prod_{1 \leq i < j \leq n} (y_j - y_i),$$

$$t(1 - t^2)^{n-1} (-1)^{n(n-1)/2} \prod_{i=2}^n x_i \prod_{2 \leq i < j \leq n} (x_j - x_i) \prod_{1 \leq i < j \leq n} (y_j - y_i),$$

respectively. These computation yield the coefficients  $c_0$  and  $c_1$ :

$$c_0 = (-1)^{n(n+1)/2}(1-t^2)^{n-1}, \quad c_1 = -(-1)^{n(n+1)/2}(1-t^2)^{n-1}t^2 \prod_{i=2}^n x_i \prod_{i=1}^n y_i,$$

which complete the proof of (19).  $\square$

**Lemma 4.3.** Let  $d_n(t)$  be the polynomial given by

$$d_n(t) = \begin{cases} (1-t)^m(1+t)^m & \text{if } n = 2m \text{ is even,} \\ (1-t)^m(1+t)^{m+1} & \text{if } n = 2m + 1 \text{ is odd.} \end{cases}$$

Then we have

$$\det W^n(x_1, \dots, x_n; t, \dots, t) = d_n(t) \prod_{1 \leq i < j \leq n} (x_j - x_i), \quad (21)$$

$$\det W^n(x_1, \dots, x_n; tx_1, \dots, tx_n) = d_{n-1}(t) \left( 1 - (-1)^n t \prod_{i=1}^n x_i \right) \prod_{1 \leq i < j \leq n} (x_j - x_i). \quad (22)$$

**Proof.** The idea of the proof is the same as Lemma 4.2, so we leave it to the reader.  $\square$

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