A Simple Proof of the Aztec Diamond Theorem

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Abstract

Based on a bijection between domino tilings of an Aztec diamond and non-intersecting lattice paths, a simple proof of the Aztec diamond theorem is given in terms of Hankel determinants of the large and small Schröder numbers.

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1 Introduction

The *Aztec diamond* of order n, denoted by $\mathrm{Az}(n)$, is defined as the union of all the unit squares with integral corners (x, y) satisfying $|x| + |y| \leq n + 1$. A *domino* is simply a 1-by-2 or 2-by-1 rectangles with integral corners. A *domino tiling* of a region R is a set of non-overlapping dominos the union of which is R. Figure 1 shows the Aztec diamond of order 3 and a domino tiling. The Aztec diamond theorem, which is first proved by Elkies *et al.* in [4], indicates that the number a_n of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$. They gave four proofs by relating the tilings to alternating sign matrices, monotone triangles, representations of general linear groups, and domino shuffling. Other approaches to this theorem appeared in [2, 3, 6]. Ciucu [3] derived the recurrence relation $a_n = 2^n a_{n-1}$ by means of perfect matchings of celluar graphs. Kuo [6] developed a method, called graphical condensation, to derive the recurrence relation $a_n a_{n-2} = 2a_{n-1}^2$, for $n \geq 3$. Recently, Brualdi and Kirkland [2] gave a proof by considering a matrix of order $n(n+1)$ the determinant of which gives a_n. In this note we give a proof in terms

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of Hankel determinants of the large and small Schröder numbers based on a bijection between the domino tilings of an Aztec diamond and non-intersecting lattice paths.

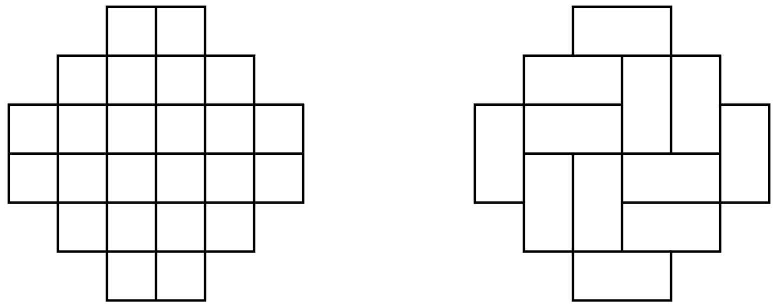

Figure 1: the Az(3) and a domino tiling

Recall the *large Schröder numbers* $\{r_n\}_{n\geq 0} := \{1, 2, 6, 22, 90, 394, 1806, \ldots\}$ and the *small Schröder numbers* $\{s_n\}_{n\geq 0} := \{1, 1, 3, 11, 45, 197, 903, \ldots\}$. Among many other combinatorial structures, the n-th large Schröder number r_n counts the number of lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0, 0)$ to $(2n, 0)$ using *up* steps $(1, 1)$, *down* steps $(1, -1)$, and *level* steps $(2, 0)$ that never pass below the x-axis. Such a path is called a *large Schröder path* of length n (or a *large n-Schröder path* for short). Let U, D, and L denote an up, down, and level step, respectively. Note that the terms of $\{r_n\}_{n\geq 1}$ are twice of those in $\{s_n\}_{n\geq 1}$. Consequently, the n-th small Schröder number s_n counts the number of large n-Schröder paths without level steps on the x-axis, for $n \geq 1$. Such a path is called a *small n-Schröder path*. Refer to [7, Exercise 6.39] for more information.

Our proof relies on the determinants of the following *Hankel matrices* of the large and small Schröder numbers

$$H_n^{(1)} := \begin{bmatrix} r_1 & r_2 & \cdots & r_n \\ r_2 & r_3 & \cdots & r_{n+1} \\ \vdots & \vdots & & \vdots \\ r_n & r_{n+1} & \cdots & r_{2n-1} \end{bmatrix}, \quad G_n^{(1)} := \begin{bmatrix} s_1 & s_2 & \cdots & s_n \\ s_2 & s_3 & \cdots & s_{n+1} \\ \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n-1} \end{bmatrix}.$$

Note that $H_n^{(1)} = 2G_n^{(1)}$. Using a method of Gessel and Viennot [5], we associate the determinants of $H_n^{(1)}$ and $G_n^{(1)}$ with the numbers of n-tuples of non-intersecting large and small Schröder paths, respectively. How to derive the determinants of $H_n^{(1)}$ and $G_n^{(1)}$ and how to establish bijections between domino tilings of an Aztec diamond and non-intersecting large Schröder paths are given in the next section.

2 A proof of the Aztec diamond theorem

Let Π_n (resp. Ω_n) denote the set of n-tuples $(\pi_1, \ldots, \pi_n)$ of large Schröder paths (resp. small Schröder paths) satisfying the following two conditions.

(A1) The path π_i goes from $(-2i+1, 0)$ to $(2i-1, 0)$, for $1 \le i \le n$, and

(A2) any two paths π_i and π_j do not intersect.

There is an immediate bijection ϕ between Π_{n-1} and Ω_n, for $n \ge 2$, which carries $(\pi_1, \ldots, \pi_{n-1}) \in \Pi_{n-1}$ into $\phi((\pi_1, \ldots, \pi_{n-1})) = (\omega_1, \ldots, \omega_n) \in \Omega_n$, where $\omega_1 = \mathsf{UD}$ and $\omega_i = \mathsf{UU}\pi_{i-1}\mathsf{DD}$ (i.e., ω_i is obtained from π_{i-1} with 2 up steps attached in the beginning and 2 down steps attached in the end, and then rises above the x-axis), for $2 \le i \le n$. For example, on the left of Figure 2 is a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$. The corresponding quadruple $(\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_4$ is shown on the right. Hence, for $n \ge 2$, we have

$$|\Pi_{n-1}| = |\Omega_n|. \tag{1}$$

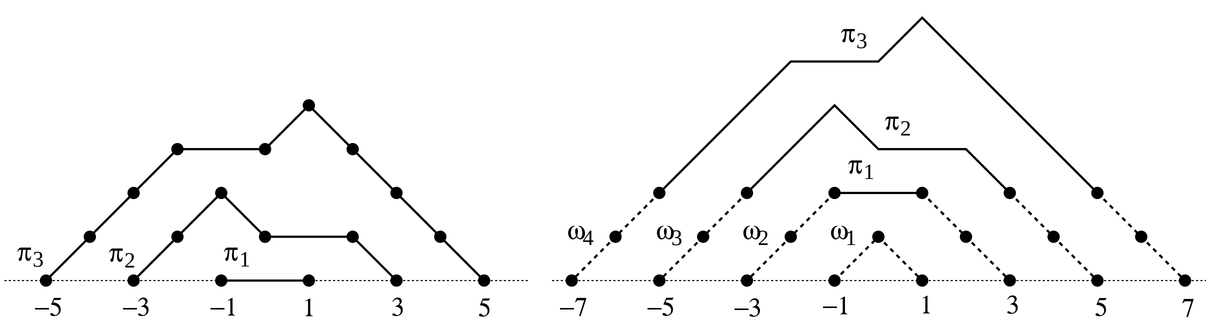

Figure 2: a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ and the corresponding quadruple $(\omega_1, \omega_2, \omega_3, \omega_4) \in \Omega_4$

For a permutation $\sigma = z_1 z_2 \cdots z_n$ of $\{1, \ldots, n\}$, the *sign* of σ, denoted by $\mathrm{sgn}(\sigma)$, is defined by $\mathrm{sgn}(\sigma) := (-1)^{\mathrm{inv}(\sigma)}$, where $\mathrm{inv}(\sigma) := \mathrm{Card}\{(z_i, z_j) \,|\, i < j \text{ and } z_i > z_j\}$ is the number of *inversions* of σ. Using the technique of a sign-reversing involution over a signed set, we prove that the cardinalities of Π_n and Ω_n coincide with the determinants of $H_n^{(1)}$ and $G_n^{(1)}$, respectively. Following the same steps as [8, Theorem 5.1], a proof is given here for completeness.

Proposition 2.1 *For $n \geq 1$, we have*

(i) $|\Pi_n| = \det(H_n^{(1)}) = 2^{n(n+1)/2}$, *and*

(ii) $|\Omega_n| = \det(G_n^{(1)}) = 2^{n(n-1)/2}$.

Proof: For $1 \leq i \leq n$, let A_i denote the point $(-2i+1, 0)$ and let B_i denote the point $(2i-1, 0)$. Let h_{ij} denote the (i,j)-entry of $H_n^{(1)}$. Note that $h_{ij} = r_{i+j-1}$ is equal to the number of large Schröder paths from A_i to B_j. Let P be the set of ordered pairs $(\sigma, (\tau_1, \ldots, \tau_n))$, where σ is a permutation of $\{1, \ldots, n\}$, and $(\tau_1, \ldots, \tau_n)$ is an n-tuple of large Schröder paths such that τ_i goes from A_i to $B_{\sigma(i)}$. According to the sign of σ, the ordered pairs in P are partitioned into P^+ and P^-. Then

$$\det(H_n^{(1)}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} h_{i,\sigma(i)} = |P^+| - |P^-|.$$

If there exists a sign-reversing involution φ on P, then $\det(H_n^{(1)})$ is equal to the number of fixed points of φ. Let $(\sigma, (\tau_1, \ldots, \tau_n)) \in P$ be such a pair that at least two paths of $(\tau_1, \ldots, \tau_n)$ intersect. Choose the first pair $i < j$ in lexical order such that τ_i intersects τ_j. Construct new paths τ_i' and τ_j' by switching the tails after the last point of intersection of τ_i and τ_j. Now τ_i' goes from A_i to $B_{\sigma(j)}$ and τ_j' goes from A_j to $B_{\sigma(i)}$. Since $\sigma \circ (ij)$ carries i into $\sigma(j)$, j into $\sigma(i)$, and k into $\sigma(k)$, for $k \neq i, j$, we define

$$\varphi((\sigma, (\tau_1, \ldots, \tau_n))) = (\sigma \circ (ij), (\tau_1, \ldots, \tau_i', \ldots, \tau_j', \ldots, \tau_n)).$$

Clearly, φ is sign-reversing. Since the first intersecting pair $i < j$ is not affected by φ, φ is an involution. The fixed points of φ are the pairs $(\sigma, (\tau_1, \ldots, \tau_n)) \in P$ such that σ is the identity, and $\tau_1, \ldots, \tau_n$ do not intersect, i.e., $(\tau_1, \ldots, \tau_n) \in \Pi_n$. Hence $\det(H_n^{(1)}) = |\Pi_n|$. By the same argument, we have $\det(G_n^{(1)}) = |\Omega_n|$. It follows from (1) and the identity $H_n^{(1)} = 2G_n^{(1)}$ that

$$|\Pi_n| = \det(H_n^{(1)}) = 2^n \cdot \det(G_n^{(1)}) = 2^n |\Omega_n| = 2^n |\Pi_{n-1}|.$$

Note that $|\Pi_1| = 2$, and hence, by induction, the assertions (i) and (ii) follow. $\square$

To prove the Aztec diamond theorem, we shall establish a bijection between Π_n and the set of domino tilings of $\mathrm{Az}(n)$ based on an idea, due to D. Randall, mentioned in [7, Solution of Exercise 6.49].

Proposition 2.2 *There is a bijection between the set of domino tilings of the Aztec diamond of order n and the set of n-tuples $(\pi_1, \ldots, \pi_n)$ of large Schröder paths satisfying the conditions (A1) and (A2).*

Proof: Given a tiling T of $\mathrm{Az}(n)$, we associate T with an n-tuple $(\tau_1, \ldots, \tau_n)$ of non-intersecting paths as follows. Let the rows of $\mathrm{Az}(n)$ be indexed by $1, 2, \ldots, 2n$ from bottom to top. For $1 \leq i \leq n$, define a path τ_i from the center of the left-hand edge of the i-th row to the center of the right-hand edge of the i-th row. Namely, each step of the path is from the center of a domino edge (where a domino is regarded as having six edges of unit length) to the center of another edge of the some domino D, such that the step is symmetric with respect to the center of D. One can check that for each tiling there is a unique such an n-tuple $(\tau_1, \ldots, \tau_n)$ of paths, moreover, any two paths τ_i, τ_j of which do not intersect. Conversely any such n-tuple of paths corresponds to a unique domino tiling of $\mathrm{Az}(n)$ (note that any domino not on these paths is horizontal).

To establish a mapping ψ, for $1 \leq i \leq n$, we form a large Schröder path π_i from τ_i with $i-1$ up steps attached in the beginning of τ_i and with $i-1$ down steps attached in the end (and then raise π_i above the x-axis), and define $\psi(T) = (\pi_1, \ldots, \pi_n)$. One can verify that the n-tuple $(\pi_1, \ldots, \pi_n)$ of large Schröder paths satisfies the conditions (A1) and (A2), and hence $\psi(T) \in \Pi_n$. To find ψ^{-1}, we can retrieve an n-tuple $(\tau_1, \ldots, \tau_n)$ of non-intersecting paths, which corresponds to a unique domino tiling of $\mathrm{Az}(n)$, from each n-tuple $(\pi_1, \ldots, \pi_n)$ of large Schröder paths satisfying the conditions (A1) and (A2) by a reverse procedure. $\square$

For example, on the left of Figure 3 is a tiling T of $\mathrm{Az}(3)$ and the associated triple (τ_1, τ_2, τ_3) of non-intersecting paths. On the right of Figure 3 is the corresponding triple $\psi(T) = (\pi_1, \pi_2, \pi_3) \in \Pi_3$ of large Schröder paths.

By Propositions 2.1 and 2.2, we deduce the Aztec diamond theorem anew.

Theorem 2.3 (Aztec diamond theorem) *The number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$.*

Remark: The proof of Proposition 2.1 relies on the recurrence relation $\Pi_n = 2^n \Pi_{n-1}$ essentially, which is derived by means of the determinants of the Hankel matrices $H_n^{(1)}$ and $G_n^{(1)}$. We are interested to hear a purely combinatorial proof of this recurrence relation.

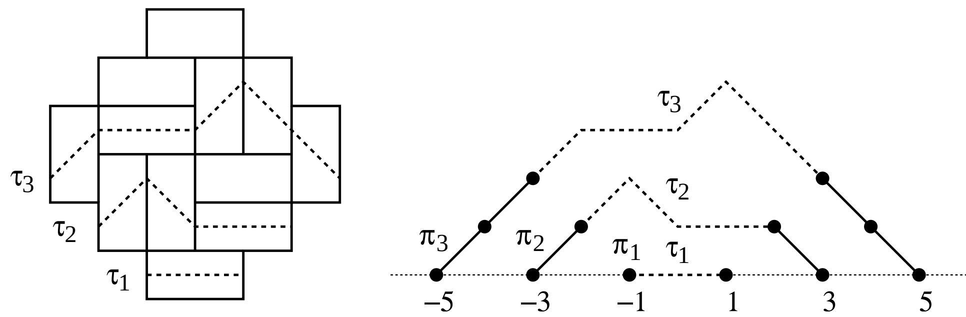

Figure 3: a tiling of Az(3) and the corresponding triple of non-intersecting Schröder paths

In a similar manner we derive the determinants of the Hankel matrices of large and small Schröder paths of the forms

$$H_n^{(0)} := \begin{bmatrix} r_0 & r_1 & \cdots & r_{n-1} \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & & \vdots \\ r_{n-1} & r_n & \cdots & r_{2n-2} \end{bmatrix}, \quad G_n^{(0)} := \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \\ s_1 & s_2 & \cdots & s_n \\ \vdots & \vdots & & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-2} \end{bmatrix}.$$

Proposition 2.4 *For $n \ge 1$, $\det(H_n^{(0)}) = \det(G_n^{(0)}) = 2^{n(n-1)/2}$.*

Proof: Let Π_n^* (resp. Ω_n^*) be the set of n-tuples $(\mu_0, \mu_1, \dots, \mu_{n-1})$ of large Schröder paths (resp. small Schröder paths) satisfying the two conditions (i) the path μ_i goes from $(-2i, 0)$ to $(2i, 0)$, for $0 \le i \le n-1$, and (ii) any two paths μ_i and μ_j do not intersect. Note that μ_0 degenerates into a single point and that Π_n^* and Ω_n^* are identical since for any $(\mu_0, \mu_1, \dots, \mu_{n-1}) \in \Pi_n^*$ all of the paths μ_i have no level steps on the x-axis. By a similar argument of Proposition 2.1, we have $\det(H_n^{(0)}) = |\Pi_n^*| = |\Omega_n^*| = \det(G_n^{(0)})$. Moreover, there is a bijection ρ between Π_{n-1} and Π_n^*, for $n \ge 2$, which carries $(\pi_1, \dots, \pi_{n-1}) \in \Pi_{n-1}$ into $\rho((\pi_1, \dots, \pi_{n-1})) = (\mu_0, \mu_1, \dots, \mu_{n-1}) \in \Pi_n^*$, where μ_0 is the origin and $\mu_i = \mathsf{U}\pi_i\mathsf{D}$, for $1 \le i \le n-1$. The assertion follows from Proposition 2.1(i). □

For example, on the left of Figure 4 is a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ of non-intersecting large Schröder paths. The corresponding quadruple $(\mu_0, \mu_1, \mu_2, \mu_3) \in \Pi_4^*$ is shown on the right.

Hankel matrices $H_n^{(0)}$ and $H_n^{(1)}$ may be associated with any given sequence of real numbers. As noted by Aigner in [1] that the sequence of determinants

$$\det(H_1^{(0)}), \det(H_1^{(1)}), \det(H_2^{(0)}), \det(H_2^{(1)}), \dots$$

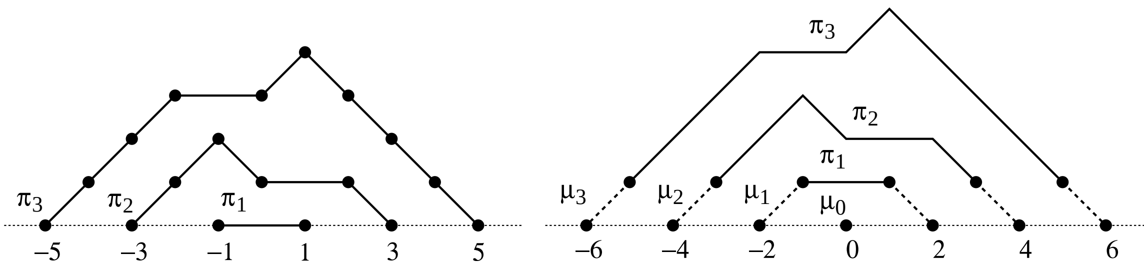

Figure 4: a triple $(\pi_1, \pi_2, \pi_3) \in \Pi_3$ and the corresponding quadruple $(\mu_0, \mu_1, \mu_2, \mu_3) \in \Pi_4^*$

uniquely determines the original number sequence provided that $\det(H_n^{(0)}) \neq 0$ and $\det(H_n^{(1)}) \neq 0$, for all $n \geq 1$, we have a characterization of large and small Schröder numbers.

Corollary 2.5 *The following results hold.*

(i) *The large Schröder numbers* $\{r_n\}_{n\geq 0}$ *are the unique sequence with the Hankel determinants* $\det(H_n^{(0)}) = 2^{n(n-1)/2}$ *and* $\det(H_n^{(1)}) = 2^{n(n+1)/2}$, *for all* $n \geq 1$.

(ii) *The small Schröder numbers* $\{s_n\}_{n\geq 0}$ *are the unique sequence with the Hankel determinants* $\det(G_n^{(0)}) = \det(G_n^{(1)}) = 2^{n(n-1)/2}$, *for all* $n \geq 1$.

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