

RIGIDITY OF INVARIANT CONVEX SETS IN SYMMETRIC SPACES

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ABSTRACT. The main result implies that a proper convex subset of an irreducible higher rank symmetric space cannot have Zariski dense stabilizer.

1. INTRODUCTION

In this paper we study convex subsets of symmetric spaces, and their stabilizers. The main results show that in the higher rank case convex sets are strongly restricted, and under mild assumptions can only arise from rank 1 constructions. This rigidity phenomenon for convex subsets is yet another example of a rigidity property enjoyed by higher rank symmetric spaces that has no analog for rank 1 symmetric spaces.

One can generate a supply of convex subsets of any Hadamard space by starting with geodesic segments, geodesic rays, complete geodesics, and horoballs, and then taking tubular neighborhoods and intersections. When X is a Hadamard manifold with pinched negative curvature convex subsets are abundant: by a theorem of Anderson [And83], any closed subset A of the geometric boundary $\partial_\infty X$ is the limit set of a closed convex subset $Y \subset X$. On the other hand, for general Hadamard spaces (or manifolds) it can be difficult to control the convex hull of even “small” subsets, like the union of three rays.

A group Γ of isometries of a Hadamard space X is *convex cocompact* if there is a Γ -invariant convex subset $C \subset X$ with compact quotient C/Γ . Discrete convex cocompact subgroups of the isometry group of hyperbolic 3-space are an important class in the theory of Kleinian groups; basic examples are uniform lattices, Schottky groups and quasi-Fuchsian groups. Analogous examples exist in $\text{Isom}(H^n)$, as well as the isometry groups of other rank 1 symmetric spaces. In a higher rank symmetric space of noncompact type, one can produce examples by taking products of uniform lattices and rank 1 convex cocompact groups. In 1994, Corlette asked if this was essentially the only way

Date: October 26, 2018.

Supported by NSF grant DMS-0204506.

to produce discrete convex cocompact groups. The answer is yes, see Theorem 1.3 below; in fact the theorem is proved by reducing it to the case of convex subsets with Zariski dense stabilizer:

Theorem 1.1. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type, and let $X = \mathbb{E}^n \times Y_1 \times Y_{\geq 2}$ denote the decomposition of X into the Euclidean factor, the product of the irreducible rank 1 factors, the product of the higher rank factors. Suppose $\Gamma \subset \text{Isom}(X) = \text{Isom}(\mathbb{E}^n) \times \text{Isom}(Y)$ is a subgroup whose projection to $\text{Isom}(Y)$ is Zariski dense in the identity component $\text{Isom}_o(Y)$, and whose projection to $\text{Isom}(\mathbb{E}^n)$ does not preserve a proper affine subspace of \mathbb{E}^n . If $C \subset X := \mathbb{E}^n \times Y$ is a Γ -invariant closed convex set, then $C = \mathbb{E}^n \times C_1 \times Y_{\geq 2}$, where $C_1 \subset Y_1$ is a closed convex subset. Furthermore, for each de Rham factor X_i of Y_1 , there is a Γ -invariant subset $\hat{C}_i \subset X_i$ such that*

- \hat{C}_i is the closed convex hull of its limit set.
- $|\partial_\infty C_i| = \infty$,
- $\hat{C}_1 := \prod_i \hat{C}_i \subset C_1$.
- $\partial_\infty \hat{C}_1 = \partial_\infty C_1$.

We recall that by convention, a symmetric space of noncompact type has no Euclidean de Rham factor. Note that a subgroup of $\text{Isom}_o(Y)$ is Zariski dense if and only if it neither fixes a point in the Tits boundary $\partial_T Y$ nor preserves a proper symmetric subspace of Y .

Corollary 1.2. *If X is a symmetric space of noncompact type with no rank 1 de Rham factors and $\Gamma \subset \text{Isom}_o(X)$ is a Zariski dense subgroup, then X contains no proper closed Γ -invariant convex subsets.*

For discrete convex cocompact groups, we have the following structural result:

Theorem 1.3. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type. Suppose $\Gamma \subset \text{Isom}(X) = \text{Isom}(\mathbb{E}^n) \times \text{Isom}(Y)$ is a discrete subgroup acting cocompactly on a closed convex subset $C \subset X$, and assume Γ does not preserve any proper symmetric of X . Then Γ projects to a subgroup of $\text{Isom}(Y)$ which is Zariski dense in $\text{Isom}_o(Y)$, and the conclusions of Theorem 1.1 apply to C .*

If a convex cocompact subgroup $\Gamma \subset \text{Isom}(X)$ preserves a proper symmetric subspace $Z \subset X$, then it acts convex cocompactly on Z – just intersect a sufficiently big tubular neighborhood of a Γ -invariant convex set with Z . Therefore there is no loss of generality in assuming X contains no proper Γ -invariant symmetric subspace.

Corollary 1.4. *If X is a symmetric space of noncompact type with no rank 1 de Rham factors, and $\Gamma \subset \text{Isom}(X)$ is a discrete subgroup acting cocompactly on a closed convex subset $C \subset X$, then either $C = X$ and Γ is a uniform lattice in $\text{Isom}(X)$, or Γ preserves a proper symmetric subspace of X .*

We give a brief outline of the proof of Theorem 1.1 in the case the Euclidean factor is absent, and Y is an irreducible higher rank symmetric space. The first step is to apply a Theorem of Benoist [Ben97], which implies one may find an open neighborhood U of a pair of antipodal points ξ_1, ξ_2 in the Tits boundary $\partial_T X$, such that U is contained in the limit set of Γ . Applying a result from [KL97], we deduce that the geometric boundary of C is a top dimensional subbuilding B of the Tits boundary of X , which is a closed subset with respect to the topology of the geometric boundary $\partial_\infty X$. The main step in the paper, implemented in Theorem 3.1, is to show that any such building is contained in the geometric boundary of a proper symmetric subspace Y , unless it coincides with $\partial_T X$; the Zariski density assumption rules out the former possibility in the case at hand. We remark that Theorem 3.1 applies to products of symmetric spaces and Euclidean buildings, and may be of independent interest.

In view of the results in this paper one may wonder whether sufficiently large convex sets in symmetric spaces of noncompact type or in spherical buildings (such as Tits boundaries of symmetric spaces) are rigid.

Question 1.5. Suppose $C \subset B$ is a convex subset of a spherical building. If C does not have circumradius $\leq \frac{\pi}{2}$, must C itself be a spherical building?

It is unclear what one should expect here. A. Balsler and A. Lytchak [BL04] proved a partial result regarding convex subsets invariant under a group action, namely if $\dim(C) \leq 2$ and C is not a spherical building then $\text{Isom}(C)$ has a fixed point in C .

After the first version of this paper was written, Quint informed the authors of very interesting related work [Qui04] on Zariski dense subgroups of semi-simple groups. His paper addresses an alternate definition of convex cocompact groups which is equivalent to the usual definition for rank 1 symmetric spaces but differs from ours in the higher rank case; for this reason it is difficult to make a direct comparison between the results of [Qui04] and the theorems above. We mention that his main result also applies to discrete subgroups of semi-simple p -adic groups.

The authors proved slightly weaker versions of Theorems 1.1 and 1.3 in 1998, and spoke publicly about them in the subsequent year.

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2. PRELIMINARIES

2.1. Hadamard spaces. We recommend [Bal95, BH99, KL97] as references for Hadamard space facts.

The term *Hadamard space* is a synonym for a $CAT(0)$ -space.

If X is a Hadamard space, we denote the geometric boundary by $\partial_\infty X$, the Tits boundary by $\partial_T X$, and the Tits angle between $\xi_1, \xi_2 \in \partial_T X$ by $\angle_T(\xi_1, \xi_2)$.

Recall that the set underlying $\partial_\infty X$ is the set of asymptote classes of geodesic rays, and that this may be identified with the set of rays leaving a given basepoint $p \in X$. If $x_1, x_2 \in X$, $Y \subset X$ is a subset, $y_i \in Y$ is a sequence with $\lim_{i \rightarrow \infty} d(y_i, p) = \infty$, then the segments $\overline{x_1 y_i}$ converge to a ray $\overline{x_1 \xi}$ iff the segments $\overline{x_2 y_i}$ converge to a ray $\overline{x_2 \xi}$. Thus the set of rays which can be obtained as limits in this fashion, as $\{y_i\}$ ranges over all such sequences, is a collection of asymptote classes and therefore determines a subset of $\partial_\infty X$, the *limit set of Y* , which we denote by $\Lambda(Y)$.

Lemma 2.1. *If $C \subset X$ is a closed convex subset, and $p \in C$, then every ray \overline{px} is contained in C , for $\xi \in \Lambda(C)$.*

Proof. This follows from the convexity of C and the definition of the limit set, since we are at liberty to select the basepoint. \square

Definition 2.2. A subset Y of a $CAT(1)$ space Z is *convex* if it contains every segment of the form $\overline{\xi_1 \xi_2}$, where $\xi_1, \xi_2 \in Y$ and $d_Z(\xi_1, \xi_2) < \pi$.

Lemma 2.3. *Let X be a proper Hadamard space, and let $C \subset X$ be a closed convex subset. Then the limit set of C in $\partial_\infty X$ determines a*

convex subset of $\partial_T X$, which is isometric to the Tits boundary of C , viewed as a Hadamard space.

Proof. The isometric embedding $C \rightarrow X$ of Hadamard spaces induces an isometric embedding $\partial_T C \rightarrow \partial_T X$ of Tits boundaries. Since $\partial_T C$ is a $CAT(1)$ space, the image of the embedding is convex. \square

Lemma 2.4. *If $\Gamma \curvearrowright X$ is a discrete, cocompact, isometric action on a Hadamard space X , and Γ fixes a point $\xi \in \partial_T X$, then there is a geodesic $\gamma \subset X$ such that $\xi \in \partial_T \gamma$ and the parallel set $\mathbb{P}(\gamma) \subset X$ is Γ -invariant.*

Proof. We may assume that X contains no proper, closed, convex, Γ -invariant nonempty subset, by applying Zorn's lemma.

Note that any element $g \in Z(\Gamma)$ is semi-simple and its minimum displacement set, $\min(g) \subset X$, is a closed, convex, and Γ -invariant subset; therefore by assumption we have $\min(g) = X$. Thus elliptic elements in $Z(\Gamma)$ act trivially on X and nonelliptic elements act by Clifford translations, i.e. they translate along the \mathbb{R} -factor of a product splitting $X = \mathbb{R} \times Z$. Hence X admits a product structure

$$(2.5) \quad X = \mathbb{E}^n \times Y$$

where $Z(\Gamma)$ acts by translations on \mathbb{E}^n and trivially on Y .

Pick $p \in X$, and a finite generating set $\Sigma \subset \Gamma$. Let $C := \max_{\sigma \in \Sigma} d(\sigma p, p)$. Note that the ray $\overline{p\xi} \subset X$ lies in the closed convex set

$$\Delta := \{x \in X \mid \text{For all } \sigma \in \Sigma, d(\sigma x, x) \leq C\},$$

since for all $g \in \Gamma$ and every $x \in \overline{p\xi}$, we have $d(gx, x) \leq d(gp, p)$ because $\overline{p\xi}$ and $\overline{(gp)\xi}$ are asymptotic rays. By a standard argument the centralizer, $Z(\Sigma) = Z(G)$, of the set Σ acts cocompactly on Δ , which implies that $\overline{p\xi}$ is contained in a finite tubular neighborhood of an n -flat $\mathbb{E}^n \times \{y\}$ of the product decomposition (2.5). Hence $\xi \in \partial_T \mathbb{E}^n$, and this implies the lemma. \square

2.2. Affine and concave functions on convex sets.

Lemma 2.6. *Let Z be a geodesic metric space with extendible geodesics. Then any concave function $Z \rightarrow [0, \infty)$ is constant.*

Proof. Trivial. \square

Lemma 2.7. *Let Z be a $CAT(-1)$ -space whose ideal boundary $\partial_\infty Z$ consists of at least two points. Suppose that there is no proper closed convex subset of Z whose ideal boundary equals $\partial_\infty Z$. Then any continuous concave function $f : Z \rightarrow [0, \infty)$ is constant.*

Proof. We first observe that f is constant along each complete geodesic. Furthermore, f is non-decreasing along each geodesic ray, and the restriction of f to a compact geodesic segment assumes its minimum at one of the endpoints.

Note that, by assumption, Z contains at least one complete geodesic. Let l be a complete geodesic and $z \in Z$ be an arbitrary point. Denote by $\rho_1, \rho_2 : [0, \infty) \rightarrow Z$ the rays emanating from z and asymptotic to the two ends of l . Then f is $\geq f(z)$ along each segment connecting $\rho_1(t)$ to $\rho_2(t)$ for $t \geq 0$. Since Z is CAT(-1) these segments converge to the line l . The continuity of f then implies that $f(l) \geq f(z)$. Thus f assumes on l its maximum which we denote by m .

It follows that f equals m on the union H_1 of all lines in Z . Consider the ascending sequence of subsets $H_n \subset Z$ defined inductively by requiring that H_{n+1} is the union of all segments with endpoints in H_n . Then the sequence of suprema $\sup(f|_{H_n})$ is non-decreasing. Hence $m = \sup(f|_{H_1}) \leq \sup(f|_{H_n}) \leq m$ and $f \equiv m$ on the closure of $\cup_{n \in \mathbb{N}} H_n$. This closure is a closed convex subset of Z with the same ideal boundary and, by assumption, equals Z . \square

By an *affine* function on a geodesic metric space we mean a function whose restriction to each geodesic segment is an affine function.

Lemma 2.8. *Let Z be a CAT(-1)-space whose ideal boundary $\partial_\infty Z$ consists of at least three points. Then any affine continuous function $f : Z \rightarrow \mathbb{R}$ is constant.*

Proof. We first observe that the slope of f along a geodesic ray depends only on the ideal point represented by it. Indeed, let $\rho_1, \rho_2 : [0, \infty) \rightarrow Z$ be two rays parametrized by unit speed. Since the geodesic segments connecting $\rho_1(0)$ with $\rho_2(t)$ converge to the ray ρ_1 it follows using continuity that the slope of f along ρ_1 equals its slope along ρ_2 .

Since any two ideal points in $\partial_\infty Z$ may be connected by a complete geodesic in Z it follows that the slopes of f at any two ideal points have opposite sign. Since $\partial_\infty Z$ contains at least three points the slopes of f must be zero at all ideal points, i.e. f is constant along every geodesic ray.

The same reasoning as in the proof of Lemma 2.7 above shows that for any point z and any complete geodesic l in Z we have $f(z) = f(l)$. Thus f is constant. \square

Lemma 2.9. *Let Z be a symmetric space of noncompact type and higher rank without Euclidean de Rham factor. Then any affine continuous function $f : Z \rightarrow \mathbb{R}$ is constant.*

Proof. We may apply Lemma 2.8 to (nonflat) totally geodesic subspaces of rank one and get that f is constant on any such subspace.

Let F be a maximal flat. Then $f|_F$ is affine. The previous remark implies that the gradient of $f|_F$ at a point $z \in F$ must be tangent to every singular hyperplane H through z because the lines in F perpendicular to H lie in a rank one subspace. Since Z has no Euclidean factor the intersection of all these hyperplanes H is just the point z . We conclude that f is constant along every maximal flat; since any two points lie in a maximal flat, this implies that f is constant on Z . \square

2.3. Asymptotic slopes of convex functions. Let Z be a Hadamard space and $f : Z \rightarrow \mathbb{R}$ a continuous convex function. For a unit speed geodesic ray $\rho : [0, \infty) \rightarrow Z$ we define the *asymptotic slope* of f along ρ as $\text{slope}_f(\rho) := \lim_{t \rightarrow \infty} \frac{f(\rho(t))}{t} \in \mathbb{R} \cup \{\infty\}$.

Lemma 2.10. *For any two asymptotic unit speed rays ρ_1 and ρ_2 , $\text{slope}_f(\rho_1) = \text{slope}_f(\rho_2)$.*

Proof. Since the segments connecting $\rho_2(0)$ with $\rho_1(t)$ Hausdorff converge to ρ_2 one estimates using the continuity of f that $f(\rho_2(t)) \leq C + \text{slope}_f(\rho_1) \cdot t$ for $t \geq 0$ and hence $\text{slope}_f(\rho_2) \leq \text{slope}_f(\rho_1)$. Symmetry implies equality. \square

Thus we may speak of the asymptotic slope, $\text{slope}_f(\xi)$, at an ideal point $\xi \in \partial_\infty Z$.

Lemma 2.11. *$\text{slope}_f : \partial_\infty Z \rightarrow \mathbb{R} \cup \{\infty\}$ is lower semicontinuous with respect to the cone topology.*

Proof. Consider a sequence of unit speed rays ρ_n with same initial point which Hausdorff converges to the ray ρ . Since $\text{slope}_f(\rho_n) \geq \frac{f(\rho_n(t)) - f(\rho_n(0))}{t}$ for $t \geq 0$ we obtain

$$\liminf_{n \rightarrow \infty} \text{slope}_f(\rho_n) \geq \frac{f(\rho(t)) - f(\rho(0))}{t} \xrightarrow{t \nearrow \infty} \text{slope}_f(\rho).$$

\square

As a consequence, slope_f attains a minimum if Z is locally compact.

Proposition 2.12. *If $\text{slope}_f : \partial_\infty Z \rightarrow \mathbb{R} \cup \{\infty\}$ assumes negative values then it has a unique minimum.*

Proof. Let $\xi_1, \xi_2 \in \partial_\infty Z$ be ideal points with $\text{slope}_f(\xi_i) \leq -a < 0$ and $\angle_T(\xi_1, \xi_2) \geq \epsilon > 0$. Let ρ_i be unit speed rays emanating from the same point $o \in Z$ and asymptotic to the ideal points ξ_i . For the midpoints $m(t)$ of the segments $\overline{\rho_1(t)\rho_2(t)}$ holds

$$\limsup_{t \rightarrow \infty} \frac{d(o, m(t))}{t} \leq \cos \frac{\angle_T(\xi_1, \xi_2)}{2} \leq \cos \frac{\epsilon}{2}.$$

Moreover $f(m(t)) \leq \text{const} - at$. The segments $\overline{om(t)}$ Hausdorff converge to a ray μ which therefore satisfies $\text{slope}_f(\mu) \leq -a(\cos \frac{\epsilon}{2})^{-1}$.

It follows that any sequence (ξ_n) in $\partial_\infty Z$ with $\text{slope}_f(\xi_n) \searrow \inf \text{slope}_f$ is a Cauchy sequence with respect to the Tits metric. Hence slope_f has a unique minimum on $\partial_\infty Z$. \square

2.4. Spherical buildings. We refer the reader to [KL97, Ron89, Tit74] for further discussion of the material here.

We will be using the geometric definition of spherical buildings from [KL97], which we now recall.

Let (S, W) be a spherical Coxeter complex, so S is a Euclidean sphere and W is a finite group generated by reflections acting on S . A *spherical building modelled on (S, W)* is a $CAT(1)$ -space B together with a collection \mathcal{A} of isometric embeddings $\iota : S \rightarrow B$, called *charts*, which satisfies properties SB1-2 described below and which is closed under precomposition with isometries in W . An *apartment* in B is the image of a chart $\iota : S \rightarrow B$; ι is a chart of the apartment $\iota(S)$.

SB1: Plenty of apartments. Any two points in B are contained in a common apartment.

Let ι_{A_1}, ι_{A_2} be charts for apartments A_1, A_2 , and let $C = A_1 \cap A_2$, $C' = \iota_{A_2}^{-1}(C) \subset S$. The charts ι_{A_i} are *W-compatible* if $\iota_{A_1}^{-1} \circ \iota_{A_2}|_{C'}$ is the restriction of an isometry in W .

SB2: Compatible apartments. The charts are W -compatible.

2.5. Root groups. If B is spherical building, and $a \subset B$ is a root, then the *root group of a* is the collection U_a of building automorphisms of B which fix a pointwise, as well as any chamber $\sigma \subset B$ such that $\sigma \cap a$ is a panel π which is not contained in the wall ∂a . The building B is *Moufang* if for every root $a \subset B$, the group U_a acts transitively on the set of roots opposite a .

Properties of root groups:

- When all the join factors of B have dimension at least 1, then U_a acts freely on the collection of roots opposite a .

- When X is a symmetric space of noncompact type and $B := \partial_T X$, then B is a Moufang building and $G := \text{Isom}_o(X)$ acts effectively on B by building automorphisms, so we may view G as a subgroup of $\text{Aut}(B)$. Each root group of B is contained in G , and is a unipotent subgroup [Tit74, pp. 77-78]. Furthermore, G is generated by the root groups of B .

2.6. Groups acting on symmetric spaces. Let X be a symmetric space of noncompact type, and let $G := \text{Isom}_o(X)$. We will require the following well known facts [Mos55, BT71]:

- A subgroup $H \subset G$ is Zariski dense if and only if H neither fixes a point in $\partial_T X$ nor preserves a proper symmetric subspace.
- A proper subgroup $H \subsetneq G$ with finitely many connected components is not Zariski dense; in particular H must either fix a point in $\partial_T X$ or preserve a proper symmetric subspace.

Remark 2.13. If a Zariski dense subgroup of a real simple group is not dense in the usual topology, then it must be discrete.

3. TOP DIMENSIONAL SUBBUILDINGS IN THE BOUNDARY OF A SYMMETRIC SPACE

In this section we prove:

Theorem 3.1. *Suppose*

$$(3.2) \quad X = X_1 \times \dots \times X_k$$

is a product of irreducible symmetric spaces of noncompact type, irreducible Euclidean buildings with discrete affine Weyl groups, and Euclidean spaces. Let $B \subset \partial_T X$ be a top dimensional subbuilding which is closed with respect to the topology of the geometric boundary $\partial_\infty X$, and which is not contained in the boundary of any proper subspace $Y \subset X$ of the form $Y = Y_1 \times \dots \times Y_k$, where $Y_i \subset X_i$ is either a totally geodesic subspace or a subbuilding, according to the type of X_i . Then there is a join decomposition

$$B = B_1 \circ \dots \circ B_k$$

where $B_i := B \cap \partial_T X_i$, such that $B_i = \partial_T X_i$ unless X_i is an irreducible rank 1 symmetric space of noncompact type.

We begin the proof by observing that if there is more than one factor in the product decomposition (3.2), then by [KL97, Prop. 3.3.1], B and $\partial_T X$ will admit corresponding compatible join decompositions

$$B = B_1 \circ \dots \circ B_k, \quad \partial_T X = \partial_T X_1 \circ \dots \circ \partial_T X_k,$$

and hence it is sufficient to prove the theorem for the irreducible factors X_i separately. So henceforth we will assume that X is irreducible. If X is Euclidean, then $\partial_T X$ is the only top dimensional subbuilding of $\partial_T X$, and so this case is trivial.

3.1. The case when X is a Euclidean building. Let $Y \subset X$ be the union of the collection of apartments $A \subset X$ such that $\partial_T A \subset B$.

Lemma 3.3. *Any two chambers $\sigma_1, \sigma_2 \subset Y$ lie in an apartment $A \subset X$ which is entirely contained in Y .*

Proof. By the definition of Y , for $i = 1, 2$ there exists an apartment $A_i \subset X$ such that $\partial_T A_i \subset B$ and $\sigma_i \subset A_i$. For $i = 1, 2$, choose an interior point $p_i \in \sigma_i$, and consider the geodesic segment $\overline{p_1 p_2} \subset X$. By perturbing p_2 slightly, if necessary, we may assume that the Δ_{mod} -direction of $\overline{p_1 p_2}$ is regular. We may prolong $\overline{p_1 p_2}$ to a complete regular geodesic $\gamma \subset X$ by concatenating it with rays $\overline{p_1 \xi_1} \subset A_1$, $\overline{p_2 \xi_2} \subset A_2$. Since $\partial_T \gamma = \{\alpha_1, \alpha_2\}$ where $\alpha_i \subset \partial_T A_i \subset B$ are regular, there is a unique apartment $\partial_T A \subset \partial_T X$ containing $\partial_\infty \gamma$, and it is contained in B . Then by the definition of Y we have $A \subset Y$, and since γ is regular and $\partial_T \gamma \subset \partial_T A$, we get $\gamma \subset A$. This implies that $\sigma_i \subset A$, since $\sigma_i \cap A \supset \{p_i\} \neq \emptyset$ is a subcomplex of X . \square

The lemma implies that Y is a subbuilding of X with Tits boundary B . By assumption we must therefore have $B = \partial_T X$, which proves Theorem 3.1 in this case.

3.2. X is an irreducible symmetric space of noncompact type.

We will assume that X has rank at least two, since otherwise there is nothing to prove. The strategy of the proof is to use B to produce a subgroup $H \subset G$ which has no fixed point in $\partial_T X$, which can be used to tie B closely with X . When B is irreducible, H is generated using “restricted” root groups, and when B is reducible H is generated by transvections, and decomposes as a product.

We let W denote the Weyl group of X . Thus $\partial_T X$ is a spherical building modelled on a spherical Coxeter complex (S, W) . We let $W_B \subset W$ denote the sub-Coxeter group defining a *thick* building structure on B , see [KL97, sec. 3.7]; thus each W_B -wall in B lies in at least 3 roots (or half-apartments) of B .

Case 1. The subbuilding B is irreducible. Our first step is to show that the Moufang property restricts to top dimensional irreducible subbuildings. Let $a \subset B$ be a W_B -root in B . Let $U_a \subset \text{Aut}(\partial_T X)$ denote the root group of a (see section 2.5).

Definition 3.4. The *restricted root group* of a is defined to be the subgroup $U_a^B \subset U_a$ which preserves the subbuilding $B \subset \partial_T X$.

Lemma 3.5. U_a^B acts transitively on the collection of roots in B opposite to a .

Proof. Pick two W_B -roots $a_1, a_2 \subset B$ opposite a . Since $\partial_T X$ is Moufang, there is a unique $g \in U_a$ such that $g(a_1) = a_2$. Let $B' := B \cap g^{-1}(B)$. Note that $B' \subset B$ is a convex subset (see Definition 2.2) containing the apartment $a \cup a_1$; therefore by [KL97, Prop. 3.10.3], B' is a top dimensional subbuilding of B . Let $\sigma \subset a$ be a W -chamber disjoint from the boundary ∂a , and for $i = 1, 2$ let $\sigma_i \subset a_i$ be the chamber in a_i opposite σ ; likewise, let $\pi \subset \sigma$ be a panel (a codimension 1 face) of σ , and for $i = 1, 2$ let $\pi_i \subset \sigma_i$ be the opposite panel in a_i . Now for each chamber $\sigma' \subset B$ incident to σ along π , for each $i = 1, 2$ there is a unique chamber σ'_i incident to σ_i along π_i , which corresponds to σ' under the correspondence of [KL97, Prop. 3.6.4]; clearly $g(\sigma_1) = \sigma_2$, and hence $g(\sigma'_1) = \sigma'_2$. This implies that $\sigma'_1 \subset B'$. Now we may argue as in the proof of [KL97, Prop. 3.12.2] to see that $B' = B$, and therefore $g(B) \subset B$; applying the same reasoning to g^{-1} we conclude that $g(B) = B$. Thus we have shown that U_a^B acts transitively on the roots in B opposite a . \square

Now pick a W_B -wall $\omega \subset B$, and let $\partial_T X(\omega) \subset \partial_T X$ be the subbuilding consisting of the union of the apartments containing ω ; similarly, let $B(\omega)$ be the subbuilding of B determined by ω . Thus if $F \subset X$ is a singular flat with $\partial_T F = \omega$, then the parallel set $\mathbb{P}(F)$ has Tits boundary $\partial_T X(\omega)$, the product splitting $\mathbb{P}(F) = F \times Y$ induces a join decomposition $\partial_T X(\omega) = \omega \circ \partial_T Y$, and $Y \subset X$ is a rank 1 symmetric subspace of dimension > 1 . This join decomposition induces a join decomposition $B(\omega) = \omega \circ \Lambda$, where $\Lambda := \partial_T Y \cap B$.

Lemma 3.6. *Λ is a compact connected manifold of positive dimension.*

Proof. We observe that for each root $a \subset \partial_T X$ with $\partial a = \omega$, the root group U_a acts freely transitively by homeomorphisms on $\partial_\infty Y \setminus \{\xi\}$, where $a = \omega \circ \xi$. Thus if we choose $\xi' \in \partial_T Y \setminus \{\xi\}$ and let $a' := \omega \circ \xi'$, then the map $\phi : U_a \rightarrow \partial_\infty Y \setminus \{\xi\}$ defined by $\phi(g) := g\xi'$ is a continuous bijection between manifolds, and is therefore a homeomorphism. Now suppose $\xi, \xi' \in \Lambda$, so that $a, a' \subset B$. The restricted root group $U_a^B \subset U_a$ acts simply transitively on $\Lambda \setminus \{\xi\}$, so ϕ restricts to a homeomorphism $U_a^B \rightarrow \Lambda \setminus \{\xi\}$. Thus U_a^B is a closed subgroup of U_a , and is therefore a manifold, which means that $\Lambda \setminus \{\xi\}$ is also a manifold. Since $\xi \in \Lambda$ was chosen arbitrarily, it follows that the group generated by the collection of restricted root groups $\{U_a^B \mid a = \omega \circ \xi, \xi \in \Lambda\}$, acts transitively on Λ . Thus Λ is a compact manifold.

Note that $|\Lambda| \geq 3$, since Λ is in bijection with the roots of B containing ω . Since U_a is unipotent, every $g \in U_a^B \setminus \{e\}$ has infinite order. This implies that Λ is an infinite set; being a compact manifold, it must have positive dimension.

If $\xi \in \Lambda$ and $a := \omega \circ \xi$, then U_a^B acts transitively on $\Lambda \setminus \{\xi\}$ while preserving the connected component of Λ containing ξ . It follows that Λ is connected. \square

Let H be the subgroup of G generated by the restricted root groups U_a^B , where a ranges over all W_B -roots in B . H is a connected subgroup of the Lie group G since it is generated by connected subgroups.

Our next objective is to show that H does not fix any point in $\partial_T X$.

Lemma 3.7. *Suppose $\sigma_0 \subset \partial_T X$ is a chamber intersecting B in a panel π_0 . Then there is a sequence $h_k \in H$ such that $h_k \sigma_0$ converges in $\partial_\infty X$ to a chamber $\sigma_1 \subset B$.*

Proof. We may assume that $\sigma_0 \not\subset B$. Therefore $\pi_0 = \sigma_0 \cap B$ is contained in a unique W_B -panel $\pi_1 \subset B$. Hence there is a W_B -chamber $\sigma_1 \subset B$ containing π_1 , and π_1 is contained in a W_B -wall $\omega \subset B$. The chambers σ_0 and σ_1 lie in roots $a_0 \subset \partial_T X(\omega)$, and $a_1 \subset B(\omega)$ respectively. Using the notation above, we have join decompositions $\partial_T X(\omega) = \omega \circ \partial_T Y$, $\partial_\infty X(\omega) = \omega \circ \partial_\infty Y$. For $i = 0, 1$ let $\xi_0, \xi_1 \in \partial_\infty Y$ be the element such that $a_i = \omega \circ \xi_i$. The unipotent root group $U_{a_1}^B$ acts by parabolic homeomorphisms on $\partial_\infty Y$ fixing $\xi_1 \in \partial_\infty Y$. If $g \in U_{a_1}^B \setminus \{e\}$, it has infinite order, and $g^k \xi_0$ converges as $|k| \rightarrow \infty$ to ξ_1 with respect to the topology of $\partial_\infty Y$. It follows that $g^k \sigma_0$ converges to σ_1 as $|k| \rightarrow \infty$. \square

Lemma 3.8. *For every chamber $\sigma_1 \subset \partial_T X$, there is a sequence $h_k \in H$ such that $h_k \sigma_1$ converges to a chamber in B .*

Proof. Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be a gallery in the spherical building $\partial_T X$, where $\sigma_k \subset B$. By Lemma 3.7, the Lemma holds if $k = 1$, so assume $k > 1$, pick $1 < j \leq k$, and suppose the lemma holds for σ_{j-1} . Thus there is a sequence $g_k \in H$ such that $g_k \sigma_{j-1}$ converges to a chamber $\tau \subset B$, and after passing to a subsequence, we may assume that $g_k \sigma_j$ converges to a chamber $\tau' \subset \partial_T X$ meeting τ along a W -panel π (at least). We are done if $\tau' \subset B$, so we assume $\tau \not\subset B$, which implies that π is contained in a W_B -panel π' . Let $U \subset \partial_\infty X$ be an open subset containing B . Applying Lemma 3.7, there is an $h \in H$ such that $h\tau' \subset U$. Then for large k we have $hg_k \sigma_j \subset U$. The open set U was arbitrary, so the lemma follows by induction. \square

The lemma implies that any point in $\partial_T X$ fixed by H must lie in B ; since H acts transitively on the set of W_B -chambers of the irreducible building B , no such fixed point exists.

Now suppose H preserved a proper symmetric subspace $Y \subset X$. Then $\partial_T Y \subset \partial_T X$ would be a proper H -invariant subbuilding which defined a closed subset of $\partial_\infty X$. Then Lemma 3.8 forces $B \subset \partial_T Y$, which

contradicts the assumption that B is not contained in the boundary of a proper symmetric subspace of X . Thus H is a connected subgroup of G which neither fixes a point in $\partial_T X$ nor preserves a proper symmetric subspace of X , and so we conclude that $H = G$, see section 2.6. Therefore $B = \partial_T X$.

Case 2. The subbuilding B is reducible.

Lemma 3.9. *B cannot have a nontrivial spherical join factor.*

Proof. Let $S \subset B$ be a maximal spherical join factor of B , and let $F \subset X$ be a flat with $\partial_T F = S$. Then the boundary of the parallel set $\mathbb{P}(F)$ contains B . By our assumption we may conclude that $X = \mathbb{P}(F)$. However, X is an irreducible symmetric space of noncompact type, so this is a contradiction. \square

Let

$$B = B_1 \circ \dots \circ B_l$$

be the unique join decomposition of B into irreducible nonspherical join factors. By case 1 above we are done if there is only one factor, so we assume that $l > 1$.

For each i , we let $H_i \subset G$ be the connected Lie group generated by transvections along geodesics whose ideal endpoints lie in B_i . Since transvections along parallel geodesics coincide, and transvections along geodesics lying in a single flat commute, it follows that H_i commutes with H_j when $i \neq j$.

Let $H := H_1 \times \dots \times H_l$.

Lemma 3.10. *H does not fix any point in $\partial_T X$.*

Proof. Pick a maximal flat $F \subset X$ such that $\partial_T F \subset B$. As H contains the full transvection group of F , we get $\text{Fix}(H, \partial_T X) \subset \partial_T F$. This means that the fixed point set of H is contained in the intersection S of the apartments of B ; this intersection is empty since B has no spherical join factor. \square

We must therefore have

$$H = H_1 \times \dots \times H_l = G,$$

see section 2.6. This contradicts the fact that G is a simple Lie group.

4. CONVEX SETS PRESERVED BY ZARISKI DENSE GROUPS

Theorem 4.1. *Let X be a symmetric space of noncompact type with de Rham decomposition $X = X_1 \times \dots \times X_k$, let $\pi_i : X \rightarrow X_i$ be the projection map, and $G = \text{Isom}_o(X)$ be the associated connected semi-simple Lie group. We denote by $X = Y_1 \times Y_{\geq 2}$ the decomposition of X*

into (the product of the) rank 1 and the higher rank factors. Suppose $\Gamma \subset G$ is a Zariski dense subgroup which preserves a closed convex subset $C \subset X$. Then C is of the form

$$(4.2) \quad C_1 \times Y_{\geq 2},$$

where $C_1 \subset Y_1$ is convex. Furthermore, for each de Rham factor X_i of Y_1 , there is a Γ -invariant subset $\hat{C}_i \subset X_i$ such that

- \hat{C}_i is the closed convex hull of its limit set.
- $|\partial_\infty C_i| = \infty$,
- $\hat{C}_1 := \prod_i \hat{C}_i \subset C_1$.
- $\partial_\infty \hat{C}_1 = \partial_\infty C_1$.

Proof. By Lemma 2.3, the limit set $\Lambda(C) = \partial_\infty C$ is a (cone topology) closed convex subset containing the limit set of Γ . By Benoist [Ben97], the limit set of Γ contains an open neighborhood (with respect to the topology of $\partial_T X$) of a pair of antipodal regular points $\xi, \hat{\xi} \in \partial_T X$. Hence $\partial_T C$ contains an apartment in $\partial_\infty X$. By [KL97, Prop. 3.10.3] it follows that $\partial_T C$ is a top dimensional subbuilding of $\partial_T X$.

Suppose $\partial_T C \subset \partial_T Y$ for some proper symmetric subspace $Y \subset X$. For every apartment $A \subset \partial_T C$, there is a unique maximal flat $F \subset X$ with $\partial_T F = A$, and so $F \subset Y$; likewise, we have $F \subset gY$ for all $g \in \Gamma$ which implies that $F \subset \bigcap_{g \in \Gamma} gY$. Since A was chosen arbitrarily, we conclude that $\bigcap_{g \in \Gamma} gY \subset X$ is a Γ -invariant proper symmetric subspace, which contradicts the Zariski density of Γ .

Theorem 3.1 applies, so the Tits boundary $\partial_T C$ splits as a join $\partial_T C = B_1 \circ \dots \circ B_k$, where $B_i = \partial_T X_i$ when X_i has rank at least two, and $|B_i| = \infty$ for each i , by the Zariski density of Γ .

Applying Lemma 2.1, it follows that C splits as in (4.2).

Define $\hat{C}_i \subset X_i$ to be the closed convex hull of B_i ; when $\text{Rank}(X) \geq 2$ then $\hat{C}_i = X_i$. Applying Lemma 2.1, it follows that $\hat{C}_1 := \prod_i \hat{C}_i \subset C_1$. \square

5. INVARIANT CONVEX SUBSETS IN SYMMETRIC SPACES WITH EUCLIDEAN DERHAM FACTORS

Theorem 5.1. *Let Y be a symmetric space of noncompact type without Euclidean de Rham factor, and suppose $\Gamma \subset \text{Isom}(Y) \times \text{Isom}(\mathbb{E}^n)$ is a subgroup whose projection $\pi_Y(\Gamma) \subset \text{Isom}(Y)$ is Zariski dense in the identity component $\text{Isom}_o(Y)$. If $C \subset X := Y \times \mathbb{E}^n$ is a Γ -invariant closed convex set, then either $C = \pi_Y(C) \times \mathbb{E}^n$ or there is a proper Γ -invariant affine subspace $A \subset \mathbb{E}^n$.*

Proof. We denote by $\text{Sh} := \pi_Y(C)$ the shadow of C in Y . For every point $y \in \text{Sh}$ we consider the slice $(\{y\} \times \mathbb{E}^n) \cap C =: C_y$. Since C is closed, the boundary at infinity $\partial_T C_y$ does not depend on y and it is a closed convex subset D of the round $(n-1)$ -sphere $\partial_T \mathbb{E}^n$. We may assume that it is a proper subset because otherwise $C = \text{Sh} \times \mathbb{E}^n$ and we are done.

If the C_y split off an \mathbb{R}^k -factor, $1 \leq k < n$, then C itself splits off an \mathbb{R}^k -factor. If $E' \subset \mathbb{E}^n$ is the maximal Euclidean factor and $\mathbb{E}^n = E' \times E''$ a splitting then this splitting is preserved by Γ . We can therefore reduce to the case that the C_y have no Euclidean factor.

Case 1: The slices C_y are unbounded. The set $D \subset \partial_T \mathbb{E}^n$ has diameter $< \pi$ and hence a well-defined center ζ which must be fixed by Γ . Let b_ζ denote the Busemann function on X associated to ζ . For every $\gamma \in \Gamma$ the difference $b_\zeta(\gamma \cdot) - b_\zeta$ equals a constant $\rho(\gamma)$ and the map $\rho : \Gamma \rightarrow \mathbb{R}$ is a group homomorphism.

The restriction of b_ζ to C_y is bounded above because $\partial_T C_y$ is contained in the open ball $B_{\frac{\pi}{2}}(\zeta)$. We may therefore assign to each $y \in \text{Sh}$ the bottom height of the slice C_y in the direction ζ defined as $h(y) := \min(-b_\zeta|_{C_y})$. The function $h : \text{Sh} \rightarrow \mathbb{R}$ is convex. We consider the asymptotic slope function $\text{slope}_h : \partial_T \text{Sh} \rightarrow \mathbb{R} \cup \{\infty\}$, see section 2.3. It is Γ -invariant. If the homomorphism ρ is nontrivial then slope_h assumes also negative values, and by Proposition 2.12 it has a unique minimum. This minimum must be fixed by Γ , a contradiction to the Zariski density of $\pi_Y(\Gamma)$ in $\text{Isom}(Y)$. Therefore ρ must be trivial, and the level sets of b_ζ yield Γ -invariant hyperplanes in \mathbb{E}^n .

Case 2: The slices C_y are bounded. We pick an ideal point $\zeta \in \partial_T \mathbb{E}^n$. As above, measuring the height in the direction of ζ , we can consider the convex function $\text{bot} : \text{Sh} \rightarrow \mathbb{R}$ given by $\text{bot}(y) := \min(-b_\zeta|_{C_y})$ and the concave function $\text{top} : \text{Sh} \rightarrow \mathbb{R}$ given by $\text{top}(y) := \max(-b_\zeta|_{C_y})$. Both functions are continuous because C is closed.

We now use the structure Theorem 4.1 for convex sets invariant under a Zariski dense group. It implies that $\partial_T \text{Sh}$ splits as the spherical join of the boundaries of the higher rank factors and of infinite subsets in the boundaries of the rank one factors. In particular, $\partial_\infty \text{Sh}$ has a well-defined and therefore $\pi_Y(\Gamma)$ -invariant convex hull $\text{CH}(\partial_\infty \text{Sh})$ in Y which is the product of the higher rank factors of Y with the closed convex hulls of the subsets in the boundaries of the rank one factors.

Lemma 2.6 applied to the higher rank factors and Lemma 2.7 applied to the rank one factors imply that the continuous concave function $\text{top} - \text{bot} : \text{Sh} \rightarrow [0, \infty)$ is constant on $\text{CH}(\partial_\infty \text{Sh})$. It follows that

the restrictions of top and bot to $\text{CH}(\partial_\infty \text{Sh})$ are affine. According to Lemmas 2.9 and 2.8 both functions are constant on $\text{CH}(\partial_\infty \text{Sh})$.

Since the values of $\text{top}(y)$ (or $\text{bot}(y)$) for all directions ζ determine the slice C_y it follows that the slices C_y equal the same compact set $B \subset E^n$ for all y in the $\pi_Y(\Gamma)$ -invariant subset $\text{CH}(\partial_\infty \text{Sh})$. In particular, the action of Γ on \mathbb{E}^n has bounded orbits and therefore a fixed point. \square

6. THE CONVEX COCOMPACT CASE

In this section we prove:

Lemma 6.1. *Let $X = \mathbb{E}^n \times Y$, where Y is a symmetric space of noncompact type. If $\Gamma \subset \text{Isom}(X)$ is a discrete convex cocompact group which does not preserve any proper symmetric subspace of X , then the fixed point set of Γ in $\partial_T X$ is contained in the Tits boundary of the Euclidean factor \mathbb{E}^n .*

Proof. Let C be a Γ -invariant closed convex set on which Γ acts cocompactly. Suppose Γ fixes a point $\xi \in \partial_\infty X \setminus \partial_\infty \mathbb{E}^n$. The Γ -action respects the join structure of $\partial_T X$, so we may assume without loss of generality that $\xi \in \partial_\infty Y$.

Recall that since Γ fixes ξ , the Γ -translates of the Busemann function b_ξ differ by a constant, and the map $\Gamma \ni g \mapsto g_*(b_\xi) - b_\xi$ defines a homomorphism $\rho : \Gamma \rightarrow \mathbb{R}$.

Suppose first that the homomorphism ρ is trivial, i.e. b_ξ is Γ -invariant. Then $b_\xi|_C$ is bounded and attains a minimum. The minimum set of $b_\xi|_C$ is a convex subset $C_1 \subset C$ lying in a horosphere. By triangle comparison one concludes that if $p_1, p_2 \in C_1$, then the ideal geodesic triangle $\overline{\xi p_1} \cup \overline{p_1 p_2} \cup \overline{p_2 \xi}$ bounds a flat half-strip. Thus C_1 is contained in the parallel set $\mathbb{P}(\gamma)$ of a geodesic $\gamma \subset \mathbb{E}^n \times Y$ which is parallel to the Y factor. Since C_1 is Γ -invariant it follows that Γ preserves a proper symmetric subspace of X , which is a contradiction. Therefore ρ is a nontrivial homomorphism and $b_\xi(C) = \mathbb{R}$.

Consider a group element $g \in \Gamma$ which translates the Busemann function b_ξ . We may assume that $b_\xi(gx) = b_\xi(x) - a$ for all $x \in X$ with $a > 0$. As the action is discrete, Γ acts on C by semi-simple isometries, and so g is an axial isometry. Pick a point $x_0 \in C$ and let $\rho : [0, \infty) \rightarrow X$ be the unit speed ray starting in x_0 and asymptotic to ξ . Then for $x_n = g^n x_0$ holds $b_\xi(x_n) = b_\xi(\rho(na))$. We obtain that $d(x_n, \rho(na)) \leq n d(x_1, \rho(a))$ and $\angle_{\rho(na)}(x_n, x_0) \geq \frac{\pi}{2}$. Triangle comparison implies that

$$\tan \angle_T(\xi_1, \xi) \leq \frac{d(x_1, \rho(a))}{a} > 0$$

and thus $\angle_T(\xi_1, \xi) < \frac{\pi}{2}$.

Since $\angle_T(\xi, \partial_\infty C) < \frac{\pi}{2}$ there is a unique $\eta \in \partial_\infty C$ at minimum Tits distance from ξ , and so η is fixed by Γ . As $\angle_T(\eta, \xi) < \frac{\pi}{2}$, it follows that η does not lie in $\partial_T \mathbb{E}^n$.

We now apply Lemma 2.4 to the convex set C . We obtain that the convex set C contains a Γ -invariant parallel set (with respect to C) $Z := \mathbb{P}(\gamma) \subset C$, where $\partial_T \gamma \ni \eta$. Therefore Γ preserves the parallel set of γ in X , which is a contradiction. \square

7. THE PROOF OF THEOREMS 1.1 AND 1.3

Proof of Theorem 1.1. This follows immediately from Theorem 5.1 and Theorem 4.1.

Proof of Theorem 1.3. By Lemma 6.1 and the fact that X contains no proper Γ -invariant symmetric subspace, the fixed point set of Γ is contained in \mathbb{E}^n . Therefore the projection of Γ to $\text{Isom}(Y)$ is Zariski dense in $\text{Isom}_o(Y)$, since otherwise it would preserve a proper symmetric subspace $Y' \subset Y$, contradicting our assumption on X . The theorem then follows from Theorem 1.1. \square

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