# QUANTIZATION OF FORMAL CLASSICAL DYNAMICAL *r*-MATRICES: THE REDUCTIVE CASE

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ABSTRACT. In this paper we prove the existence of a formal dynamical twist quantization for any triangular and non-modified formal classical dynamical r-matrix in the reductive case. The dynamical twist is constructed as the image of the dynamical r-matrix by a  $L_{\infty}$ -quasi-isomorphism. This quasiisomorphism also allows us to classify formal dynamical twist quantizations up to gauge equivalence.

### INTRODUCTION

In [Fe], Felder introduced dynamical versions of both classical and quantum Yang-Baxter equations which has been generalized to the case of a nonabelian base in [EV] for the classical part and in [X3] for the quantum part. Naturally this leads to quantization problems which have been formulated in terms of twist quantization à la Drinfeld ([Dr1]) in [X2, X3, EE1, EE2].

Let us formulate this problem in the general context. Consider an inclusion  $\mathfrak{h} \subset \mathfrak{g}$  of Lie algebras equipped with an element  $Z \in (\wedge^3 \mathfrak{g})^{\mathfrak{g}}$ . A *(modified) classical* dynamical r-matrix for  $(\mathfrak{g}, \mathfrak{h}, Z)$  is a regular (meaning  $C^{\infty}$ , meromorphic, formal, ... depending on the context)  $\mathfrak{h}$ -equivariant map  $\rho : \mathfrak{h}^* \to \wedge^2 \mathfrak{g}$  which satisfies the (modified) classical dynamical Yang-Baxter equation (CDYBE)

(1) 
$$\operatorname{CYB}(\rho) - \operatorname{Alt}(d\rho) = Z$$

where  $\text{CYB}(\rho) := [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] = \frac{1}{2}[\rho, \rho]$  and

$$\operatorname{Alt}(d\rho) := \sum_{i} \left( h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda^i} - h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda^i} + h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda^i} \right)$$

Here  $(h_i)$  and  $(\lambda^i)$  are dual basis of  $\mathfrak{h}$  and  $\mathfrak{h}^*$ .

Let  $\Phi = 1 + O(\hbar^2) \in (U\mathfrak{g}^{\otimes 3})^{\mathfrak{g}}[[\hbar]]$  be an associator quantizing Z (of which the existence was proved in [Dr2, proposition 3.10]). A dynamical twist quantization of a (modified) classical dynamical r-matrix  $\rho$  associated to  $\Phi$  is a regular h-equivariant map  $J = 1 + O(\hbar) \in \operatorname{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$  such that  $\operatorname{Alt} \frac{J-1}{\hbar} = \rho \mod \hbar$  and which satisfies the (modified) dynamical twist equation (DTE)

(2) 
$$J^{12,3}(\lambda) * J^{1,2}(\lambda + \hbar h^3) = \Phi^{-1} J^{1,23}(\lambda) * J^{2,3}(\lambda)$$

where \* denotes the PBW star-product of functions on  $\mathfrak{h}^*$  and

$$J^{1,2}(\lambda + \hbar h^3) := \sum_{k \ge 0} \frac{\hbar^k}{k!} \sum_{i_1, \dots, i_k} (\partial_{\lambda^{i_1}} \cdots \partial_{\lambda^{i_k}} J)(\lambda) \otimes (h_{i_1} \cdots h_{i_k})$$

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Now observe that many (modified) classical dynamical r-matrices can be viewed as formal ones by taking their Taylor expansion at 0. In this paper we are interested in the following conjecture:

**Conjecture 0.1** ([EE1]). Any (modified) formal classical dynamical r-matrix admits a dynamical twist quantization.

Let us reformulate DTE in the formal framework. A formal (modified) dynamical twist is an element  $J(\lambda) = 1 + O(\hbar) \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S}\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  which satisfies DTE, and  $J^{1,2}(\lambda + \hbar h^3) \in (U\mathfrak{g}^{\otimes 3} \hat{\otimes} \hat{S}\mathfrak{h})[[\hbar]]$  is equal to  $(\mathrm{id}^{\otimes 2} \otimes \tilde{\Delta})(J)$  where  $\tilde{\Delta} : \hat{S}\mathfrak{h} \to (U\mathfrak{g} \hat{\otimes} \hat{S}\mathfrak{h})[[\hbar]]$  is induced by  $\mathfrak{h} \ni x \mapsto \hbar x \otimes 1 + 1 \otimes x$ . Then define  $K := J(\hbar\lambda) \in (U\mathfrak{g}^{\otimes 2} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  which we view as an element of  $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  using the symmetrization map  $S\mathfrak{h} \to U\mathfrak{h}$ . Since J is a solution of DTE K satisfies the (modified) algebraic dynamical twist equation (ADTE)

(3) 
$$K^{12,3,4}K^{1,2,34} = (\Phi^{-1})^{1,2,3}K^{1,23,4}K^{2,3,4}$$

Moreover and by construction,  $K = 1 + \sum_{n \ge 1} \hbar^n K_n$  has the  $\hbar$ -adic valuation property. Namely,  $U\mathfrak{h}$  is filtered by  $(U\mathfrak{h})_{\le n} = \ker (\operatorname{id} - \eta \circ \varepsilon)^{\otimes n+1} \circ \Delta^{(n)}$  where  $\varepsilon : U\mathfrak{h} \to \mathbf{k}$  and  $\eta : \mathbf{k} \to U\mathfrak{h}$  are the counit and unit maps, and  $K_n \in (U\mathfrak{h})_{\le n-1}$ . Conversely, any algebraic dynamical twist having the  $\hbar$ -adic valuation property can be obtained from a unique formal dynamical twist by this procedure.

This paper, in which we always assume Z = 0 and  $\Phi = 1$  (non-modified case), is organized as follow.

In section 1 we define two differential graded Lie algebras (dgla's) respectively associated to classical dynamical *r*-matrices and algebraic dynamical twists. Then we formulate the main theorem of this paper which states that if  $\mathfrak{h}$  admits an ad $\mathfrak{h}$ -invariant complement (the *reductive* case) then these two dgla's are  $L_{\infty}$ -quasiisomorphic and we prove that it implies Conjecture 0.1 in this case, which generalizes Theorem 5.3 of [X2]:

**Theorem 0.2.** In the reductive case, any formal classical dynamical r-matrix for  $(\mathfrak{g}, \mathfrak{h}, 0)$  admits a dynamical twist quantization (associated to the trivial associator).

The second section is devoted to the proof of the main theorem of section 1: using an equivariant formality theorem for homogeneous spaces which is obtain from [Do], we construct a  $L_{\infty}$ -quasi-isomorphism which we then modify in order to obtain the desired one. We use this  $L_{\infty}$ -quasi-isomorphism to classify formal dynamical twist quantizations up to gauge equivalence for the reductive case in section 3. In section 4 we prove that if  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  for  $\mathfrak{h}$  abelian and  $\mathfrak{m}$  a Lie subalgebra then the results of sections 1 and 2 are still true in this situation. We conclude the paper with some open questions, and recall basic results for  $L_{\infty}$ -algebras in an appendix.

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# 1. Definitions and results

Let  $\mathfrak{h} \subset \mathfrak{g}$  be an inclusion of Lie algebras.

1.1. Algebraic structures associated to CDYBE. Let us consider the following graded vector space

$$\underline{\mathrm{CDYB}} := \wedge^* \mathfrak{g} \otimes S \mathfrak{h} = \bigoplus_{k \geq 0} \wedge^k \mathfrak{g} \otimes S \mathfrak{h}$$

equipped with the differential d defined by

(4) 
$$d(x_1 \wedge \dots \wedge x_k \otimes h_1 \dots h_l) := -\sum_{i=1}^l h_i \wedge x_1 \wedge \dots \wedge x_k \otimes h_1 \dots h_l \hat{h}_i$$

With the exterior product  $\wedge$  it becomes a differential graded commutative associative algebra. Moreover, one can define a graded Lie bracket of degree -1 on <u>CDYB</u> which is the Lie bracket of  $\mathfrak{g}$  extended to <u>CDYB</u> in the following way:

(5) 
$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|-1)|b|} b \wedge [a, c]$$

Thus one can observe that polynomial solutions to CDYBE are exactly elements  $\rho \in \underline{\text{CDYB}}$  of degree 2 such that  $d\rho + \frac{1}{2}[\rho, \rho] = 0$ . We would like to say that such a  $\rho$  is a Maurer-Cartan element but ( $\underline{\text{CDYB}}[1], d, [,]$ ) is not a differential graded Lie algebra (dgla).

Instead, remember that we are interested in  $\mathfrak{h}$ -equivariant solutions of CDYBE (i.e., dynamical *r*-matrices) and thus consider the subspace  $\mathfrak{g}_1 = (\underline{\text{CDYB}})^{\mathfrak{h}}$  of  $\mathfrak{h}$ -invariants with the same differential and bracket.

**Proposition 1.1.**  $(\mathfrak{g}_1[1], d, [,])$  is a dgla. Moreover  $(\mathfrak{g}_1, d, \wedge, [,])$  is a Gerstenhaber algebra.

*Proof.* Let  $a = x_1 \wedge \cdots \wedge x_k \otimes h_1 \cdots h_s$  and  $b = y_1 \wedge \cdots \wedge y_l \otimes m_1 \cdots m_t$  be  $\mathfrak{h}$ -invariant elements in  $\mathfrak{g}_1$ . We want to show that

(6) 
$$d[a,b] = [da,b] + (-1)^{k-1}[a,db]$$

The l.h.s. of (6) is equal to

$$-\Big(\sum_{i=1}^{s} h_i \wedge [x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_l] \otimes h_1 \dots h_s m_1 \dots m_t \hat{h}_i \\ + \sum_{j=1}^{t} m_j \wedge [x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_l] \otimes h_1 \dots h_s m_1 \dots m_t \hat{m}_j\Big)$$

The first term in the r.h.s. of (6) gives

$$\sum_{i=1}^{s} \left( (-1)^{k-1} x_1 \wedge \dots \wedge x_k \wedge [h_i, y_1 \wedge \dots \wedge y_l] - h_i \wedge [x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_l] \right) \otimes h_1 \dots h_s m_1 \dots m_t \hat{h}_i$$

and for the second term we obtain

$$\sum_{j=1}^{t} \left( (-1)^{k-1} [m_j, x_1 \wedge \dots \wedge x_k] \wedge y_1 \wedge \dots \wedge y_l - m_j \wedge [x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_l] \right) \otimes h_1 \cdots h_s m_1 \cdots m_t \hat{m}_j$$

Thus the difference between the l.h.s. and the r.h.s. of (6) is equal to

$$(-1)^{k} \Big( \sum_{i=1}^{k} x_{1} \wedge \dots \wedge x_{k} \wedge [h_{i}, y_{1} \wedge \dots \wedge y_{l}] \otimes h_{1} \dots h_{s} m_{1} \dots m_{t} \hat{h}_{i} \\ + \sum_{j=1}^{l} [m_{j}, x_{1} \wedge \dots \wedge x_{k}] \wedge y_{1} \wedge \dots \wedge y_{l} \otimes h_{1} \dots h_{s} m_{1} \dots m_{t} \hat{m}_{j} \Big)$$

Then using  $\mathfrak{h}$ -invariance of a and b one obtains

$$(-1)^{k-1}\sum_{i,j}x_1\wedge\cdots\wedge x_k\wedge y_1\wedge\cdots\wedge y_l\otimes (h_1\cdots h_s m_1\cdots m_t([h_i,m_j]-[m_j,h_i])\hat{h}_i\hat{m}_j)=0$$

The second statement of the proposition is obvious from the definition (5) of the bracket.  $\hfill \Box$ 

Let  $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S} \mathfrak{h})^{\mathfrak{h}}$  be a formal classical dynamical *r*-matrix. Since  $\rho$  satisfies CDYBE,  $\alpha := \hbar \rho(\hbar \lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$  is a Maurer-Cartan element (i.e.  $d\alpha + \frac{1}{2}[\alpha, \alpha] = 0$ ).

1.2. Algebraic structures associated to ADTE. Let us now consider the graded vector space

$$\underline{\mathrm{ADT}} := T^*U\mathfrak{g} \otimes U\mathfrak{h} = \bigoplus_{k \ge 0} \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

equipped with the differential b given by

(7) 
$$b(P) := P^{2,\dots,k+2} + \sum_{i=1}^{k+1} (-1)^i P^{1,\dots,ii+1,\dots,k+2} \text{ for } P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$$

**Remark 1.2.** This is just the coboundary operator of Hochschild's cohomology with value in a comodule; and  $b^2 = 0$  follows directly from an easy calculation.

One can define on <u>ADT</u> an associative product  $\cup$  (the *cup* product) which is given on homogeneous elements  $P \in \otimes^k U\mathfrak{g} \otimes U\mathfrak{h}$  and  $Q \in \otimes^l U\mathfrak{g} \otimes U\mathfrak{h}$  by

$$P \cup Q := P^{1,\ldots,k,k+1\ldots k+l+1}Q^{k+1,\ldots,k+l+1}$$

**Proposition 1.3.** (ADT, b,  $\cup$ ) is a differential graded associative algebra.

*Proof.* The cup product is obviously associative. Thus the only thing we have to check is that

$$b(P \cup Q) = bP \cup Q + (-1)^{|P|}P \cup bQ$$

Let k = |P| and l = |Q|. The l.h.s. of (8) is equal to

$$P^{2,\dots,k+1,k+2\dots,k+l+2}Q^{k+2,\dots,k+l+2} + \sum_{i=1}^{k} (-1)^{i}P^{1,\dots,ii+1,\dots,k+1,k+2\dots,k+l+2}Q^{k+2,\dots,k+l+2} + \sum_{i=k+1}^{k+l+1} (-1)^{i}P^{1,\dots,k,k+1\dots,k+l+2}Q^{k+1,\dots,ii+1,\dots,k+l+2}$$

The first line of this expression is equal to

$$bP \cup Q - (-1)^{k+1} P^{1,\dots,k,k+1\dots,k+l+2} Q^{k+2,\dots,k+l+2}$$

and the last term of the same expression gives

 $(-1)^{k} (P \cup bQ - P^{1,\dots,k,k+1\dots,k+l+2}Q^{k+2,\dots,k+l+2})$ 

(8)

The proposition is proved.

Recall that in the case  $\mathfrak{h} = \{0\}$  one can define a brace algebra structure on  $(T^*U\mathfrak{g})[1]$  (see [Ge]). Unfortunately we are not able to extend this structure to ADT in general. Since we deal with  $\mathfrak{h}$ -equivariant solutions of ADTE we can consider the subspace  $\mathfrak{g}_2 = (\underline{ADT})^{\mathfrak{h}}$  of  $\mathfrak{h}$ -invariants. Let us now define a collection of linear homogeneous maps of degree zero  $\{-|-,\ldots,-\}: \mathfrak{g}_2[1] \otimes \mathfrak{g}_2[1]^{\otimes m} \to \mathfrak{g}_2[1]$  indexed by  $m \ge 0$ , and  $\{P|Q_1, \ldots, Q_m\}$  is given by

$$\sum_{\substack{0 \le i_1, i_m + k_m \le n \\ i_l + k_l \le i_{l+1}}} (-1)^{\epsilon} P^{1, \dots, i_1 + 1 \dots i_1 + k_1, \dots, i_m + 1 \dots i_m + k_m, \dots, n+1} \prod_{s=i}^m Q_s^{i_s + 1, \dots, i_s + k_s, i_s + k_s + 1 \dots n+1}$$

where  $k_s = |Q_s|$ ,  $n = |P| + \sum_s k_s - m$  and  $\epsilon = \sum_s (k_s - 1)i_s$ .

**Proposition 1.4.**  $(g_2[1], \{-|-, ..., -\})$  is a brace algebra.

*Proof.* Since we work with  $\mathfrak{h}$ -invariant elements one can remark that if  $i_s + k_s \leq i_t$  then  $Q_s^{i_s+1,\ldots,i_s+k_s,i_s+k_s+1\ldots n+1}$  and  $Q_t^{i_t+1,\ldots,i_t+k_t,i_t+k_t+1\ldots n+1}$  commute. Using this the proof becomes identical to the case when  $\mathfrak{h} = 0$  (see [Ge] for example).

Now observe that since  $m = 1^{\otimes 3} \in (\otimes^2 U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}$  is such that  $\{m|m\} = 0$  one obtains a  $B_{\infty}$ -algebra structure ([Ba]) on  $\mathfrak{g}_2$  (see [Kh]). More precisely, we have a differential graded bialgebra structure on the cofree tensorial coalgebra  $T(\mathfrak{g}_2[1])$  of which structure maps  $a^n, a^{p,q}$  are given by

• 
$$a^{1}(P) = bP = (-1)^{|P|-1}[m, P]_{G}$$
, where  
 $[P, Q]_{G} := \{P|Q\} - (-1)^{(|P|-1)(|Q|-1)}\{Q|P\}$   
•  $a^{2}(P, Q) = \{m|P, Q\} = P \cup Q$   
•  $a^{0,1} = a^{1,0} = \mathrm{id}$   
•  $a^{1,n}(P; Q_{1}, \dots, Q_{n}) = \{P|Q_{1}, \dots, Q_{n}\}$  for  $n \ge 1$   
• all other maps are zero

In particular, we have

**Proposition 1.5.**  $(\mathfrak{g}_2[1], b, [,]_G)$  is a dgla.

**Remark 1.6.** Since that for any graded vector space V, dg bialgebra structures on the cofree coassociative coalgebra  $T^{c}V$  are in one-to-one correspondence with dg Lie bialgebra structures on the cofree Lie coalgebra  $L^{c}V$  (see [Ta], section 5), then  $L^{c}(\mathfrak{g}_{2}[1])$  becomes a dg Lie bialgebra with differential and Lie bracket given by maps  $l^{n}, l^{p,q}$  such that  $l^{1} = b$  and  $l^{1,1} = [,]_{G}$ . Therefore  $d_{2} := \sum_{i \geq 0} l^{i} + \sum_{p,q \geq 0} l^{p,q}$ :  $C^{c}(L^{c}(\mathfrak{g}_{2}[1])) \rightarrow C^{c}(L^{c}(\mathfrak{g}_{2}[1]))$  defines a  $G_{\infty}$ -algebra structure on  $\mathfrak{g}_{2}$   $(d_{2} \circ d_{2} = 0)$ since  $d_2$  is just the Chevalley-Eilenberg differential on the dg Lie algebra  $L^c(\mathfrak{g}_2[1]))$ .

1.3. Main result and proof of theorem 0.2. First of all, observe that <u>CDYB</u>,  $\mathfrak{g}_1$  and  $\mathcal{G}_1 := C^c(\mathfrak{g}_1[2])$  have a natural grading induced by the one of Sh. In the same way <u>ADT</u>,  $\mathfrak{g}_2$  and  $\mathcal{G}_2 := C^c(\mathfrak{g}_2[2])$  have a natural filtration induced by the one of  $U\mathfrak{h}$ . Our main goal is to prove the following theorem, which is sufficient to obtain algebraic dynamical twists from formal dynamical *r*-matrices.

**Theorem 1.7.** In the reductive case, there exists a  $L_{\infty}$ -quasi-isomorphism

 $\Psi: (\mathcal{G}_1, \mathbf{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$ 

with the following two filtration properties:

(F1)  $\forall X \in (\mathfrak{g}_1)_k, \Psi^1(X) = (\text{alt} \otimes \text{sym})(X) \mod (\mathfrak{g}_2)_{\leq k-1}$ (F2)  $\forall X \in (\Lambda^n \mathfrak{g}_1)_k, \Psi^n(X) \in (\mathfrak{g}_2)_{\leq n+k-1}$ 

Proof of Theorem 0.2. Now consider a formal solution  $\rho(\lambda) \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S} \mathfrak{h})^{\mathfrak{h}}$  to CDYBE. Let us define  $\alpha := \hbar \rho(\hbar \lambda) \in \hbar \mathfrak{g}_1[[\hbar]]$  which is a Maurer-Cartan element in  $\hbar \mathfrak{g}_1[[\hbar]]$ . The  $L_{\infty}$ -morphism property implies that  $\tilde{\alpha} := \sum_{n=1}^{\infty} \frac{1}{n!} \Psi^n(\Lambda^n \alpha)$  is a Maurer-Cartan element in  $\hbar \mathfrak{g}_2[[\hbar]]$ ; this exactly means that  $K := 1 + \tilde{\alpha} \in (\otimes^2 \mathfrak{U} \mathfrak{g} \otimes \mathfrak{U} \mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  satisfies ADTE. Moreover, due to (F2) the coefficient  $K_n$  of  $\hbar^n$  in K lies in  $(\mathfrak{g}_2)_{\leq n-1}$ . It means that there exists  $J \in (U\mathfrak{g}^{\otimes 2} \hat{\otimes} \hat{S} \mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  satisfying DTE and such that  $K = (\mathrm{id}^{\otimes 2} \otimes \mathrm{sym})(J(\hbar \lambda))$ . Finally, property (F1) obviously implies that the semiclassical limit condition  $\frac{J-J^{\mathrm{op}}}{\hbar} = \rho \mod \hbar$  is satisfied.  $\Box$ 

### 2. Proof of theorem 1.7

In this section we assume that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  with  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ . Let us denote by  $p: \mathfrak{g} \to \mathfrak{m}$  the projection on  $\mathfrak{m}$  along  $\mathfrak{h}$ ; it is  $\mathfrak{h}$ -equivariant.

2.1. **Resolutions.** Let us first observe that the bilinear map  $[,]_{\mathfrak{m}} := (\wedge p) \circ [,]$  defines a graded Lie bracket of degree -1 on  $(\wedge^*\mathfrak{m})^{\mathfrak{h}}$ . Then we prove

**Proposition 2.1.** The natural map  $p_1 : (\mathfrak{g}_1[1], \mathrm{d}, [,]) \to ((\wedge^*\mathfrak{m})^{\mathfrak{h}}[1], 0, [,]_{\mathfrak{m}})$  is a morphism of dgla's. Moreover, there exists an operator  $\delta : \mathfrak{g}_1^* \to \mathfrak{g}_1^{*-1}$  such that  $\delta \mathrm{d} + \mathrm{d}\delta = \mathrm{id} - p_1, \ \delta \circ \delta = 0$  and  $\delta((\mathfrak{g}_1)_k) \subset (\mathfrak{g}_1)_{k+1}$ . In particular,  $p_1$  induces an isomorphism in cohomology.

*Proof.* The projection  $p_1 := (\land p) \otimes \varepsilon : (\underline{CDYB}, d) \to (\land^* \mathfrak{m}, 0)$  is a  $\mathfrak{h}$ -equivariant morphism of complexes, and it obviously restricts to a morphism of (differential) graded Lie algebras at the level of  $\mathfrak{h}$ -invariants.

Moreover,  $\wedge^n \mathfrak{g} \otimes S\mathfrak{h} \cong \bigoplus_{p+q=n} \wedge^p \mathfrak{m} \otimes \wedge^q \mathfrak{h} \otimes S\mathfrak{h}$  as a  $\mathfrak{h}$ -module; and under this identification d becomes  $-\mathrm{id} \otimes d_K$ , where  $d_K : \wedge^* \mathfrak{h} \otimes S\mathfrak{h} \to \wedge^{*+1} \mathfrak{h} \otimes S\mathfrak{h}$  is Koszul's coboundary operator, and  $p_1$  corresponds to the projection on the part of zero antisymmetric and symmetric degrees in  $\mathfrak{h}$ . Let us define  $\delta = \mathrm{id} \otimes \delta_K$  with  $\delta_K : \wedge^* \mathfrak{h} \otimes S^* \mathfrak{h} \to \wedge^{*-1} \mathfrak{h} \otimes S^{*+1} \mathfrak{h}$  defined by

$$\delta_K(x_1 \wedge \dots \wedge x_n \otimes h_1 \dots h_m) = \begin{cases} \frac{1}{m+n} \sum_i (-1)^i x_1 \wedge \dots \cdot \hat{x_i} \dots \wedge x_n \otimes h_1 \dots h_m x_i & \text{if } m+n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Finally remark  $\delta$  is a  $\mathfrak{h}$ -equivariant homotopy operator:  $\delta d + d\delta = id - p_1$  and  $\delta \circ \delta = 0$ . The proposition is proved.

Now we prove a similar result for  $\mathfrak{g}_2$ . Let us first define  $U\mathfrak{m} := \operatorname{sym}(S\mathfrak{m}) \subset U\mathfrak{g}$ ; this is a sub-coalgebra of  $U\mathfrak{g}$  and thus  $T^*U\mathfrak{m}$  equipped with its Hochschild's coboundary operator  $b_{\mathfrak{m}}$  becomes a cochain subcomplex of the Hochschild complex  $(T^*U\mathfrak{g}, b_{\mathfrak{g}})$  of  $U\mathfrak{g}$ . We also have the following

**Lemma 2.2.**  $U\mathfrak{g} = U\mathfrak{g}\cdot\mathfrak{h}\oplus U\mathfrak{m}$  as a filtered  $\mathfrak{h}$ -module. Moreover  $[,]_{G,\mathfrak{m}} := (\otimes p)\circ[,]$  defines a graded Lie bracket of degree -1 on  $(T^*U\mathfrak{m})^{\mathfrak{h}}$ 

*Proof.* See [He, Ch.II §4.2] for the first statement. The second statement follows from a direct computation.  $\Box$ 

Then we prove the

**Proposition 2.3.** The natural map  $p_2 : (\mathfrak{g}_2[1], b, [,]_G) \to ((T^*U\mathfrak{m})^{\mathfrak{h}}[1], b_{\mathfrak{m}}, [,]_{G,\mathfrak{m}})$ is a morphism of dgla's. Moreover, there exists an operator  $\kappa : \mathfrak{g}_2^* \to \mathfrak{g}_2^{*-1}$  such that  $\kappa b + b\kappa = \mathrm{id} - p_2, \ \kappa \circ \kappa = 0$  and  $\kappa((\mathfrak{g}_2)_{\leq k}) \subset (\mathfrak{g}_2)_{\leq k+1}$ . In particular,  $p_2$  induces an isomorphism in cohomology.

Proof. The projection  $p_2 := (\otimes p) \otimes \varepsilon : (\underline{ADT}, b) \to (T^*U\mathfrak{m}, b_{\mathfrak{m}})$  is a  $\mathfrak{h}$ -equivariant morphism of complexes, and it obviously restricts to a morphism of dgla's at the level of  $\mathfrak{h}$ -invariants (by lemma 2.2).

Remember that  $\mathfrak{g}_2$  has a natural filtration induced by the one of  $U\mathfrak{h}$ . Then one obtains a spectral sequence of which we compute the first terms:

$$\begin{aligned} E_0^{*,*} &= (T^* \mathcal{U}\mathfrak{g} \otimes S^* \mathfrak{h})^{\mathfrak{h}} \quad d_0 = b_{\mathfrak{g}} \otimes \mathrm{id} \\ E_1^{*,*} &= (\wedge^* \mathfrak{g} \otimes S^* \mathfrak{h})^{\mathfrak{h}} \qquad d_1 = \mathrm{d} \\ E_2^{*,*} &= E_2^{*,0} = (\wedge^* \mathfrak{m})^{\mathfrak{h}} \qquad d_2 = 0 \end{aligned}$$

Then the proposition follows from proposition 2.1.

2.2. Inverting  $p_2$ . In this subsection, taking our inspiration from [Mo, appendix], we prove the following

# **Proposition 2.4.** There exists a $L_{\infty}$ -quasi-isomorphism

$$\mathcal{Q}_2: (C^c((T^*U\mathfrak{m})^{\mathfrak{h}}[2]), b_{\mathfrak{m}} + [,]_{G,\mathfrak{m}}) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$$

such that  $\mathcal{Q}_2^1$  is the natural inclusion and  $\mathcal{Q}_2^n$  takes values in  $(\mathfrak{g}_2)_{\leq n-1}$ .

*Proof.* Let  $(N, b_N) \subset (\mathfrak{g}_2, b)$  be the kernel of the surjective morphism of complexes  $p_2: (\mathfrak{g}_2, b) \to ((T^*U\mathfrak{m})^{\mathfrak{h}}, b_{\mathfrak{m}})$ . It follows from the proofs of propositions 2.1 and 2.3 that there exists an operator  $H: N^* \to N^{*-1}$  such that  $H \circ H = 0, b_N H + H b_N = \mathrm{id}$  and  $H(N_{\leq n}) \subset N_{\leq n+1}$ .

Now let us construct a  $L_{\infty}$ -isomorphism

$$\mathcal{F}: \left(C^{c}(\mathfrak{g}_{2}[2]), b+[,]_{G}\right) \xrightarrow{\sim} \left(C^{c}((T^{*}U\mathfrak{m})^{\mathfrak{h}}[2] \oplus N[2]), b_{\mathfrak{m}}+b_{N}+[,]_{G,\mathfrak{m}}\right)$$

with structure maps  $\mathcal{F}^n : \Lambda^n \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1-n]$  such that

- $\mathcal{F}^1$  is the sum of  $p_2$  with the projection on N along  $(T^*U\mathfrak{m})^{\mathfrak{h}}$  (in some sense  $\mathcal{F}^1$  is the identity),
- for any n > 1 and  $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$ ,  $\mathcal{F}^n(X) \in N_{\leq n+k-1}$ .

Let us prove it by induction on n. First  $\mathcal{F}^1$  is a morphism of complexes by definition. Then let us define  $\mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \to ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$  by

$$\mathcal{K}_2(x\Lambda y) = [\mathcal{F}^1(x), \mathcal{F}^1(y)]_{G,\mathfrak{m}} - \mathcal{F}^1([x,y]_G)$$

It takes values in N[1] and is such that  $b_N \mathcal{K}_2(x, y) + \mathcal{K}_2(bx, y) + \mathcal{K}_2(x, by) = 0$ . Consequently  $\mathcal{F}^2 := H \circ \mathcal{K}_2 : \Lambda^2 \mathfrak{g}_2 \to N$  is such that

$$b_N \mathcal{F}^2(x,y) - \mathcal{F}^2(bx,y) - \mathcal{F}^2(x,by) = \mathcal{K}_2(x,y)$$
 ( $L_\infty$ -condition for  $\mathcal{F}^2$ )

and for any  $X \in (\Lambda^2 \mathfrak{g}_2)_{\leq k}$ ,  $\mathcal{F}^2(X) \in N_{\leq k+1}$ . After this, suppose we have constructed  $\mathcal{F}^1, \ldots, \mathcal{F}^n$  and let us define

$$\mathcal{K}_{n+1} := [,]_{G,\mathfrak{m}} \circ \mathcal{F}^{\leq n} - \mathcal{F}^{\leq n} \circ [,]_{G} : \Lambda^{2}\mathfrak{g}_{2} \to ((T^{*}U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$$

It obviously takes values in N[1] and is such that  $b_N K_{n+1} + K_{n+1}b = 0$ . Consequently  $\mathcal{F}^{n+1} := H \circ K_{n+1}$  satisfies the  $L_{\infty}$ -condition

$$b_N \mathcal{F}^{n+1} - \mathcal{F}^{n+1}b = b_N H K_{n+1} - H K_{n+1}b = (b_N H + H b_N) K_{n+1} = K_{n+1}$$

and for any  $X \in (\Lambda^n \mathfrak{g}_2)_{\leq n+1}$ ,  $\mathcal{F}^{n+1}(X) \in N_{\leq n+k}$  (since  $\mathcal{K}_{n+1}(X) \in N_{\leq n+k-1}$ ).

#### DAMIEN CALAQUE

Now let  $\mathcal{H}$  be the inverse of the isomorphism  $\mathcal{F}$ , it is such that for any  $n \geq 1$ and  $X \in (\Lambda^n \mathfrak{g}_2)_{\leq k}$ ,  $\mathcal{H}^n(X) \in N_{\leq n+k-1}$ . Finally we obtain  $\mathcal{Q}_2$  by composing  $\mathcal{H}$ with the inclusion of dgla's  $(T^*U\mathfrak{m})^{\mathfrak{h}}[1] \hookrightarrow ((T^*U\mathfrak{m})^{\mathfrak{h}} \oplus N)[1]$ .  $\Box$ 

2.3. End of the proof. Recall from [He, Ch.II §4.2] that  $(T^*U\mathfrak{m})^{\mathfrak{h}} = \operatorname{Diff}^*(G/H)^G$ and  $(\wedge^*\mathfrak{m})^{\mathfrak{h}} = \Gamma(G/H, \wedge^*T(G/H))^G$  as dgla's. Remember also from [No, Ch.II §8] that G-invariant connections on G/H are in one-to-one correspondence with  $\mathfrak{h}$ -equivariant linear maps  $\alpha : \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ , and that the torsion tensor is given by  $\alpha - \alpha^{21} - \mathfrak{p} \circ [,]$ . Thus G/H is equipped with a G-invariant torsion free connection  $\nabla$ , corresponding to the map  $\alpha := \frac{1}{2}\mathfrak{p} \circ [,]$ . Then using a theorem of Dolgushev, see [Do, theorem 5], we obtain a G-equivariant  $L_{\infty}$ -quasi-isomorphism  $\phi : \Gamma(G/H, \wedge^*T(G/H)) \to \operatorname{Diff}^*(G/H)$  with first structure map  $\phi^1 = \operatorname{alt}$ , which restricts to a  $L_{\infty}$ -quasi-isomorphism at the level of G-invariants. Let us define  $\psi := \mathcal{Q}_2 \circ \phi \circ p_1 : (C^c(\mathfrak{g}_1[2]), d + [,]) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$ ; it is a  $L_{\infty}$ -quasiisomorphism with first structure map  $\psi^1 = (\operatorname{alt} \otimes 1) \circ (\wedge^{\circ}\mathfrak{p} \otimes \varepsilon)$ .

Finally define  $V := (\operatorname{alt} \otimes \operatorname{sym}) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$  and use lemma A.3 to construct a  $L_{\infty}$ -quasi-morphism  $\Psi : (C^c(\mathfrak{g}_1[2]), d + [,]) \to (C^c(\mathfrak{g}_2[2]), b + [,]_G)$  with first structure map  $\Psi^1 = \psi^1 + b \circ V + V \circ d$ . Since for any  $X \in (\mathcal{G}_1)_k$ , then

$$b \circ (\operatorname{alt} \otimes \operatorname{sym})(X) = (\operatorname{alt} \otimes \operatorname{sym}) \circ \operatorname{d}(X) \mod (\mathfrak{g}_2)_{\leq k-1}$$

and thu

thus 
$$\Psi^{1}(X) = \psi^{1}(X) + bV(X) + V(dX)$$
  
=  $(alt \otimes sym) \circ (p_{1} + d\delta + \delta d)(X) \mod (\mathfrak{g}_{2})_{\leq k-1}$   
=  $(alt \otimes sym)(X) \mod (\mathfrak{g}_{2})_{\leq k-1}$ 

Consequently  $\Psi$  satisfies (F1). Moreover, it follows from remark A.4 that  $\Psi$  also satisfies (F2).  $\Box$ 

# 3. CLASSIFICATION

Theorem 1.7 implies a stronger result than just the existence of the twist quantization. Namely, since  $\Psi$  is a  $L_{\infty}$ -quasi-isomorphism there is a bijection between the moduli spaces of Maurer-Cartan elements of dgla's  $(\mathfrak{g}_1[1])[[\hbar]]$  and  $(\mathfrak{g}_2[1])[[\hbar]]$ .

3.1. Classification of algebraic and formal dynamical twists. Following [EE1], two dynamical twists  $J(\lambda)$  and  $J'(\lambda)$  are said to be gauge equivalent if there exists a regular  $\mathfrak{h}$ -equivariant map  $T(\lambda) = \exp(q) + O(\hbar) \in \operatorname{Reg}(\mathfrak{h}^*, U\mathfrak{g})^{\mathfrak{h}}[[\hbar]]$ , with  $q \in \operatorname{Reg}(\mathfrak{h}^*, \mathfrak{g})^{\mathfrak{h}}$  such that q(0) = 0, and satisfying

(9) 
$$J'(\lambda) = T^{12}(\lambda) * J(\lambda) * T^2(\lambda)^{-1} * T^1(\lambda + \hbar h^2)^{-1}$$

Dealing with formal functions one can easily derive an equivalence relation for the corresponding algebraic dynamical twists  $K = J(\hbar \lambda)$  and  $K' = J'(\hbar \lambda)$ :

(10) 
$$K' = Q^{12,3} K(Q^{2,3})^{-1} (Q^{1,23})^{-1}$$

in  $(U\mathfrak{g}^{\otimes 2} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ , with  $Q = 1 + O(\hbar) \in (U\mathfrak{g} \otimes U\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  given by  $T(\hbar\lambda)$ .

Assume now we are in the reductive case.

Since the composition  $\mathcal{Q}_2 \circ \phi : (C^c((\wedge \mathfrak{m})^{\mathfrak{h}}[2]), [,]_{\mathfrak{m}}) \to (C^c(\mathfrak{g}_2[2]), b+[,]_G)$  in the previous section is a  $L_{\infty}$ -quasi-isomorphism then we have a bijective correspondence

(11) 
$$\frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}}$$

where  $G_0$  is the prounipotent group corresponding to the Lie algebra  $\hbar \mathfrak{m}^{\mathfrak{h}}[[\hbar]]$ . Moreover, since the structure maps  $\mathcal{Q}_2^n$  take values in  $(\mathfrak{g}_2)_{\leq n-1}$  then it appears that any algebraic dynamical twist is gauge equivalent to a one with the  $\hbar$ -adic valuation property and thus we have a bijection

(12) 
$$\frac{\{\text{algebraic dynamical twists}\}}{\text{gauge equivalence (10)}} \longleftrightarrow \frac{\{\text{formal dynamical twists}\}}{\text{gauge equivalence (9)}}$$

3.2. Classical counterpart. Assume that we are in the reductive case. Since  $p_1$  is a  $L_{\infty}$ -quasi-isomorphism by proposition 2.1 then we have a bijection

$$\frac{\{\alpha \in \hbar(\wedge^2 \mathfrak{g} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } d\alpha + \frac{1}{2}[\alpha, \alpha] = 0\}}{G_1} \longleftrightarrow \frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0}$$

where  $G_1$  is a prounipotent group and its action (by affine transformations) is given by the exponentiation of the infinitesimal action of its Lie algebra  $\hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]$ :

(13)  $q \cdot \alpha = \mathrm{d}q + [q, \alpha] \qquad \left(q \in \hbar(\mathfrak{g} \otimes S\mathfrak{h})^{\mathfrak{h}}[[\hbar]]\right)$ 

Then going along the lines of subsection 2.2 one can prove the following

**Proposition 3.1.** There exists a  $L_{\infty}$ -quasi-isomorphism

$$\mathcal{Q}_1: (C^c((\wedge^*\mathfrak{m})^{\mathfrak{h}}[2]), [,]_{\mathfrak{m}}) \to (C^c(\mathfrak{g}_1[2]), d+[,])$$

such that  $\mathcal{Q}_1^1$  is the natural inclusion and  $\mathcal{Q}_1^n$  takes values in  $(\mathfrak{g}_1)_{\leq n-1}$ .

Consequently any Maurer-Cartan element in  $(\mathfrak{g}_1[1])[[\hbar]]$  is equivalent to a one of the form  $\hbar\rho_{\hbar}(\hbar\lambda)$ , where  $\rho_{\hbar} \in (\wedge^2 \mathfrak{g} \hat{\otimes} \hat{S} \mathfrak{h})^{\mathfrak{h}}[[\hbar]]$  satisfies CDYBE. In other words  $\rho_{\hbar}$ is  $\hbar$ -dependant formal dynamical *r*-matrix. On such a  $\rho_{\hbar}$  the infinitesimal action (13) becomes

(14) 
$$q \cdot \rho_{\hbar} = -\sum_{i} h_{i} \wedge \frac{\partial q}{\partial \lambda^{i}} + [q, \rho_{\hbar}] \qquad (q \in \mathfrak{g} \hat{\otimes} \hat{S} \mathfrak{h})^{\mathfrak{h}}[[\hbar]]$$

This action integrates in an affine action of some group  $\widetilde{G}_1$  of  $\mathfrak{h}$ -equivariant formal maps with values in the Lie group G of  $\mathfrak{g}$ . And then we have a bijection

(15) 
$$\frac{\{\pi \in \hbar(\wedge^2 \mathfrak{m})^{\mathfrak{h}}[[\hbar]] \text{ s.t. } [\pi, \pi]_{\mathfrak{m}} = 0\}}{G_0} \longleftrightarrow \frac{\{\text{form. dynam. } r\text{-matrices}/\mathbb{R}[[\hbar]]\}}{\widetilde{G_1}}$$

**Remark 3.2.** This bijection has to be compared with Proposition 2.13 in [X2] and section 3 of [ES]

Finally, combining (15), (11) and (12) we obtain the following generalization of Theorem 6.11 in [X2] to the case of a nonabelian base:

**Theorem 3.3.** Let  $\pi \in (\wedge^2 \mathfrak{m})^{\mathfrak{h}}$  such that  $[\pi, \pi]_{\mathfrak{m}} = 0$ . Then there are bijective correspondences between

- (1) the set of  $\hbar$ -dependant and G-invariant Poisson structures  $\pi_{\hbar} = \hbar \pi \mod \hbar^2$ on G/H, modulo the action of  $G_0$ ,
- (2) the set of  $\hbar$ -dependant formal dynamical r-matrices  $\rho_{\hbar}(\lambda)$  such that  $\rho_{\hbar}(0) = \pi \mod \hbar \ in \wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$ , modulo the action (14) of  $\widetilde{G}_1$ ,
- (3) the set of formal dynamical twists  $J(\lambda)$  satisfying  $Alt \frac{J(0)-1}{\hbar} = \pi \mod \hbar$  in  $\wedge^2(\mathfrak{g}/\mathfrak{h})[[\hbar]]$ , modulo gauge equivalence (9).

#### 4. Another case when the twist quantization exists

In this section we assume that  $\mathfrak{h}$  is abelian and admits a Lie subalgebra  $\mathfrak{m}$  as complement.

Note that since  $\mathfrak{h}$  is abelian and  $\mathfrak{m}$  a Lie subalgebra, the projection  $p: \mathfrak{g} \to \mathfrak{g}$  on  $\mathfrak{m}$  along  $\mathfrak{h}$  extends to a morphsim of graded Lie algebras  $\wedge p: (\wedge \mathfrak{g})^{\mathfrak{h}} \to (\wedge \mathfrak{g})^{\mathfrak{h}}$  at the level of  $\mathfrak{h}$ -invariants. And thus  $\wedge p \otimes \varepsilon : (\mathfrak{g}_1[1], d, [,]) \to ((\wedge \mathfrak{g})^{\mathfrak{h}}[1], 0, [,])$  is a morphism of dgla's. Then the natural inclusion  $\mathrm{id} \otimes 1 : (T^*U\mathfrak{g})^{\mathfrak{h}} \to \mathfrak{g}_2$  obviously allows one to consider  $(T^*U\mathfrak{g})^{\mathfrak{h}}[1]$  as a sub-dgla of  $\mathfrak{g}_2[1]$ . Finally recall from [Ca, section 3.3] that there exists a  $L_{\infty}$ -quasi-isomorphism  $\mathcal{F}: C^c((\wedge^*\mathfrak{g})^{\mathfrak{h}}[2]) \to C^c((T^*U\mathfrak{g})^{\mathfrak{h}}[2])$  with  $\mathcal{F}^1$  = alt. By composing these maps one obtains a  $L_{\infty}$ -morphism

$$\mathcal{F}: (\mathcal{G}_1, \mathrm{d} + [,]) \to (\mathcal{G}_2, b + [,]_G)$$

with values in  $(\mathcal{G}_2)_{\leq 0}$  and first structure map  $\widetilde{\mathcal{F}}^1 = (\operatorname{alt} \otimes 1) \circ (\wedge p \otimes \varepsilon)$ .

**Theorem 4.1.** There exists a  $L_{\infty}$ -quasi-iomorphism

 $\Psi: (\mathcal{G}_1, d+[,]) \to (\mathcal{G}_2, b+[,]_G)$ 

with properties (F1) and (F2) of Theorem 1.7.

*Proof.* First observe that since  $\mathfrak{h}$  is abelian then  $\mathfrak{g}_1 \cong ((\wedge \mathfrak{g})^{\mathfrak{h}} \cap \wedge \mathfrak{m}) \otimes \wedge \mathfrak{h} \otimes S\mathfrak{h}$  as a vector space. Thus if  $\delta_K$  is as in the proof of proposition 2.1 then  $\delta := \mathrm{id} \otimes \delta_K$  is a homotopy operator:  $\delta d + d\delta = \mathrm{id} - \wedge p \otimes \varepsilon$  and  $\delta \circ \delta = 0$ .

Now we proceed like in subsection 2.3: use lemma A.3 to construct a  $L_{\infty}$ morphism  $\Psi$  with first structure map  $\Psi^1 = \widetilde{\mathcal{F}}^1 + b \circ V + V \circ d$ , where V :=  $(alt \otimes sym) \circ \delta : \mathfrak{g}_1 \to \mathfrak{g}_2[-1].$ 

It remains to prove that  $\Psi$  is a quasi-isomorphism. It follows from the first observation in this proof that  $H^*(\mathfrak{g}_1, d) = (\wedge \mathfrak{g})^{\mathfrak{h}} \cap \wedge \mathfrak{m}$ , which also equals  $H^*(\mathfrak{g}_2, b)$  due to the spectral sequence argument. Consequently  $\widetilde{\mathcal{F}}^1$  is a quasi-isomorphism of complexes, and so is  $\Psi^1$ .

Finally using the same argumentation as in the proof of theorem 0.2 (subsection 1.3) one obtains the

**Theorem 4.2.** If  $\mathfrak{h}$  is an abelian subalgebra of  $\mathfrak{g}$  with a Lie subalgebra as a complement, then any formal classical dynamical r-matrix for  $(\mathfrak{g}, \mathfrak{h}, 0)$  admits a dynamical twist quantization (associated to the trivial associator).

**Example 4.3.** In particular, this allows us to quantize dynamical *r*-matrices arizing from semi-direct products  $\mathfrak{g} = \mathfrak{m} \ltimes \mathbb{C}^n$  like in [EN, example 3.7].

### CONCLUDING REMARKS

Let us first observe that if  $\mathfrak{h}$  is abelian then  $(\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1]$  (resp.  $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\mathrm{sym}(S\mathfrak{m})[1]$ ) inherits a dgla structure from the one of  $\mathfrak{g}_1[1]$  (resp.  $\mathfrak{g}_2[1]$ ) and  $H^*(\mathfrak{g}_1, \mathrm{d}) = (\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m} = H^*(\mathfrak{g}_2, b)$ , for any complement  $\mathfrak{m}$  of  $\mathfrak{h}$ . Thus I conjecture that there exists a  $L_{\infty}$ -quasi-isomorphism between  $(\wedge^*\mathfrak{g})^{\mathfrak{h}} \cap \wedge^*\mathfrak{m}[1]$  and  $(T^*U\mathfrak{g})^{\mathfrak{h}} \cap T^*\mathrm{sym}(S\mathfrak{m})[1]$  which generalizes together  $\phi$  of subsection 2.3 and  $\mathcal{F}$  of section 4. In particular this would imply conjecture 0.1 in the abelian (and non-modified) case.

Let us then mention that one can consider a non-triangular (i.e., non-antisymmetric) version of non-modified classical dynamical *r*-matrices. Namely, **h**-equivariant maps

 $r \in \operatorname{Reg}(\mathfrak{h}^*, \mathfrak{g} \otimes \mathfrak{g})$  such that  $\operatorname{CYB}(r) - \operatorname{Alt}(dr) = 0$ . According to [X3], a quantization of such a r is a  $\mathfrak{h}$ -equivariant map  $R = 1 + \hbar r + O(\hbar^2) \in \operatorname{Reg}(\mathfrak{h}^*, U\mathfrak{g}^{\otimes 2})[[\hbar]]$  that satisfies the quantum dynamical Yang-Baxter equation (QDYBE)

(16) 
$$R^{1,2}(\lambda) * R^{1,3}(\lambda + \hbar h^2) * R^{2,3}(\lambda) = R^{2,3}(\lambda + \hbar h^1) * R^{1,3}(\lambda) * R^{1,2}(\lambda + \hbar h^3)$$

Question 4.4. Does such a quantization always exist?

The most famous example of non-triangular dynamical *r*-matrices was found in [AM] by Alekseev and Meinrenken, then extended successively to a more general context in [EV, ES, EE1], and quantized in [EE1].

Following [EE1], remark that for any non-triangular dynamical *r*-matrix *r* such that  $r + r^{\text{op}} = t \in (S^2 \mathfrak{g})^{\mathfrak{g}}$  (quasi-triangular case) one can define  $\rho := r - t/2$  and  $Z := \frac{1}{4}[t^{1,2}, t^{2,3}]$ . Then  $\rho$  is a modified dynamical *r*-matrix for  $(\mathfrak{g}, \mathfrak{h}, Z)$ ; morever the assignment  $r \mapsto \rho$  is a bijective correspondence between quasi-triangular dynamical *r*-matrices for  $(\mathfrak{g}, \mathfrak{h}, t)$  and modified dynamical *r*-matrices for  $(\mathfrak{g}, \mathfrak{h}, Z)$ . Now observe that if  $J(\lambda)$  is a dynamical twist quantizing  $\rho$ , then  $R(\lambda) = J^{\text{op}}(\lambda)^{-1} * e^{\hbar t/2} * J(\lambda)$  is a quantum dynamical *R*-matrix quantizing *r*.

In this paper we have constructed such a dynamical twist in the triangular case t = 0. One can ask

**Question 4.5.** Does such a dynamical twist exist for any quasi-triangular dynamical r-matrix? At least in the reductive and abelian cases?

This question seems to be more reasonable than the previous one. More generally one can ask if conjecture 0.1 (and its smooth and meromorphic versions) is true in general. A positive answer was given in [EE1] when  $\mathfrak{h} = \mathfrak{g}$ ; but unfortunately it is not known in general, even for the non-dynamical case  $\mathfrak{h} = \{0\}$  (which is the last problem of Drinfeld [Dr1]: quantization of coboundary Lie bial-gebras).

Finally let me mention that if  $r(\lambda)$  is a triangular dynamical *r*-matrix for  $(\mathfrak{g}, \mathfrak{h})$ , then the bivector field

$$\pi := \overrightarrow{r(\lambda)} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} + \pi_{\mathfrak{h}^*}$$

is a  $G \times H$ -biinvariant Poisson structure on  $G \times \mathfrak{h}^*$  and the projection  $p : G \times \mathfrak{h}^* \to \mathfrak{h}^*$ is a momentum map. Moreover, according to [X3] any dynamical twist quantization  $J(\lambda)$  of  $r(\lambda)$  allows us to define a  $G \times H$ -biinvariant star-product \* quantizing  $\pi$  on  $G \times \mathfrak{h}^*$  as follows:

$$\begin{array}{ll} f*g = f*_{PBW}g & \text{if} & f,g \in C^{\infty}(\mathfrak{h}^{*}) \\ f*g = fg & \text{if} & f \in C^{\infty}(G), g \in C^{\infty}(\mathfrak{h}^{*}) \\ f*g = \exp\left(\hbar \sum_{i} \frac{\partial}{\partial \lambda^{i}} \otimes \overrightarrow{h_{i}}\right) \cdot (f \otimes g) & \text{if} & f \in C^{\infty}(\mathfrak{h}^{*}), g \in C^{\infty}(G) \\ f*g = \overrightarrow{J(\lambda)}(f \otimes g) & \text{if} & f,g \in C^{\infty}(G) \end{array}$$

This way the map  $p^* : (\operatorname{Fct}(\mathfrak{h}^*)[[\hbar]], *_{PBW}) \to (\operatorname{Fct}(G \times \mathfrak{h}^*)[[\hbar]], *)$  becomes a quantum momentum map in the sens of [X1].

So there may be a way to see momentum maps and their quantum analogues as Maurer-Cartan elements in dgla's.

## APPENDIX A. HOMOTOPY LIE ALGEBRAS

See [HS] for a detailed discussion of the theory.

Recall that a  $L_{\infty}$ -algebra structure on a graded vector space  $\mathfrak{g}$  is a degree 1 coderivation Q on the cofree cocommutative coalgebra  $C^{c}(\mathfrak{g}[1])$  such that  $Q \circ Q = 0$ . By cofreeness, such a coderivation Q is uniquely determined by structure maps  $Q^{n}: \Lambda^{n}\mathfrak{g} \to \mathfrak{g}[2-n]$  which satisfy an infinite collection of equations. In particular  $(\mathfrak{g}, Q^{1})$  is a cochain complex.

**Example A.1.** Any dgla  $(\mathfrak{g}, d, [,])$  is canonically a  $L_{\infty}$ -algebra. Namely, Q is given by structure maps  $Q^1 = d$ ,  $Q^2 = [,]$  and  $Q^n = 0$  for n > 2.

A  $L_{\infty}$ -morphism between two  $L_{\infty}$ -algebras  $(\mathfrak{g}_1, Q_1)$  and  $(\mathfrak{g}_2, Q_2)$  is a degree 0 morphism of coalgebras  $F : C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$  such that  $F \circ Q_1 = Q_2 \circ F$ . Again by cofreeness, such a morphism is uniquely determined by structure maps  $F^n : \Lambda^n \mathfrak{g}_1 \to \mathfrak{g}_2[1-n]$  which satisfy an infinite collection of equations. In particular  $F^1 : \mathfrak{g}_1 \to \mathfrak{g}_2$  is a morphism of complexes; when it induces an isomorphism in cohomology we say that F is a  $L_{\infty}$ -quasi-isomorphism.

**Example A.2.** Any morphism of dgla's is a  $L_{\infty}$ -morphism with all structure maps equal to zero except the first one.

In this paper we use many times the following

**Lemma A.3** ([Do]). Let  $F : C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$  be a  $L_\infty$ -morphism. For any linear map  $V : \mathfrak{g}_1 \to \mathfrak{g}_2[-1]$  there exists a  $L_\infty$ -morphism  $\Psi : C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$ with first structure map  $\Psi^1 = F^1 + Q_2^1 \circ V + V \circ Q_1^1$ . Moreover, if F is a  $L_\infty$ -quasiisomorphism then  $\Psi$  is also.

*Proof.* First remark that V extends uniquely to a linear map  $C^c(\mathfrak{g}_1[1]) \to C^c(\mathfrak{g}_2[1])$  of degree -1 such that

$$\Delta_2 \circ V = \left(F \otimes V + V \otimes F + \frac{1}{2}V \otimes (Q_2 \circ V + V \circ Q_1) + \frac{1}{2}(Q_2 \circ V + V \circ Q_1) \otimes V\right) \circ \Delta_1$$

where  $\Delta_1$  and  $\Delta_2$  denote comultiplications in  $C^c(\mathfrak{g}_1[1])$  and  $C^c(\mathfrak{g}_2[1])$ , respectively. Then define  $\Psi := F + Q_2 \circ V + V \circ Q_1$ .

**Remark A.4.** Assume that in the previous lemma  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are filtrated, F is such that  $F^n$  takes values in  $(\mathfrak{g}_2)_{\leq n-1}$ , and  $V((\mathfrak{g}_1)_{\leq k}) \subset (\mathfrak{g}_2)_{\leq k+1}$ . Then one can obviously check that for any  $X \in (\Lambda^n \mathfrak{g}_1)_{\leq k}$ ,  $F^n(X) \in (\mathfrak{g}_2)_{\leq n+k-1}$ .

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