Decompositions of Reflexive Bimodules over Maximal Abelian Selfadjoint Algebras

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Abstract

We generalize the notion of 'diagonal' from the class of CSL algebras to masa bimodules. We prove that a reflexive masa bimodule decomposes as a sum of two bimodules, the diagonal and a module generalizing the w*-closure of the Jacobson radical of a CSL algebra. The latter module turns out to be reflexive, a result which is new even for CSL algebras. We show that the projection onto the direct summand contained in the diagonal is contractive and preserves compactness and reduces rank of operators. Stronger results are obtained when the module is the reflexive hull of its rank-one subspace.

Keywords: Operator algebras, reflexivity, TRO, masa-bimodules MSC (2000) : Primary 47L05; Secondary 47L35, 46L10

1 Introduction

In this paper we attempt a generalisation of the concept of the diagonal of a CSL algebra to reflexive spaces of operators which are modules over maximal abelian selfadjoint algebras (masas).

Recall [\[2\]](#page-33-0) that a CSL algebra is an algebra A of operators on a Hilbert space H which can be written in the form

$$
\mathcal{A} = \{ A \in B(H) : AP = PAP \text{ for all } P \in \mathcal{S} \}
$$

where S is a commuting family of projections. Note that A contains any masa containing S'' .

More generally, a reflexive masa bimodule U of operators from H to another Hilbert space K can be written in the form

$$
\mathcal{U} = \{ T \in B(H, K) : TP = \phi(P)TP \text{ for all } P \in \mathcal{S} \}
$$

where S is a commuting family of projections on H and ϕ maps them to commuting projections on K (see below for details).

The diagonal $A \cap A^*$ of a CSL algebra A is a von Neumann algebra, which equals the commutant

$$
\mathcal{S}' = \{ A \in B(H) : AP = PA \text{ for all } P \in \mathcal{S} \}
$$

of the corresponding invariant projection family. The natural corresponding object for a reflexive masa bimodule $\mathcal U$ is a ternary ring of operators (TRO)

$$
\Delta(\mathcal{U}) = \{ T \in B(H, K) : TP = \phi(P)T \text{ for all } P \in \mathcal{S} \}
$$

which is also a reflexive masa bimodule.

This 'diagonal' $\Delta(U)$ is the primary object of study of the present paper.

We decompose U as a sum $U_0 + \Delta(U)$, where U_0 also turns out to be reflexive (Theorem 5.2). This is new even for the case of CSL algebras; note, however, that for nest algebras reflexivity of w*-closed bimodules is automatic [\[7\]](#page-33-1). An analogous decomposition for the case of nest subalgebras of von Neumann algebras is in [\[11\]](#page-34-0).

We also prove (Corollary 5.3) that the bimodule \mathcal{U}_0 has in our context the role corresponding to the w^* closure of the Jacobson radical of a CSL algebra.

The diagonal $\Delta(U)$ is proved to be generated by a partial isometry and natural von Neumann algebras assosiated to \mathcal{U} (Theorem 4.1).

The above decomposition may be further refined to a direct sum: $\mathcal{U} =$ $\mathcal{U}_0 \oplus \mathcal{M}$ where $\mathcal M$ is a TRO ideal of the diagonal $\Delta(\mathcal{U})$ (Theorem 3.3), containing the compact operators of the diagonal (Proposition 6.3).

In case $\mathcal U$ is strongly reflexive (that is, coincides with the reflexive hull of the rank one operators it contains) we show (Theorem 7.4) that $\mathcal M$ coincides with the w^{*}-closed linear span of the finite rank operators of the diagonal, an equality which fails in general.

As in the case of von Neumann algebras, we show that every TRO decomposes in an 'atomic' and a 'nonatomic' part. The 'atomic' part of the diagonal $\Delta(\mathcal{U})$ is contained (properly in general) in M (Proposition 6.3).

We also study the projection $\theta : \mathcal{U} \longrightarrow \mathcal{M}$ defined by the above direct sum decomposition. We prove that it is contractive and maps compact operators to compact operators and finite rank operators to operators of at most the same rank.

In case U is strongly reflexive, we show that $\theta = D|_{\mathcal{U}}$, where D is the natural projection onto the 'atomic' part of the diagonal $\Delta(\mathcal{U})$.

A main tool used to obtain these results is an appropriate sequence of projections (U_n) on $B(H, K)$ which depend on $\mathcal U$. This sequence behaves analogously to the net of 'diagonal sums' used in nest algebras (see for example [\[2\]](#page-33-0)).

In nest algebra theory, the net of diagonal sums of a compact operator converges in norm to a compact operator in the 'atomic' part of the diagonal. This has been generalised to CSL algebras by Katsoulis [\[10\]](#page-34-1). Here we show (Proposition 6.10) that for every compact operator K, the sequence $(U_n(K))$ converges in norm to $D(K)$.

We present some definitions and concepts we use in this work. All Hilbert spaces will be assumed separable.

If S is a set of operators then $R_1(S)$ denotes the subset of S which contains the rank 1 operators and the zero operator. If H is a Hilbert space and $S \subset B(H)$, the set of orthogonal projections of S is denoted by $\mathcal{P}(S)$.

If H_1, H_2 are Hilbert spaces, $C_1(H_1, H_2)$ are the trace class operators and R a subset of $C_1(H_1, H_2)$, we denote by \mathcal{R}^0 the set of operators which are annihilated by \mathcal{R} :

$$
\mathcal{R}^0 = \{ T \in B(H_2, H_1) : tr(TS) = 0 \text{ for all } S \in \mathcal{R} \}.
$$

Let H_1, H_2 be Hilbert spaces and U a subset of $B(H_1, H_2)$. Then the **reflexive hull** of \mathcal{U} is defined [\[12\]](#page-34-2) to be the space

$$
\text{Ref}(\mathcal{U}) = \{ T \in B(H_1, H_2) : Tx \in \overline{[\mathcal{U}x]} \text{ for each } x \in H_1 \}.
$$

Simple arguments show that

 $\text{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : \text{for all projections } E, F : E \mathcal{U} F = 0 \Rightarrow E T F = 0\}$

A subspace U is called **reflexive** if $\mathcal{U} = \text{Ref}(\mathcal{U})$. It is called **strongly re**flexive if there exists a set $L \subset B(H_1, H_2)$ of rank 1 operators such that $\mathcal{U} =$ $\mathrm{Ref}(L)$.

Now we present some concepts introduced by Erdos [\[5\]](#page-33-2).

Let $\mathcal{P}_i = \mathcal{P}(B(H_i)), i = 1, 2$. Define $\phi = \text{Map}(\mathcal{U})$ to be the map ϕ : $P_1 \rightarrow P_2$ which associates to every $P \in \mathcal{P}_1$ the projection onto the subspace

 $[TPy : T \in \mathcal{U}, y \in H_1]$ ⁻. The map ϕ is ∨-continuous (that is, it preserves arbitrary suprema) and 0 preserving.

Let $\phi^* = \text{Map}(\mathcal{U}^*), \mathcal{S}_{1,\phi} = \{\phi^*(P)^{\perp} : P \in \mathcal{P}_2\}, \mathcal{S}_{2,\phi} = \{\phi(P) : P \in \mathcal{P}_1\}.$ Erdos has proved that $S_{1,\phi}$ is meet complete and contains the identity projection, $S_{2,\phi}$ is join complete and contains the zero projection, while $\phi|_{S_{1,\phi}}$: $S_{1,\phi} \to S_{2,\phi}$ is a bijection. In fact

$$
(\phi|_{\mathcal{S}_{1,\phi}})^{-1}(Q) = \phi^*(Q^{\perp})^{\perp} \tag{1.1}
$$

for all $Q \in \mathcal{S}_{2,\phi}$ and

$$
\text{Ref}(\mathcal{U}) = \{ T \in B(H_1, H_2) : \phi(P)^{\perp} T P = 0 \text{ for each } P \in \mathcal{S}_{1, \phi} \}.
$$

We call the families $S_{1,\phi}$, $S_{2,\phi}$ the semilattices of U.

A C.S.L. is a complete abelian lattice of projections which contains the identity and the zero projection.

If $\mathcal{A}_1 \subset B(H_1)$, and $\mathcal{A}_2 \subset B(H_2)$ are algebras, a subspace $\mathcal{U} \subset B(H_1, H_2)$ is called an A_1, A_2 –bimodule if $A_2 \mathcal{U} A_1 \subset \mathcal{U}$.

A subspace M of $B(H_1, H_2)$ is called a **ternary ring of operators** (TRO) if $MM^*M \subset M$. Katavolos and Todorov [\[9\]](#page-34-3) have proved that a TRO $\mathcal M$ is w^* closed if and only if it is wot closed if and only if it is reflexive. In this case, if $\chi = \text{Map}(\mathcal{M})$, then

$$
\mathcal{M} = \{ T \in B(H_1, H_2) : TP = \chi(P)T \text{ for all } P \in \mathcal{S}_{1,\chi} \}.
$$

They also proved that if M is a strongly reflexive TRO, then there ex- $\sum_{n=1}^{\infty} \bigoplus E_n B(H_1, H_2) F_n$. We present a new proof of this result in Corollary ist families of mutually orthogonal projections $(F_n), (E_n)$ such that $\mathcal{M} =$ 6.9.

The following proposition is easily proved.

Proposition 1.1 *Let* H_1, H_2 *be Hilbert spaces,* $A_1 \subset B(H_1), A_2 \subset B(H_2)$ *masas and* U *a* A_1 *,* A_2 −*bimodule. Then*

 $\text{Ref}(\mathcal{U}) = \{T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{A}_2), F \in \mathcal{P}(\mathcal{A}_1), E \mathcal{U} F = 0 \Rightarrow ETF = 0\}.$

The next section contains some preliminary results.

2 Decomposition of a reflexive TRO.

In this section we show that a w^* -closed TRO decomposes into a 'nonatomic' and a 'totaly atomic' part.

Let H_1, H_2 be Hilbert spaces, $\mathcal{M} \subset B(H_1, H_2)$ be a w^{*}-closed TRO and $\mathcal{B}_1=(\mathcal{M}^*\mathcal{M})'',\mathcal{B}_2=(\mathcal{M}\mathcal{M}^*)''.$

Remark 2.1 *We suppose that* \mathcal{M}_0 *is a* w^* -closed TRO ideal of \mathcal{M} ; *namely,* M⁰ *is a linear subspace of* M *and*

$$
\mathcal{M}_0\mathcal{M}^*\mathcal{M}\subset \mathcal{M}_0,\ \mathcal{M}\mathcal{M}^*\mathcal{M}_0\subset \mathcal{M}_0.
$$

It follows that $\mathcal{MM}^*_0\mathcal{M} \subset \mathcal{M}_0$ [\[4\]](#page-33-3)*.*

Now, we observe that there exist projections Q_i in the centre of \mathcal{B}_i , $i = 1, 2$ *such that* $\mathcal{M}_0 = \mathcal{M}Q_1 = Q_2\mathcal{M}$. *Hence* \mathcal{M}_0 *is a* $\mathcal{B}_1, \mathcal{B}_2$ *-bimodule.*

Proof

Let $\mathcal{J}_1 = [\mathcal{M}_0^*\mathcal{M}_0]^{-w^*}$ and $\mathcal{J}_2 = [\mathcal{M}_0\mathcal{M}_0^*]^{-w^*}$.

We can easily verify that \mathcal{J}_i is an ideal of \mathcal{B}_i , $i = 1, 2$. Hence there is a projection Q_i in the centre of \mathcal{B}_i so that $\mathcal{J}_i = \mathcal{B}_i Q_i, i = 1, 2$.

One easily checks that

$$
\mathcal{MB}_1 \subset \mathcal{M}, \ \ \mathcal{B}_2 \mathcal{M} \subset \mathcal{M},
$$

$$
\mathcal{MI}_1 \subset \mathcal{M}_0, \ \ \mathcal{J}_2 \mathcal{M} \subset \mathcal{M}_0
$$

We observe that $\mathcal{M}Q_1 \subset \mathcal{M}J_1 \subset \mathcal{M}_0$.

Let $T \in \mathcal{M}_0$ then $T^*T \in \mathcal{J}_1$, so $T^*T = T^*TQ_1$ and thus $T = TQ_1$. Hence $T \in \mathcal{MQ}_1$. We conclude that $\mathcal{M}_0 \subset \mathcal{MQ}_1$ and hence equality holds.

Similarly one shows that $\mathcal{M}_0 = Q_2 \mathcal{M}$. \Box

Since $[R_1(\mathcal{M})]^{-w^*}$ is a strongly reflexive TRO, by Proposition 3.5 in [\[9\]](#page-34-3) there exist mutually orthogonal projections (F_n) in the centre of \mathcal{B}_1 and (E_n) in the centre of \mathcal{B}_2 such that $[R_1(\mathcal{M})]^{-w^*} = \sum_{n=1}^{\infty} \bigoplus E_n B(H_1, H_2) F_n$. We write $E = \vee_n E_n, F = \vee_n F_n$.

Theorem 2.2 *The space* M *decomposes in the following direct sum*

$$
\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M})^*)^0) \oplus [R_1(\mathcal{M})]^{-w^*}.
$$

The spaces $\mathcal{M} \cap (R_1(\mathcal{M})^*)^0$ *and* $[R_1(\mathcal{M})]^{-w^*}$ *are TRO ideals of* \mathcal{M} *. Moreover*

$$
[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}F = E\mathcal{M} = E\mathcal{M}F
$$

$$
\mathcal{M} \cap (R_1(\mathcal{M})^*)^0 = \mathcal{M}F^\perp = E^\perp \mathcal{M} = E^\perp \mathcal{M}F^\perp.
$$

Proof

We observe that $[R_1(\mathcal{M})]^{-w^*}$ is a TRO ideal of \mathcal{M} .

By Remark 2.1 there exists projection Q in the centre of \mathcal{B}_1 such that $[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}Q.$

For every $m \in \mathbb{N}$, we have $E_m B(H_1, H_2) F_m \subset \mathcal{M}Q$. It follows that $E_m B(H_1, H_2)F_m = E_m B(H_1, H_2)F_m Q$, so $F_m = F_m Q$. We conclude that $\vee_m F_m = F \leq Q$. Since $F \in \mathcal{B}_1$ we get $\mathcal{M}F \subset \mathcal{M}$, therefore $\mathcal{M}F = \mathcal{M}FQ \subset \mathcal{M}Q$. It follows that

$$
[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}Q \supset \mathcal{M}F \supset [R_1(\mathcal{M})]^{-w^*}F = [R_1(\mathcal{M})]^{-w^*}.
$$

We proved that $[R_1(\mathcal{M})]^{-w^*} = \mathcal{M}F$.

If $M \in \mathcal{M}$ and $R \in R_1(\mathcal{M})$, then $R = RF$ so $tr(MF^{\perp}R^*)$ $= tr(M(RF^{\perp})^*) = tr(M0) = 0.$

We conclude that

$$
\mathcal{M}F^{\perp} \subset \mathcal{M} \cap (R_1(\mathcal{M})^*)^0
$$

Hence $\mathcal{M} = \mathcal{M} F^{\perp} + \mathcal{M} F \subset \mathcal{M} \cap (R_1(\mathcal{M})^*)^0 + [R_1(\mathcal{M})]^{-w^*} \subset \mathcal{M}.$

It follows that

$$
\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M})^*)^0) + [R_1(\mathcal{M})]^{-w^*}.
$$

We shall prove that this sum is direct.

If $T \in [R_1(\mathcal{M})]^{-w^*} \cap (R_1(\mathcal{M})^*)^0$ then $T = \sum_{n=1}^{\infty} E_n T F_n$. If R is a rank 1 operator then $tr(TR) = \sum_{n=1}^{\infty} tr(E_n T F_n R) = \sum_{n=1}^{\infty} tr(T F_n R E_n).$

But for every $n \in \mathbb{N}$, $tr(TF_nRE_n) = tr(T(E_nR^*F_n)^*) = 0$ since $E_nR^*F_n \in$ $R_1(\mathcal{M})$ and $T \in (R_1(\mathcal{M})^*)^0$.

Thus $tr(TR) = 0$ for every rank 1 operator R, hence $T = 0$. This shows that $[R_1(\mathcal{M})]^{-w^*} \cap (R_1(\mathcal{M})^{\perp})^* = 0.$

We have shown that $\mathcal{M} = (\mathcal{M} \cap (R_1(\mathcal{M})^*)^0) \oplus [R_1(\mathcal{M})]^{-w^*}.$

Since $\mathcal{M} = \mathcal{M} F^{\perp} \oplus \mathcal{M} F$, $[R_1(\mathcal{M})]^{-w^*} = \mathcal{M} F$ and $\mathcal{M} F^{\perp} \subset \mathcal{M} \cap (R_1(\mathcal{M})^*)^0$ we conclude that

$$
\mathcal{M} F^{\perp} = \mathcal{M} \cap (R_1(\mathcal{M})^*)^0.
$$

The equalities $E^{\perp} \mathcal{M} = \mathcal{M} \cap (R_1(\mathcal{M})^*)^0, E\mathcal{M} = [R_1(\mathcal{M})]^{-w^*}$ are proved similarly. \square

Proposition 2.3 Let θ : M \longrightarrow M be the projection onto $[R_1(\mathcal{M})]^{-w^*}$ *defined by the decomposition in Theorem 2.2. Then* $\theta(T) = \sum_{n=1}^{\infty} E_n T F_n$ *for every* $T \in \mathcal{M}$.

Proof

Since M decomposes as the direct sum of the $\mathcal{B}_1, \mathcal{B}_2$ -bimodules $\mathcal{M} \cap (R_1(\mathcal{M})^*)^0$ and $[R_1(\mathcal{M})]^{-w^*}$, θ is a $\mathcal{B}_1, \mathcal{B}_2$ -bimodule map:

$$
\theta(B_2TB_1)=B_2\theta(T)B_1
$$

for every $T \in \mathcal{M}, B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$.

Since $(E_n) \subset \mathcal{B}_1$, $(F_n) \subset \mathcal{B}_2$ we have that:

$$
\theta(T) = \sum_{n=1}^{\infty} E_n \theta(T) F_n = \sum_{n=1}^{\infty} \theta(E_n T F_n) = \sum_{n=1}^{\infty} E_n T F_n.
$$

3 Decomposition of a reflexive masa bimodule

Let H_1 , H_2 be Hilbert spaces, $\mathcal{P}_i = \mathcal{P}(B(H_i)), i = 1, 2, \mathcal{D}_i \subset B(H_i), i = 1, 2$ be masas, $U \subset B(H_1, H_2)$ be a reflexive $\mathcal{D}_1, \mathcal{D}_2$ -bimodule. Write

$$
\phi = \text{Map}(\mathcal{U}), \qquad \phi^* = \text{Map}(\mathcal{U}^*), \mathcal{S}_{2,\phi} = \phi(\mathcal{P}_1), \quad \mathcal{S}_{1,\phi} = \{P^\perp : P \in \phi^*(\mathcal{P}_2)\} \mathcal{A}_2 = (\mathcal{S}_{2,\phi})', \qquad \mathcal{A}_1 = (\mathcal{S}_{1,\phi})'.
$$

Observe that $\mathcal{S}_{i,\phi} \subset \mathcal{D}_i$ hence $\mathcal{D}_i \subset \mathcal{A}_i, i = 1, 2$. We define

$$
\mathcal{U}_0 = [\phi(P)TP^{\perp} : T \in \mathcal{U}, P \in \mathcal{S}_{1,\phi}]^{-w^*},
$$

$$
\Delta(\mathcal{U}) = \{ T : TP = \phi(P)T \text{ for all } P \in \mathcal{S}_{1,\phi} \}.
$$

We remark that \mathcal{U}_0 and $\Delta(\mathcal{U})$ are $\mathcal{D}_1,\mathcal{D}_2-$ bimodules contained in \mathcal{U} and $\Delta(\mathcal{U})$ is a reflexive TRO. We call $\Delta(\mathcal{U})$ the **diagonal** of \mathcal{U} .

Theorem 3.1 $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$.

Proof

As noted in the introduction

$$
\mathcal{U} = \{ T \in B(H_1, H_2) : \phi(P)^{\perp} T P = 0 \text{ for all } P \in \mathcal{S}_{1, \phi} \}.
$$

Since the Hilbert spaces H_1, H_2 are separable we can choose a sequence $(P_n) \subset S_{1,\phi}$ such that

$$
\mathcal{U} = \{ T \in B(H_1, H_2) : \phi(P_n)^{\perp} T P_n = 0 \text{ for all } n \in \mathbb{N} \}.
$$

We define

$$
V_n: B(H_1, H_2) \to B(H_1, H_2): V_n(T) = \phi(P_n)TP_n + \phi(P_n)^{\perp}TP_n^{\perp}, n \in \mathbb{N}.
$$

One easily checks that V_n is idempotent and a norm contraction.

We also define $U_n = V_n \circ V_{n-1} \circ \dots \circ V_1, n \in \mathbb{N}$.

Let $T \in \mathcal{U}$, then

$$
T = U_1(T) + \phi(P_1)TP_1^{\perp}
$$

$$
U_1(T) = U_2(T) + \phi(P_2)U_1(T)P_2^{\perp}
$$

by induction

$$
U_{n-1}(T) = U_n(T) + \phi(P_n)U_{n-1}(T)P_n^{\perp}
$$

for all $n \in \mathbb{N}$.

Adding the previous equalities we obtain

$$
T = U_n(T) + M_n
$$

where

$$
M_n = \phi(P_1)TP_1^{\perp} + \phi(P_2)U_1(T)P_2^{\perp} + \dots + \phi(P_n)U_{n-1}(T)P_n^{\perp} \in \mathcal{U}_0
$$

for all $n \in \mathbb{N}$.

We observe that $\phi(P_i)^{\perp}U_n(T)P_i = \phi(P_i)U_n(T)P_i^{\perp} = 0$ for $i = 1, 2, ...n$ and $||U_n(T)|| \le ||U_{n-1}(T)|| \le ... \le ||T||$ for all $n \in \mathbb{N}$.

The sequence $(U_n(T))$ is bounded, so there exists a subsequence $(U_{n_m}(T))$ that converges in the weak−∗ topology to an operator L.

Then $M_{n_m} = T - U_{n_m}(T) \stackrel{w^*}{\rightarrow} T - L = M \in \mathcal{U}_0$.

Since $\phi(P_i)^{\perp} L P_i = \phi(P_i) L P_i^{\perp} = 0$ for all $i \in \mathbb{N}$ we have $L \in \Delta(\mathcal{U})$ and $T = M + L \in \mathcal{U}_0 + \Delta(\mathcal{U}).$

Remark 3.2 *The following are equivalent:*

 $i)$ U *is a TRO ii*) $\mathcal{U} = \Delta(\mathcal{U})$ *iii*) $\mathcal{U}_0 = 0$.

Theorem 3.3 *There exist projections* $Q_i \in \mathcal{D}_i$, $i = 1, 2$ *such that:*

 $\mathcal{U} = \mathcal{U}_0 \oplus (I - Q_2) \Delta(\mathcal{U}) (I - Q_1) = \mathcal{U}_0 \oplus (I - Q_2) \Delta(\mathcal{U}) = \mathcal{U}_0 \oplus \Delta(\mathcal{U}) (I - Q_1).$

Proof

We make the following observations:

i) $\mathcal{U}\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \mathcal{U}, \ \ \Delta(\mathcal{U})^*\Delta(\mathcal{U})\mathcal{U} \subset \mathcal{U}.$ *Proof* Let $T \in \mathcal{U}, M, N \in \Delta(\mathcal{U})$. Then for every $P \in \mathcal{S}_{1,\phi}$ we have $\phi(P)^{\perp}TM^{*}NP = \phi(P)^{\perp}TM^{*}\phi(P)N = \phi(P)^{\perp}TPM^{*}N = 0M^{*}N = 0.$ Thus $TM^*N \in \mathcal{U}$. Similarly we have that $MN^*T \in \mathcal{U}$.

 $ii) \; \mathcal{U}_0 \Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset \mathcal{U}_0, \; \; \Delta(\mathcal{U}) \Delta(\mathcal{U})^* \mathcal{U}_0 \subset \mathcal{U}_0.$ *Proof* Let $T \in \mathcal{U}, M, N \in \Delta(\mathcal{U})$. Then for every $P \in \mathcal{S}_{1,\phi}$ we have $\phi(P)TP^{\perp}M^*N = \phi(P)TM^*\phi(P)^{\perp}N = \phi(P)TM^*NP^{\perp}.$ It follows by (i) that $TM^*N \in \mathcal{U}$ so $\phi(P)TP^{\perp}M^*N \in \mathcal{U}_0$. Taking the w^* closed linear span we get $SM^*N \in \mathcal{U}_0$ for all $S \in \mathcal{U}_0, M, N \in$ $\Delta(\mathcal{U})$. Similarly we have that $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{U}_0$.

iii) The space $U_0 \cap \Delta(U)$ is a TRO ideal of $\Delta(U)$. *Proof* Since $\Delta(\mathcal{U})$ is a TRO $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))\Delta(\mathcal{U})^*\Delta(\mathcal{U}) \subset \Delta(\mathcal{U})$.

Using observation (ii) we have that $(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset \mathcal{U}_0$. It follows that $(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset \mathcal{U}_0 \cap \Delta(\mathcal{U})$. Analogously we get $\Delta(\mathcal{U})\Delta(\mathcal{U})^*(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \mathcal{U}_0 \cap \Delta(\mathcal{U})$. We conclude that the space $U_0 \cap \Delta(U)$ is a TRO ideal of $\Delta(U)$.

So there exist projections $Q_i \in \mathcal{D}_i, i = 1, 2$, such that $\mathcal{U}_0 \cap \Delta(\mathcal{U}) =$ $\Delta(\mathcal{U})Q_1 = Q_2\Delta(\mathcal{U})$ (Remark 2.1).

By Theorem 3.1 we have

$$
\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U}) = \mathcal{U}_0 + \Delta(\mathcal{U})Q_1 + \Delta(\mathcal{U})(I - Q_1) = \mathcal{U}_0 + \Delta(\mathcal{U})(I - Q_1).
$$

Clearly $\mathcal{U}_0 \cap \Delta(\mathcal{U})(I - Q_1) = 0.$

Similarly one shows that $\mathcal{U} = \mathcal{U}_0 \oplus (I - Q_2) \Delta(\mathcal{U})$ and it therefore follows that $\mathcal{U} = \mathcal{U}_0 \oplus (I - Q_2) \Delta(\mathcal{U}) (I - Q_1).$

Remark 3.4 *The projection* $\theta : \mathcal{U} \to \mathcal{U}$ *onto* $(I - Q_2) \Delta(\mathcal{U})(I - Q_1)$ *defined by the decomposition in Theorem 3.3 is a contraction.*

Indeed, if $T \in \mathcal{U}$, as in Theorem 3.1 we have $T = M + S$ where $M \in$ $\Delta(\mathcal{U}), S \in \mathcal{U}_0$ and $||M|| \leq ||T||$ (see the proof). Since $\theta(T) = (I - Q_2)M(I - Q_1)$, we obtain $\|\theta(T)\| \leq \|T\|$.

Let $\mathcal{N}_i = Alg(\mathcal{S}_{i,\phi}) = \{T : P^{\perp}TP = 0 \text{ for all } P \in \mathcal{S}_{i,\phi}\}, i = 1,2, \text{ and }$ $\mathcal{L}_i = [PTP^{\perp} : T \in \mathcal{N}_i, P \in \mathcal{S}_{i,\phi}]^{-w^*}, i = 1,2.$

Lemma 3.5 *i)* $A_2\Delta(\mathcal{U})A_1 \subset \Delta(\mathcal{U})$. *ii*) $\Delta(\mathcal{U})^* \mathcal{A}_2 \Delta(\mathcal{U}) \subset \mathcal{A}_1, \Delta(\mathcal{U}) \mathcal{A}_1 \Delta(\mathcal{U})^* \subset \mathcal{A}_2.$ *iii*) $\mathcal{U} = \mathcal{N}_2 \mathcal{U} \mathcal{N}_1$. *iv)* $U_0 = \mathcal{N}_2 \mathcal{U}_0 \mathcal{N}_1$. *v)* $\mathcal{UL}_1 \subset \mathcal{U}_0$, $\mathcal{L}_2 \mathcal{U} \subset \mathcal{U}_0$. *vi*) $\Delta(\mathcal{U})^*\mathcal{U} \subset \mathcal{N}_1$, $\mathcal{U}\Delta(\mathcal{U})^* \subset \mathcal{N}_2$. *vii*) $\Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{L}_1$, $\mathcal{U}_0\Delta(\mathcal{U})^* \subset \mathcal{L}_2$.

Proof

Claims (i) , (ii) are obvious and (iii) is Lemma 1.1 in [\[9\]](#page-34-3).

iv) If
$$
N_1 \in \mathcal{N}_1
$$
, $N_2 \in \mathcal{N}_2$, $T \in \mathcal{U}$ and $P \in \mathcal{S}_{1,\phi}$ then

$$
N_2\phi(P)TP^{\perp}N_1 = \phi(P)N_2\phi(P)TP^{\perp}N_1P^{\perp} \in \mathcal{U}_0
$$

since $N_2\phi(P)TP^{\perp}N_1 \in \mathcal{U}$ by (iii). Taking the w^{*} closed linear span we get $\mathcal{N}_2\mathcal{U}_0\mathcal{N}_1\subset\mathcal{U}_0.$

v) If $N_1 \in \mathcal{N}_1, T \in \mathcal{U}$ and $P \in \mathcal{S}_{1,\phi}$ then

$$
TPN_1P^{\perp} = \phi(P)TPN_1P^{\perp} \in \mathcal{U}_0
$$

since $TPN_1 \in \mathcal{UN}_1 \subset \mathcal{U}$. Taking the w^{*} closed linear span we get $TK \in \mathcal{U}_0$ for every $K \in \mathcal{L}_1$.

The second inclusion follows by symmetry.

vi) Let $M \in \Delta(U), T \in U, P \in S_{1,\phi}$. Then $PM^*TP = M^*\phi(P)TP =$ M^*TP so $M^*T \in \mathcal{N}_1$. Similarly one shows that $TM^* \in \mathcal{N}_2$.

vii) Let $M \in \Delta(\mathcal{U}), T \in \mathcal{U}, P \in \mathcal{S}_{1,\phi}$ then $M^*\phi(P)TP^{\perp} = PM^*TP^{\perp} \in \mathcal{L}_1$ since $M^*T \in \Delta(\mathcal{U})^*\mathcal{U} \subset \mathcal{N}_1$. Taking the w^* closed linear span we get $M^*S \in$ \mathcal{L}_1 for every $S \in \mathcal{U}_0$.

Similarly one shows that $\mathcal{U}_0\Delta(\mathcal{U})^*\subset \mathcal{L}_2$. \Box

Proposition 3.6 *The following are equivalent:*

i)
$$
U = U_0
$$
.
\n*ii*) $\Delta(U)^* \Delta(U) \subset L_1 \cap A_1$.
\n*iii*) $\Delta(U) \Delta(U)^* \subset L_2 \cap A_2$.

Proof

If $\mathcal{U} = \mathcal{U}_0$ then $\Delta(\mathcal{U}) \subset \mathcal{U}_0$, hence $\Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset \Delta(\mathcal{U})^* \mathcal{U}_0 \subset \mathcal{L}_1$ by the previous lemma.

Since $\Delta(\mathcal{U})^*\Delta(\mathcal{U})\subset \mathcal{A}_1$ we get $\Delta(\mathcal{U})^*\Delta(\mathcal{U})\subset \mathcal{L}_1\cap \mathcal{A}_1$.

If conversely $\Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset \mathcal{L}_1 \cap \mathcal{A}_1$, then $\Delta(\mathcal{U})^* \Delta(\mathcal{U}) (I - Q_1) \subset \mathcal{L}_1 \cap \mathcal{A}_1$, so by the previous lemma $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U})(I-Q_1) \subset \mathcal{UL}_1 \subset \mathcal{U}_0$. $(Q_1$ is the projection in Theorem 3.3). Since $U_0 \cap \Delta(U)$ is a TRO ideal of $\Delta(U)$ (Theorem 3.3) we have that $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U})Q_1 = \Delta(\mathcal{U})\Delta(\mathcal{U})^*(\Delta(\mathcal{U})\cap\mathcal{U}_0) \subset \Delta(\mathcal{U})\cap\mathcal{U}_0 \subset \mathcal{U}_0.$ We conclude that $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U})\subset \mathcal{U}_0$. Since $\Delta(\mathcal{U})$ is a TRO its subspace $\Delta(\mathcal{U})\Delta(\mathcal{U})^*\Delta(\mathcal{U})$ is norm-dense [\[4\]](#page-33-3). Therefore $\Delta(\mathcal{U}) \subset \mathcal{U}_0$ and so $\mathcal{U} = \mathcal{U}_0$.

The equivalence $(i) \Leftrightarrow (iii)$ is proved similarly. \square

Proposition 3.7 *The following are equivalent:*

i)
$$
U = U_0 \oplus \Delta(U)
$$
.
\n*ii*) $\Delta(U) (\mathcal{L}_1 \cap \mathcal{A}_1) = 0$.
\n*iii*) $(\mathcal{L}_2 \cap \mathcal{A}_2) \Delta(U) = 0$.

Proof

Note by Lemma 3.5 that $\Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) \subset \Delta(\mathcal{U})\mathcal{A}_1 \subset \Delta(\mathcal{U})$ and $\Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) \subset \mathcal{UL}_1 \subset \mathcal{U}_0.$

Thus if the sum $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$ is direct then $\Delta(\mathcal{U})$ ($\mathcal{L}_1 \cap \mathcal{A}_1$) = 0.

Suppose conversely that $\Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) = 0$. Using again Lemma 3.5 we have that $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^* (\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \Delta(\mathcal{U})^* \Delta(\mathcal{U})$ $\subset A_1$ and $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \Delta(\mathcal{U})^*\mathcal{U}_0 \subset \mathcal{L}_1$ so $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^* (\mathcal{U}_0 \cap \Delta(\mathcal{U})) \subset \mathcal{L}_1 \cap \mathcal{A}_1$ and so $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^* (\mathcal{U}_0 \cap \Delta(\mathcal{U}))$ $\subset \Delta(\mathcal{U})(\mathcal{L}_1 \cap \mathcal{A}_1) = 0.$

But since $U_0 \cap \Delta(U)$ is a TRO (Theorem 3.3), its subspace $(\mathcal{U}_0 \cap \Delta(\mathcal{U}))(\mathcal{U}_0 \cap \Delta(\mathcal{U}))^*(\mathcal{U}_0 \cap \Delta(\mathcal{U}))$ is norm-dense [\[4\]](#page-33-3). Therefore $\mathcal{U}_0 \cap \Delta(\mathcal{U})$ = 0.

This shows that (i)and (ii) are equivalent.

The proof of the equivalence of (i) and (iii) is analogous. \Box

4 The diagonal

Let $\mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi$ be as in section 3 and $\chi = \text{Map}(\Delta(\mathcal{U})).$

Theorem 4.1 *There exists a partial isometry* $V \in \Delta(U)$ *such that* $\Delta(U)$ = $[\mathcal{A}_2 V \mathcal{A}_1]^{-w^*}$ (recall that $\mathcal{A}_i = (\mathcal{S}_{i,\phi})'.$

Proof

If $T \in \Delta(U)$ and $T = U|T|$ is the polar decomposition of T, then $U \in$ $\Delta(\mathcal{U})$ and $|T| \in \mathcal{A}_1$: Proposition 2.6 in [\[9\]](#page-34-3).

By Zorn's lemma there exists a maximal family of partial isometries $(V_n) \subset \Delta(\mathcal{U})$ such that: $V_n^* V_n \perp V_m^* V_m$, $V_n V_n^* \perp V_m V_m^*$ for $n \neq m$.

Let $V = \sum_{n=1}^{\infty} V_n$. Then V is a partial isometry in $\Delta(\mathcal{U})$.

First we show that

$$
\Delta(\mathcal{U}) = \{ T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{A}'_1), F \in \mathcal{P}(\mathcal{A}'_2), FVE = 0 \Rightarrow FTE = 0 \}
$$
\n(4.1)

Let T be such that, if $FVE = 0$ for $E \in \mathcal{P}(\mathcal{A}'_1)$ and $F \in \mathcal{P}(\mathcal{A}'_2)$, then $FTE = 0$. Since $\phi(P)^{\perp}VP = \phi(P)VP^{\perp} = 0$ for every $P \in \mathcal{S}_{1,\phi}$ and $\mathcal{S}_{i,\phi} \subset$ $\mathcal{A}'_i, i = 1,2$ we have $\phi(P)^{\perp}TP = \phi(P)TP^{\perp} = 0$ for every $P \in \mathcal{S}_{1,\phi}$ so $T \in \Delta(\mathcal{U})$.

For the converse let $T \in \Delta(U)$ and $T = U|T|$ be the polar decomposition of T.

If $E \in \mathcal{P}(\mathcal{A}'_1), F \in \mathcal{P}(\mathcal{A}'_2)$ are such that $FVE = 0$, since $|T| \in \mathcal{A}_1$, we have $FTE = FU|T|E = FUE|T|$.

Hence it suffices to show that $FUE = 0$.

We observe that:

$$
V^*V(FUE)^*FUE = (V^*V)EU^*FUE = E(V^*V)U^*FUE \quad (V^*V \in \mathcal{A}_1)
$$

=
$$
EV^*(VU^*)FUE = EV^*F(VU^*)UE \quad (VU^* \in \mathcal{A}_2)
$$

=
$$
0VU^*UE = 0
$$

hence

$$
(FUE)^* FUE \le I - V^*V.
$$
\n
$$
(4.2)
$$

Similarly, one shows that

$$
FUE(FUE)^* \le I - VV^*.
$$
\n
$$
(4.3)
$$

Since FUE is a partial isometry in $\Delta(\mathcal{U})$, the maximality of V and $(4.2),(4.3)$ imply that $FUE = 0$.

Thus claim (4.1) holds.

Let $\mathcal{M} = [\mathcal{A}_2 V \mathcal{A}_1]^{-w^*}.$

We observe that M is a TRO which is contained in $\Delta(\mathcal{U})$. Since M is w^{*} closed, it is reflexive.

If $\zeta = \text{Map}(\mathcal{M})$ then for every projection P,

$$
\zeta(P) = [A_2 V A_1 P y : A_i \in \mathcal{A}_i, i = 1, 2, y \in H_1]^{-}.
$$

We observe that $\zeta(P) \in \mathcal{A}'_2$ for every projection P so $\mathcal{S}_{2,\zeta} \subset \mathcal{A}'_2$. Similarly if $\zeta^* = \text{Map}(\mathcal{M}^*)$ then $\mathcal{S}_{2,\zeta^*} \subset \mathcal{A}'_1$ but $\mathcal{S}_{1,\zeta} = \{P^\perp : P \in \mathcal{S}_{2,\zeta^*}\}\$ so we have that $S_{1,\zeta} \subset \mathcal{A}'_1$.

Now since $V \in \mathcal{M}$ we conclude that $\zeta(P)^{\perp}VP = 0$ for every $P \in \mathcal{S}_{1,\zeta}$.

From claim (4.1) we obtain $\zeta(P)^{\perp} \Delta(\mathcal{U})P = 0$ for every $P \in \mathcal{S}_{1,\zeta}$, so since $\mathcal M$ is reflexive $\Delta(\mathcal U) \subset \mathcal M$. \square

By the previous theorem it follows that if $\mathcal M$ is a w^{*}-closed TRO masa bimodule and $\zeta = \text{Map}(\mathcal{M})$ then there exists a partial isometry $V \in \mathcal{M}$ so that $\mathcal{M} = [(\mathcal{S}_{2,\zeta})'V(\mathcal{S}_{1,\zeta})']^{-w^*}.$

But we shall prove a stronger result:

Theorem 4.2 *Let* \mathcal{M} *a* w^* -*closed TRO* masa bimodule and $\mathcal{B}_1 = [\mathcal{M}^*\mathcal{M}]^{-w^*}$ $B_2 = [MM^*]^{-w^*}$. Then there exists a partial isometry V such that $\mathcal{M} =$ $[\mathcal{B}_2 V \mathcal{B}_1]^{-w^*}.$

Proof

Let $\mathcal{D}_i \subset B(H_i), i = 1, 2$ be mass such that $\mathcal{D}_2\mathcal{M}\mathcal{D}_1 \subset \mathcal{M}$ and put $\zeta = \text{Map}(\mathcal{M}).$

We shall prove that $\mathcal{B}'_2\mathcal{MB}'_1 \subset \mathcal{M}$.

In [\[9\]](#page-34-3), Theorem 2.10 it is shown that

$$
\mathcal{B}'_2 = (\mathcal{M}\mathcal{M}^*)' \subset \mathcal{D}_2|_{\zeta(I)} \oplus B(\zeta(I)^{\perp}(H_2))
$$

and

$$
\mathcal{B}'_1 = (\mathcal{M}^*\mathcal{M})' \subset \mathcal{D}_1|_{\zeta^*(I)} \oplus B(\zeta^*(I)^{\perp}(H_1)).
$$

So it suffices to show that

$$
(\mathcal{D}_2|_{\zeta(I)}\oplus B(\zeta(I)^{\perp}(H_2)))\mathcal{M}(\mathcal{D}_1|_{\zeta^{*}(I)}\oplus B(\zeta^{*}(I)^{\perp}(H_1)))\subset \mathcal{M}.
$$

But this is true because $\mathcal{D}_2\mathcal{M}\mathcal{D}_1\subset \mathcal{M}, \zeta(I)\in \mathcal{D}_2, \zeta^*(I)\in \mathcal{D}_1$ and $\mathcal{M}=$ $\zeta(I)\mathcal{M}\zeta^*(I).$

Now, we shall follow the proof of the previous theorem: By Zorn's lemma there exists a maximal family of partial isometries $(V_n) \subset$ M such that: $V_n^* V_n \perp V_m^* V_m$, $V_n V_n^* \perp V_m V_m^*$ for $n \neq m$. Let $V = \sum_{n=1}^{\infty} V_n$. Then V is a partial isometry in M. We shall show that

$$
\mathcal{M} \subset \{ T \in B(H_1, H_2) : E \in \mathcal{P}(\mathcal{B}'_1), F \in \mathcal{P}(\mathcal{B}'_2), FVE = 0 \Rightarrow FTE = 0 \}
$$
\n
$$
(4.4)
$$

Let $T \in \mathcal{M}$ and $T = U|T|$ be the polar decomposition of T. Then $|T| \in$ $(\mathcal{M}^*\mathcal{M})''$ and $U \in \mathcal{M}$, (Proposition 2.6 in [\[9\]](#page-34-3)).

If $E \in \mathcal{P}(\mathcal{B}'_1)$, $F \in \mathcal{P}(\mathcal{B}'_2)$ are such that $FVE = 0$, since $|T| \in (\mathcal{M}^*\mathcal{M})''$ and $E \in \mathcal{B}'_1 = (\mathcal{M}^*\mathcal{M})'$, we have $FTE = FU|T|E = FUE|T|$. Hence it suffices to show that $FUE = 0$.

As in the proof of the previous theorem we have that $V^*V \perp (FUE)^*(FUE)$ and $VV^* \perp (FUE) (FUE)^*$.

But $FUE \in \mathcal{B}'_2\mathcal{MB}'_1 \subset \mathcal{M}$, so by the maximality of V we have that $FUE = 0.$

Let $W = [\mathcal{B}_2 V \mathcal{B}_1]^{-w^*}$. We observe that $W \subset \mathcal{M}$.

For the converse, we follow the proof of the previous theorem and we use the relation (4.4)

An alternative proof of the previous theorem was communicated to us by I. Todorov, based on his paper [\[14\]](#page-34-4).

Theorem 4.3 *The semilattices of* $\Delta(U)$ *are the following:*

$$
\mathcal{S}_{1,\chi} = \chi^*(I)^{\perp} \oplus \chi^*(I) \mathcal{P}((\mathcal{S}_{1,\phi})'')
$$

$$
\mathcal{S}_{2,\chi} = \chi(I) \mathcal{P}((\mathcal{S}_{2,\phi})'').
$$

The map $\chi : \mathcal{S}_{1,\chi} \longrightarrow \mathcal{S}_{2,\chi}$ *is such that*

$$
\chi(\chi^*(I)^{\perp} \oplus \chi^*(I)Q) = \chi(I)\phi(Q) \quad \text{for every } Q \in \mathcal{S}_{1,\phi}.
$$
 (4.5)

Proof

i) In Theorem [4.1](#page-11-0) we showed that there exists a partial isometry V in $\Delta(\mathcal{U})$ such that $\Delta(\mathcal{U}) = [(\mathcal{S}_{2,\phi})'V(\mathcal{S}_{1,\phi})']^{-w^*}.$

So if $P \in \mathcal{S}_{1,\chi}$ then $\chi(P)$ is the projection onto $[(\mathcal{S}_{2,\phi})^{\prime}V(\mathcal{S}_{1,\phi})^{\prime}P(H_1)]^{-}$.

We conclude that $\chi(P) \in (\mathcal{S}_{2,\phi})^{\prime\prime}$. Hence $\mathcal{S}_{2,\chi} \subset (\mathcal{S}_{2,\phi})^{\prime\prime}$.

If H is a Hilbert space, \mathcal{B} is a subset of $B(H)$ and Q a projection in \mathcal{B}' the set ${T|_{Q(H)} : T \in \mathcal{B}}$ is denoted by $\mathcal{B}|_Q$.

We have shown that $(\mathcal{S}_{2,\chi})''|_{\chi(I)} \subset (\mathcal{S}_{2,\phi})''|_{\chi(I)}$. Let $P \in \mathcal{S}_{1,\phi}$ then $\Delta(\mathcal{U})P = \phi(P)\Delta(\mathcal{U})$. Hence $\chi(P) = \phi(P)\chi(I)$. So $\chi(I)\mathcal{S}_{2,\phi} \subset \mathcal{S}_{2,\chi}$ hence, $(\mathcal{S}_{2,\phi})''|_{\chi(I)} \subset (\mathcal{S}_{2,\chi})''|_{\chi(I)}$.

We proved that

$$
(\mathcal{S}_{2,\phi})''|_{\chi(I)} = (\mathcal{S}_{2,\chi})''|_{\chi(I)}.
$$

Since $\Delta(\mathcal{U})$ is a TRO, using Theorem 2.10 in [\[9\]](#page-34-3) (see the proof) we have that

$$
\mathcal{S}_{2,\chi}|_{\chi(I)}=\mathcal{P}((\mathcal{S}_{2,\chi})''|_{\chi(I)}).
$$

It follows that

$$
\mathcal{S}_{2,\chi}=\chi(I)\mathcal{P}((\mathcal{S}_{2,\phi})'').
$$

Applying this to $\Delta(\mathcal{U})^* = \Delta(\mathcal{U}^*)$,

$$
\mathcal{S}_{2,\chi^*}=\chi^*(I)\mathcal{P}((\mathcal{S}_{2,\phi^*})'').
$$

Since $\mathcal{S}_{1,\phi} = \{Q^{\perp} : Q \in \mathcal{S}_{2,\phi^*}\}\)$, see the introduction, we have that

$$
\mathcal{S}_{2,\chi^*} = \chi^*(I) \mathcal{P}((\mathcal{S}_{1,\phi})'').
$$

But

$$
\mathcal{S}_{1,\chi} = \{Q^{\perp} : Q \in \mathcal{S}_{2,\chi^*}\} = \{(\chi^*(I)Q)^{\perp} : Q \in \mathcal{P}((\mathcal{S}_{1,\phi})'')\} \\
= \{\chi^*(I)^{\perp} \oplus \chi^*(I)Q : Q \in \mathcal{P}((\mathcal{S}_{1,\phi})'')\}.
$$

ii) If $Q \in \mathcal{S}_{1,\phi}$ then

$$
\chi(\chi^*(I)^{\perp} \oplus \chi^*(I)Q) = \chi(\chi^*(I)Q) \qquad (\chi(\chi^*(I)^{\perp}) = 0)
$$

= $\chi(Q)$ $(\Delta(\mathcal{U})\chi^*(I) = \Delta(\mathcal{U}))$
= $\phi(Q)\chi(I)$. $(\Delta(\mathcal{U})Q = \phi(Q)\Delta(\mathcal{U}))$

 \Box

Remark 4.4 *The smallest ortholattice containing the commutative family* $\chi(I)\mathcal{S}_{2,\phi}$ is easily seen to be $\chi(I)\mathcal{P}((\mathcal{S}_{2,\phi})'')$, which equals $\mathcal{S}_{2,\chi}$; similarly the $family \ \chi^*(I)^{\perp} \oplus \chi^*(I) \mathcal{S}_{1,\phi}$ generates the complete ortho-lattice $\mathcal{S}_{1,\chi}$.

Therefore, since $\chi|_{\mathcal{S}_{1,\chi}}$ *is a complete ortho-lattice isomorphism (Theorem*) *2.10 in [\[9\]](#page-34-3)) equality [\(4.5\)](#page-14-0) determines the map* χ.

Proposition 4.5 *The families* $\chi^*(I)S_{1,\phi}$ *and* $\chi(I)S_{2,\phi}$ *are complete lattices and the map*

$$
\vartheta : \chi^*(I)\mathcal{S}_{1,\phi} \to \chi(I)\mathcal{S}_{2,\phi} : \vartheta(\chi^*(I)P) = \chi(I)\phi(P)
$$

is a complete lattice isomorphism.

Proof

We use Theorem [4.3](#page-14-1) and the fact [\[9\]](#page-34-3) that the map $\chi|_{\mathcal{S}_{1,\chi}}$ is a complete ortholattice isomorphism .

Let $(P_i)_{i\in I} \subset \mathcal{S}_{1,\phi}$. We claim that

$$
\wedge_{i \in I} \chi(I)\phi(P_i) = \chi(I)\phi(\wedge_{i \in I} P_i). \tag{4.6}
$$

Indeed, by (4.5) ,

$$
\begin{aligned} \wedge_{i\in I}\chi(I)\phi(P_i) &= \wedge_{i\in I}\chi(\chi^*(I)^{\perp} \oplus \chi^*(I)P_i) \\ &= \chi(\wedge_{i\in I}(\chi^*(I)^{\perp} \oplus \chi^*(I)P_i)) = \chi(\chi^*(I)^{\perp} \oplus \chi^*(I)(\wedge_{i\in I}P_i)). \end{aligned}
$$

Since $\wedge_{i\in I} P_i \in S_{1,\phi}$ we get that $\chi(\chi^*(I)^{\perp} \oplus \chi^*(I)(\wedge_{i\in I} P_i)) = \chi(I)\phi(\wedge_{i\in I} P_i)$ again using [\(4.5\)](#page-14-0).

By [\(1.1\)](#page-3-0), there exist $(Q_i)_{i\in I} \subset S_{1,\phi^*}$ such that $\phi^*(Q_i)^{\perp} = P_i$ for every $i \in I$.

We shall prove that

$$
\vee_{i\in I}\chi^*(I)P_i = \chi^*(I)(\phi^*(\wedge_{i\in I}Q_i))^{\perp}.
$$
\n(4.7)

Since $\Delta(\mathcal{U}^*) = \Delta(\mathcal{U})^*$ we have that $\chi^* = Map(\Delta(\mathcal{U}^*))$ and so applying equation [\(4.6\)](#page-16-0) to χ^* we have that

$$
\wedge_{i\in I}\chi^*(I)\phi^*(Q_i) = \chi^*(I)\phi^*(\wedge_{i\in I}Q_i) \Rightarrow
$$

$$
\vee_{i\in I}(\chi^*(I)\phi^*(Q_i))^{\perp} = (\chi^*(I)\phi^*(\wedge_{i\in I}Q_i)^{\perp} \Rightarrow
$$

$$
\vee_{i\in I}(\chi^*(I)^{\perp} \oplus \chi^*(I)(\phi^*(Q_i))^{\perp}) = \chi^*(I)^{\perp} \oplus \chi^*(I)(\phi^*(\wedge_{i\in I}Q_i))^{\perp} \Rightarrow
$$

$$
\vee_{i\in I}(\chi^*(I)^{\perp} \oplus \chi^*(I)P_i) = \chi^*(I)^{\perp} \oplus \chi^*(I)(\phi^*(\wedge_{i\in I}Q_i))^{\perp} \Rightarrow
$$

$$
\vee_{i\in I}\chi^*(I)P_i = \chi^*(I)(\phi^*(\wedge_{i\in I}Q_i))^{\perp}.
$$

From equalities (4.6) and (4.7) we conclude that the families $\chi^*(I)S_{1,\phi}$, $\chi(I)\mathcal{S}_{2,\phi}$ are complete lattices.

Since $\chi(\chi^*(I)^{\perp} \oplus \chi^*(I)Q) = \chi(I)\phi(Q)$ for every $Q \in \mathcal{S}_{1,\phi}$ and $\chi|_{\mathcal{S}_{1,\chi}}$ is $1 - 1$ the map ϑ is a bijection.

It remains to show that ϑ is sup and inf continuous.

Let $(P_i)_{i \in I} \subset S_{1,\phi}$ and $(Q_i)_{i \in I} \subset S_{1,\phi^*}$ be such that $\phi^*(Q_i)^{\perp} = P_i$, equivalently by equation (1.1) $\phi(P_i)^{\perp} = Q_i$ for every $i \in I$. Then, since $\wedge_{i\in I}P_i\in\mathcal{S}_{1,\phi}$, by the definition of ϑ we have

$$
\vartheta(\wedge_{i\in I}\chi^*(I)P_i) = \vartheta(\chi^*(I)(\wedge_{i\in I}P_i)) = \chi(I)\phi(\wedge_{i\in I}P_i)
$$

=
$$
\wedge_{i\in I}\chi(I)\phi(P_i) = \wedge_{i\in I}\vartheta(\chi^*(I)P_i).
$$

Using equations (4.7) and (1.1) we have that

$$
\vartheta(\vee_{i\in I}\chi^*(I)P_i) = \vartheta(\chi^*(I)(\phi^*(\wedge_{i\in I}Q_i))^{\perp}) = \chi(I)\phi((\phi^*(\wedge_{i\in I}Q_i))^{\perp})
$$

= $\chi(I)(\wedge_{i\in I}Q_i)^{\perp} = \vee_{i\in I}\chi(I)Q_i^{\perp}$
= $\vee_{i\in I}\chi(I)\phi(P_i) = \vee_{i\in I}\vartheta(\chi^*(I)P_i).$

5 The space \mathcal{U}_0 is reflexive.

Let $\mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi$ be as in section 3 and $\chi = \text{Map}(\Delta(\mathcal{U})), \psi = \text{Map}(\mathcal{U}_0)$.

Lemma 5.1 *If* $\Delta(\mathcal{U})$ *is essential, i.e.* $\chi(I) = I, \chi^*(I) = I$, *then* $\mathcal{S}_{1,\psi} \subset \mathcal{S}_{1,\phi}$ *and* $\mathcal{S}_{2,\psi} \subset \mathcal{S}_{2,\phi}$.

Proof

Since $\chi(I) = I$ we have $\phi(I) = I$, so by Proposition 4.5, $\mathcal{S}_{2,\phi}$ is a C.S.L. Since $\chi^*(I) = I$, \mathcal{S}_{2,ϕ^*} is a C.S.L. and so $\mathcal{S}_{1,\phi}$ is a C.S.L.

If E is a projection, then $Alg(\mathcal{S}_{2,\phi})\mathcal{U}_0E \subset \mathcal{U}_0E$ (Lemma 3.5). It follows that $\psi(E)^{\perp}Alg(\mathcal{S}_{2,\phi})\psi(E) = 0$. Hence $\psi(E) \in Lat(Alg(\mathcal{S}_{2,\phi}))$. Since commutative subspace lattices are reflexive [\[1\]](#page-33-4), it follows that $\psi(E) \in$ $\mathcal{S}_{2,\phi}$. We get that $\mathcal{S}_{2,\psi} \subset \mathcal{S}_{2,\phi}$. Analogously $\mathcal{U}_0Alg(\mathcal{S}_{1,\phi}) \subset Alg(\mathcal{S}_{1,\phi})$ so $Alg(\mathcal{S}_{1,\phi}^{\perp})\mathcal{U}_0^* \subset \mathcal{U}_0^*$. As above we obtain $\mathcal{S}_{2,\psi^*} \subset \mathcal{S}_{1,\phi}^{\perp}$ hence $\mathcal{S}_{1,\psi} \subset \mathcal{S}_{1,\phi}$. \Box

Theorem 5.2 *The space* U_0 *is reflexive*

Proof

Firstly, we suppose that $\Delta(\mathcal{U})$ is essential $(\chi(I) = I, \chi^*(I) = I)$. Now, by Theorem 4.3 we have that $\mathcal{S}_{1,\chi} = \mathcal{P}((\mathcal{S}_{1,\phi})'')$, $\mathcal{S}_{2,\chi} = \mathcal{P}((\mathcal{S}_{2,\phi})'')$ and $\chi|_{\mathcal{S}_{1,\phi}} = \phi.$

If $E \in \mathcal{S}_{1,\phi}$, then $\phi(E), \psi(E) \in \mathcal{P}((\mathcal{S}_{2,\phi})'')$ so there exists a unique $F \in$ $\mathcal{P}((\mathcal{S}_{1,\phi})'')$ such that $\chi(F) = \phi(E) - \psi(E)$.

We observe that $\chi(F) \leq \phi(E) = \chi(E)$. Since χ is a lattice isomorphism $F \leq E$ and so $\psi(F) \leq \psi(E)$; therefore $\chi(F) \perp \psi(F)$.

Since $\chi = \text{Map}(\Delta(\mathcal{U}))$ and $\psi = \text{Map}(\mathcal{U}_0)$ we obtain that $\Delta(\mathcal{U})F(H_1)\perp \mathrm{Ref}(\mathcal{U}_0)F(H_1)$ and so $\Delta(\mathcal{U})F \cap \mathrm{Ref}(\mathcal{U}_0)F = 0.$ By Theorem 3.1 $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$, hence $\mathcal{U}F = \text{Ref}(\mathcal{U}_0)F \oplus \Delta(\mathcal{U})F$ and $\mathcal{U}F = \mathcal{U}_0F \oplus \Delta(\mathcal{U})F.$

It follows that $\mathcal{U}_0 F = \text{Ref}(\mathcal{U}_0) F$ and so $\mathcal{U}_0 F$ is reflexive.

Let

$$
P=\vee\{F\in\mathcal{P}((\mathcal{S}_{1,\phi})''):\chi(F)=\phi(E)-\psi(E), E\in\mathcal{S}_{1,\phi}\}.
$$

By the previous arguments the space \mathcal{U}_0 is reflexive. Since χ is ∨−continuous we have that

$$
\chi(P) = \vee \{ \phi(E) - \psi(E), E \in \mathcal{S}_{1,\phi} \}.
$$

Let $Q = \chi(P)^{\perp}$ then $Q\phi(E) = Q\psi(E)$ for all $E \in \mathcal{S}_{1,\phi}$. Therefore, it follows that

$$
Q\mathcal{U} = \{T : Q\phi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}_{1,\phi}\} =
$$

$$
= \{T : Q\psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}_{1,\phi}\}.
$$

Using the previous lemma $(S_{1,\psi} \subset S_{1,\phi})$ we obtain that $Q\mathcal{U}$ is contained in the space:

$$
\{T: Q\psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}_{1,\psi}\} =
$$

= $Q = \text{Ref}(\mathcal{U}_0) = \text{Ref}(Q\mathcal{U}_0) \subset Q\mathcal{U}.$

We proved that $Q\mathcal{U} = \text{Ref}(Q\mathcal{U}_0)$.

Katavolos and Todorov [\[9\]](#page-34-3) have proved that $\Delta(\mathcal{U}) \subset (\mathcal{U})_{min}$ where $(\mathcal{U})_{min}$ is the smallest w^* -closed masa bimodule such that $\text{Ref}((\mathcal{U})_{min}) = \mathcal{U}$. So $Q\Delta(\mathcal{U})$ \subset $Q(\mathcal{U})_{min}$ = $(Q\mathcal{U})_{min}$. But since $Q\mathcal{U}_0$ is a w^{*}-closed masa bimodule such that Ref(QU_0) = QU it follows that $Q\Delta(U) \subset QU_0$. Now $Q\Delta(\mathcal{U}) = \chi(P)^{\perp} \Delta(\mathcal{U}) = \Delta(\mathcal{U}) P^{\perp}$, hence $\Delta(\mathcal{U}) P^{\perp} \subset \mathcal{U}_0$.

So $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})P^{\perp} + \Delta(\mathcal{U})P = \mathcal{U}_0 + \Delta(\mathcal{U})P$ and so $\mathcal{U}P^{\perp} = \mathcal{U}_0P^{\perp}$. We conclude that $\mathcal{U}_0 P^{\perp}$ is reflexive. Since $\mathcal{U}_0 P$ is reflexive too, \mathcal{U}_0 is reflexive.

Now, relax the assumption that $\Delta(\mathcal{U})$ is essential.

Let $W = \chi(I) \mathcal{U}|_{\chi^*(I)}$. This is a masa bimodule in $B(\chi^*(I)(H_1), \chi(I)(H_2))$. We have that

$$
\mathcal{W} = \{ T : \chi(I)\phi(L)^{\perp}TL|_{\chi^*(I)} = 0 \text{ for all } L \in \mathcal{S}_{1,\phi} \}.
$$

By Proposition 4.5 the families $S_{1,\phi}|_{\chi^*(I)}, S_{2,\phi}|_{\chi(I)}$ are complete lattices and the map $S_{1,\phi}|_{\chi^*(I)} \to S_{2,\phi}|_{\chi(I)} : P|_{\chi^*(I)} \to \phi(P)|_{\chi(I)}$ is a complete lattice isomorphism.

By the Lifting theorem of J.Erdos [\[5\]](#page-33-2) it follows that the (semi)lattices of W are the families $\mathcal{S}_{1,\phi}|_{\chi^*(I)}, \mathcal{S}_{2,\phi}|_{\chi(I)}.$

Therefore, $\mathcal{W}_0 = [\chi(I)\phi(L)TL^{\perp}|_{\chi^*(I)} : T \in \mathcal{W}, L \in \mathcal{S}_{1,\phi}]^{-w^*} = \chi(I)\mathcal{U}_0|_{\chi^*(I)}.$

By the proof in the essential case we have that the spase $\chi(I)\mathcal{U}_0\chi^*(I)$ is reflexive.

But $\chi(I)^{\perp}U = \chi(I)^{\perp}U_0$ and $\mathcal{U}\chi^*(I)^{\perp} = \mathcal{U}_0\chi^*(I)^{\perp}$ so the spaces $\chi(I)^{\perp}\mathcal{U}_0$ and $\mathcal{U}_0 \chi^*(I)^\perp$ are reflexive.

Finally the space \mathcal{U}_0 is reflexive. \Box

For the rest of this section let S be a C.S.L. $\mathcal{U} = Alg(\mathcal{S}), \mathcal{J} = [PTP^{\perp} :$ $T \in \mathcal{U}, P \in \mathcal{S}]^{-\|\cdot\|}, Rad(\mathcal{U})$ be the radical of $\mathcal{U}, \mathcal{U}_0 = \mathcal{J}^{-w^*}, \psi = \text{Map}(\mathcal{U}_0)$. It is known that $\mathcal{J} \subset \text{Rad}(\mathcal{U})$. The equality $\mathcal{J} = \text{Rad}(\mathcal{U})$ is an open problem (Hopenwasser's conjecture), [\[8\]](#page-33-5), [\[3\]](#page-33-6).

I.Todorov [\[13\]](#page-34-5) has proved that $\mathcal J$ and $Rad(\mathcal U)$ have the same reflexive hull. We improve this by showing the next corollary.

Corollary 5.3 *The spaces* $\mathcal J$ *and* $Rad(\mathcal U)$ *have the same* w^* – *closure.*

Proof

$$
\mathcal{U}_0 = \mathcal{J}^{-w^*} \subset \text{Rad}(\mathcal{U})^{-w^*} \subset \text{Ref}(\text{Rad}(\mathcal{U})) = \text{Ref}(\mathcal{J}) = \mathcal{U}_0.
$$

Corollary 5.4 $Rad(\mathcal{U})^{-w^*} = \{T : \psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}\}.$

Proof

 \Box

 $Rad(\mathcal{U})^{-w^*} = \mathcal{U}_0 = \{T : \psi(E)^{\perp}TE = 0 \text{ for every projection } E\} \subset$ $\{T: \psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}\}.$ Using Lemma 5.1 the last space is contained in the space: ${T : \psi(E)^\perp TE = 0 \text{ for all } E \in \mathcal{S}_{1,\psi} } = \mathcal{U}_0 = Rad(\mathcal{U})^{-w^*}$ \Box

Now we are ready to give the form of the decomposition of $\mathcal U$ in the case that U is a C.S.L. algebra:

Proposition 5.5 *Let* $Q = \vee \{E - \psi(E) : E \in \mathcal{S}\}\$ *then*

$$
\mathcal{U} = Rad(\mathcal{U})^{-w^*} \oplus Q\mathcal{S}'.
$$

Proof

We observe that $Q^{\perp}E = Q^{\perp}\psi(E)$ for all $E \in \mathcal{S}$, so we have:

$$
Q^{\perp}U = \{T : Q^{\perp}E^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}\}
$$

$$
= \{T : Q^{\perp}\psi(E)^{\perp}TE = 0 \text{ for all } E \in \mathcal{S}\}.
$$

By the previous corollary the last space is the space $Q^{\perp}Rad(\mathcal{U})^{-w^*}$. So we have that $Q^{\perp}S' \subset Q^{\perp}Rad(\mathcal{U})^{-w^*} \subset Rad(\mathcal{U})^{-w^*}.$ Since $\mathcal{U} = Rad(\mathcal{U})^{-w^*} + \mathcal{S}'$ we have $\mathcal{U} = Rad(\mathcal{U})^{-w^*} + Q\mathcal{S}'.$

It suffices to show that $Rad(\mathcal{U})^{-w^*} \cap QS' = 0$. Let $E \in \mathcal{S}$ and $T \in \mathcal{U}_0 \cap (E - \psi(E))\mathcal{S}'$ then $T = (E - \psi(E))T$ $=\psi(E)^{\perp} E T = \psi(E)^{\perp} T E = 0$, because $T \in \mathcal{U}_0$. If $T \in \mathcal{U}_0 \cap Q\mathcal{S}' \Rightarrow (E - \psi(E))T \in \mathcal{U}_0 \cap (E - \psi(E))\mathcal{S}' = 0.$ So $(E - \psi(E))T = 0$ for all $E \in \mathcal{S}$. But $T = (\sqrt{E} - \psi(E) : E \in \mathcal{S})$ T. It follows that $T = 0$

6 Decomposition of compact operators in reflexive masa bimodules

Let $\mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi, \mathcal{D}_1, \mathcal{D}_2, Q_1$ be as in section 3 and $\chi = \text{Map}(\Delta(\mathcal{U}))$.

We denote by K the set of compact operators and by C_p the set of p-Schatten class operators in $B(H_1, H_2)$.

Proposition 6.1 *If* $T \in R_1(\mathcal{U})$ *, there exist* $L \in R_1(\Delta(\mathcal{U}))$ *and* $S \in [R_1(\mathcal{U}_0)]^{-\| \|_1}$ such that $T = L + S$.

Proof

Write $\mathcal{U} = \{X : \phi(P_n)^{\perp} X P_n = 0 \text{ for all } n \in \mathbb{N}\}\$ for an approriate sequence $(P_n) \subset S_{1,\phi}$ and let $T \in R_1(\mathcal{U})$.

As in the proof of Theorem 3.1

$$
T = L_1 + \phi(P_1)TP_1^{\perp}
$$
, where $L_1 = \phi(P_1)TP_1 + \phi(P_1)^{\perp}TP_1^{\perp}$.

Since $\phi(P_1)^{\perp}TP_1 = 0$ and T has rank 1 either $\phi(P_1)^{\perp}T = 0$ or $TP_1 = 0$, hence either $L_1 = \phi(P_1)TP_1$ or $L_1 = \phi(P_1)^{\perp}TP_1^{\perp}$.

$$
L_1 = L_2 + \phi(P_2)L_1 P_2^{\perp}, \text{where } L_2 = \phi(P_2)L_1 P_2 + \phi(P_2)^{\perp} L_1 P_2^{\perp}.
$$

Since $\phi(P_2)^{\perp}L_1P_2 = 0$, either $L_2 = \phi(P_2)L_1P_2$ or $L_2 = \phi(P_2)^{\perp}L_1P_2^{\perp}$. Similarly

$$
L_{n-1} = L_n + \phi(P_n)L_{n-1}P_n^{\perp}, \text{where } L_n = \phi(P_n)L_{n-1}P_n + \phi(P_n)^{\perp}L_{n-1}P_n^{\perp}.
$$

As before, either $L_n = \phi(P_n)L_{n-1}P_n$ or $L_n = \phi(P_n)^{\perp}L_{n-1}P_n^{\perp}$ for all $n \in \mathbb{N}$.

We conclude that there exist projections $(Q_n) \subset \mathcal{D}_2$, $(R_n) \subset \mathcal{D}_1$ such that $L_n = (\wedge_{i=1}^n Q_i) T(\wedge_{i=1}^n R_i), n \in \mathbb{N}.$

We observe that $T = L_n + M_n$ where $M_n = \phi(P_1) T P_1^{\perp} + \phi(P_2) L_1 P_2^{\perp} +$ $\ldots + \phi(P_n)L_{n-1}P_n^{\perp}, n \in \mathbb{N}.$

Since $\wedge_{i=1}^n Q_i \stackrel{soft}{\rightarrow} \wedge_{i=1}^{\infty} Q_i$, $\wedge_{i=1}^n R_i \stackrel{soft}{\rightarrow} \wedge_{i=1}^{\infty} R_i$ and T has rank 1

$$
L_n \stackrel{\parallel \parallel_1}{\rightarrow} (\wedge_{i=1}^{\infty} Q_i) T(\wedge_{i=1}^{\infty} R_i) = L
$$
, say.

Now $\phi(P_i)^{\perp} L_n P_i = \phi(P_i) L_n P_i^{\perp} = 0, i = 1, 2, \dots n$ for all $n \in \mathbb{N}$, therefore $\phi(P_i)^{\perp} L P_i = \phi(P_i) L P_i^{\perp} = 0$ for all $i \in \mathbb{N}$.

Thus $L \in R_1(\Delta(\mathcal{U}))$.

We have
$$
M_n = T - L_n \stackrel{\|\cdot\|_1}{\to} T - L = S \in [R_1(\mathcal{U}_0)]^{-\|\cdot\|_1}
$$
. \Box

Proposition 6.2 $\mathcal{U}_0 \subset (R_1(\Delta(\mathcal{U}))^*)^0$.

Proof

Let
$$
T \in \mathcal{U}
$$
, $P \in \mathcal{S}_{1,\phi}$, $R \in R_1(\Delta(\mathcal{U}))$. Then
\n
$$
tr(\phi(P)TP^{\perp}R^*) = tr(T(\phi(P)RP^{\perp})^*) = tr(T0) = 0.
$$

Taking the w^{*} closed linear span we get $tr(SR^*) = 0$ for every $S \in$ \mathcal{U}_0 . \Box

Proposition 6.3 *i*) $R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1)$. ii) $\Delta(\mathcal{U}) \cap \mathcal{K} = [R_1(\Delta(\mathcal{U}))]^{-\|\cdot\|} \subset \Delta(\mathcal{U})(I - Q_1).$

Proof

Let $R \in R_1(\Delta(\mathcal{U}))$ then as in Theorem 3.3 $RQ_1 \in \Delta(\mathcal{U})Q_1 = \mathcal{U}_0 \cap \Delta(\mathcal{U}) \subset$ \mathcal{U}_0 . By the previous proposition we have: $tr(RQ_1R^*) = 0 \Rightarrow tr(R^*RQ_1) = 0 \Rightarrow$ $RQ_1 = 0 \Rightarrow R = R(I - Q_1).$ We conclude that $R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1)$.

For part (ii), observe that if $K \in \Delta(U) \cap \mathcal{K}$ then K can be approximated in the norm topology by sums of rank 1 operators in $\Delta(\mathcal{U})$: Proposition 3.4 in [\[9\]](#page-34-3). \Box

Remark 6.4 We will see below that if U is a strongly reflexive masa bi*module then* $[R_1(\Delta(\mathcal{U}))]^{-w^*} = \Delta(\mathcal{U})(I - Q_1)$. *This is not true in general. For example take* U *to be a* T RO *which is not strongly reflexive. Then* $[R_1(\Delta(\mathcal{U}))]^{-w^*}$ is strictly contained in $\Delta(\mathcal{U})(I - Q_1) = \mathcal{U}$.

Proposition 6.5 $\Delta(\mathcal{U}) \subset (R_1(\mathcal{U}_0)^*)^0$.

Proof

Let $T \in R_1(\mathcal{U}_0)$. Then as in Proposition 6.1 we have $T = L + M$ where

$$
L \in R_1(\Delta(\mathcal{U})) \text{ and } M \in [R_1(\phi(P_n)\mathcal{U}P_n^{\perp}) : n \in \mathbb{N}]^{-\|\cdot\|_1} \subset \mathcal{U}_0.
$$

So $L = T - M \in \mathcal{U}_0 \cap R_1(\Delta(\mathcal{U}))$.

Using Proposition 6.3, $U_0 \cap R_1(\Delta(\mathcal{U})) \subset U_0 \cap \Delta(\mathcal{U})(I-Q_1)$ which vanishes by Theorem 3.3 so $L = 0$ and hence $T = M$.

We conclude that

$$
R_1(\mathcal{U}_0) \subset [R_1(\phi(P_n)\mathcal{U}P_n^{\perp}) : n \in \mathbb{N}]^{-\| \| \cdot \|_1}.
$$
 (6.1)

Let $A \in \Delta(U)$. We want to show that $tr(A^*R) = 0$ for every $R \in R_1(\mathcal{U}_0)$.

Using (6.1) it suffices to show that $tr(A^*R) = 0$ for every $R \in R_1(\phi(P_n)UP_n^{\perp}),$ and $n \in \mathbb{N}$.

Let R a rank 1 operator such that $R = \phi(P_n)RP_n^{\perp}$ then

$$
tr(A^*R) = tr(A^*\phi(P_n)RP_n^{\perp}) = tr(P_n^{\perp}A^*\phi(P_n)R)
$$

$$
= tr((\phi(P_n)AP_n^{\perp})^*R) = tr(0R) = 0.
$$

Let $P \in \mathcal{S}_{1,\phi}$. We suppose that $\forall \{\phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P)\} < \phi(P)$. Since $\mathcal{S}_{2,\phi}$ is join complete there exists $P_0 \in \mathcal{S}_{1,\phi}$ such that

$$
\phi(P_0) = \vee \{ \phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P) \}.
$$

We call the projection $P - P_0$ an atom of U and we denote the projection $\phi(P) - \phi(P_0)$ by $\delta(P - P_0)$.

Proposition 6.6 *Let* F *be an atom of* U. *i)The projection* F *is minimal in the algebra* $(S_{1,\phi})''$. *ii*)The projection $\chi(I)\delta(F)$ *is minimal in the algebra* $\chi(I)(S_{2,\phi})''$. $iii) \chi(I) \delta(F) B(H_1, H_2) F \subset \Delta(U)$. $iv)\chi(I)^{\perp}\delta(F)B(H_1, H_2)F \subset \mathcal{U}_0.$

Proof

i) Let $P, P_0 \in S_{1,\phi}$ be such that $\phi(P_0) = \vee \{ \phi(L) : L \in S_{1,\phi}, \phi(L)$ $\phi(P)$ } $<$ $\phi(P)$ and $F = P - P_0$. If $Q \in \mathcal{S}_{1,\phi}$ either $P \leq Q$ or $QP < P$. If $P \leq Q$ then $QF = F$. If $QP < P$ then (since $QP \in \mathcal{S}_{1,\phi}$ and ϕ is 1-1 on $\mathcal{S}_{1,\phi}$) $\phi(QP) < \phi(P) \Rightarrow$ $\phi(QP) \leq \phi(P_0) \Rightarrow QP \leq P_0$, so $QF = 0$.

We conclude that $QFB(H_1)F = FB(H_1)QF$ for all $Q \in \mathcal{S}_{1,\phi}$, therefore $FB(H_1)F \subset (\mathcal{S}_{1,\phi})'$, hence F is a minimal projection in $(\mathcal{S}_{1,\phi})''$.

ii)Since $P, P_0 \in S_{1,\phi}$ we have that $\phi(P) \Delta(\mathcal{U}) = \Delta(\mathcal{U})P$ and $\phi(P_0) \Delta(\mathcal{U}) =$ $\Delta(\mathcal{U})P_0$ hence

$$
\delta(F)\Delta(U) = \Delta(U)F
$$
 and so $\chi(I)\delta(F) = \chi(F)$.

Let $Q \in \mathcal{S}_{1,\phi}$.

$$
If \t QF = 0 \t then \t \chi(I)\delta(F)\phi(Q) = 0. \t (6.2)
$$

Indeed, $\delta(F)\Delta(U) = \Delta(U)F$ so $\delta(F)\Delta(U)Q = 0$ so $\delta(F)\chi(Q) = 0$ so $\chi(I)\delta(F)\phi(Q) = 0.$

$$
If \qquad QF = F \Rightarrow \chi(I)\delta(F)\phi(Q) = \chi(I)\delta(F). \tag{6.3}
$$

Indeed, $\delta(F)\Delta(U) = \Delta(U)F$ so $\delta(F)\Delta(U)Q = \Delta(U)F$ so $\delta(F)\gamma(Q) =$ $\chi(F)$ so $\chi(I)\delta(F)\phi(Q) = \chi(I)\delta(F)$.

Using equations (6.2), (6.3) as in (i) we have that $\chi(I)\delta(F)$ is a minimal projection in $\chi(I)(\mathcal{S}_{2,\phi})''$.

iii)Let $T \in B(H_1, H_2)$ and $Q \in \mathcal{S}_{1,\phi}$. From equations (6.2), (6.3) it follows that $\phi(Q)\chi(I)\delta(F)TF = \chi(I)\delta(F)TFQ$, so $\chi(I)\delta(F)TF \in \Delta(U)$.

iv)If $T \in \mathcal{U}$ then $\chi(I)^{\perp}T \in \mathcal{U}_0$. Indeed, by Theorem 3.1 there exist $T_1 \in \mathcal{U}_0$, $T_2 \in \Delta(\mathcal{U})$ so that $T = T_1 + T_2$. But $T_2 = \chi(I)T_2$ so $\chi(I)^{\perp}T = \chi(I)^{\perp}T_1 \in \mathcal{U}_0$.

Now it suffices to show that $\delta(F)B(H_1, H_2)F \subset \mathcal{U}$. Let $T \in B(H_1, H_2)$ and $Q \in \mathcal{S}_{1,\phi}$. If $FQ = 0$ then $\phi(Q)^{\perp} \delta(F) T F Q = 0$. If $FQ = F$ then $P - P_0 \le Q$ hence $\delta(F) = \phi(P) - \phi(P_0) \le \phi(P - P_0) \le \phi(Q)$ so $\phi(Q)^{\perp} \delta(F) T F Q = 0.$ We conclude that $\delta(F)TF \in \mathcal{U}$. \Box

Remark 6.7 There exists a simple example of a reflexive masa bimodule \mathcal{U} so that $\delta(F)B(H_1, H_2)F \subset \mathcal{U}_0$ for any atom F in \mathcal{U} .

(Take U to be the set of 3×3 matrixes with zero diagonal.)

This is an example of the different behaviour of algebras and bimodules: it is known that if U is a CSL algebra in a Hilbert space H and F is an atom in U then $FB(H)F \subset \Delta(\mathcal{U})$.

We thank Dr. I.Todorov for suggesting the 'atomic decomposition' in the theorem below.

Theorem 6.8 *Let* ${F_n : n \in \mathbb{N} = {F : F atom of U}.$ *Then*

$$
[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \oplus \chi(I)\delta(F_n)B(H_1, H_2)F_n.
$$

Proof

By the previous proposition it follows that

$$
[R_1(\Delta(U))]^{-w^*} \supset \sum_{n=1}^{\infty} \bigoplus \chi(I) \delta(F_n) B(H_1, H_2) F_n.
$$

Let $R = x \otimes y^* \in \Delta(\mathcal{U})$. For every $Q \in \mathcal{S}_{1,\phi}$ we have that $x \otimes (Qy)^* = (\phi(Q)x) \otimes y^*$ so $\phi(Q)x \neq 0 \Leftrightarrow Qy \neq 0 \Leftrightarrow \phi(Q)x = x \Leftrightarrow Qy = y.$

Let $P = \wedge \{Q \in \mathcal{S}_{1,\phi}: Qy = y\}$, then $P \in \mathcal{S}_{1,\phi}$. If $Q \in \mathcal{S}_{1,\phi}$ so that $\phi(Q) < \phi(P)$ then $\phi(Q)x = 0$. (If $\phi(Q)x = x$ then $Qy = y$ so $Q \ge P$).

Let $P_0 \in \mathcal{S}_{1,\phi}$ with $\phi(P_0) = \vee \{ \phi(L) : L \in \mathcal{S}_{1,\phi}, \phi(L) < \phi(P) \}.$ We observe that $\phi(P_0)x = 0$ and $\phi(P)x = x$, hence $\phi(P_0) < \phi(P)$. We conclude that $F = P - P_0$ is an atom of U . The equalities $(P - P_0)y = y$ and $(\phi(P) - \phi(P_0))x = x$ imply that $R = \delta(F)RF$. But $R = \chi(I)R$ so $R = \chi(I)\delta(F)RF$. The proof is complete. \Box

Every strongly reflexive TRO is a masa bimodule [\[9\]](#page-34-3). So using the previous theorem we have a new proof of the following result in [\[9\]](#page-34-3).

Corollary 6.9 If M is a strongly reflexive TRO, $\zeta = \text{Map}(\mathcal{M})$ and ${A_n : n \in \mathbb{N} } = {A : A atom of \mathcal{M}}, then$

$$
\mathcal{M} = \sum_{n=1}^{\infty} \oplus \zeta(A_n) B(H_1, H_2) A_n.
$$

Let $(P_n) \subset \mathcal{S}_{1,\phi}$ be a sequence such that

$$
\mathcal{U} = \{ T \in B(H_1, H_2) : \phi(P_n)^{\perp} T P_n = 0 \text{ for all } n \in \mathbb{N} \}.
$$

Let $V_n, U_n : B(H_1, H_2) \longrightarrow B(H_1, H_2), n \in \mathbb{N}$ be as in theorem 3.1. By Theorem 6.8

$$
[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \oplus E_n B(H_1, H_2) F_n,
$$

where F_n atom of U and $E_n = \chi(I)\delta(F_n)$ for all $n \in \mathbb{N}$.

Thus $[R_1(\Delta(\mathcal{U}))]^{-w^*}$ is the range of the contractive projection D defined by

$$
D: B(H_1, H_2) \longrightarrow B(H_1, H_2): D(T) = \sum_{n=1}^{\infty} E_n T F_n.
$$

Proposition 6.10 *Let* $K \in \mathcal{K}$, *then the sequence* $(U_n(K))$ *converges to* D(K) *in norm.*

Proof

We observe that $(V_n|_{C_2})$ is a commuting sequence of orthogonal projections in the Hilbert space C_2 .

Hence $(U_n|_{C_2})$ is a decreasing sequence of orthogonal projections. Therefore if $T \in C_2$ the sequence $(U_n(T))$ converges in the Hilbert-Schmidt norm $\|\cdot\|_2.$

Let $K \in \mathcal{K}$. Then for $\varepsilon > 0$ there exist $K_{\varepsilon} \in C_2$ such that $||K - K_{\varepsilon}|| < \frac{\varepsilon}{3}$ 3 and $n_0 \in \mathbb{N}$ such that $||U_n(K_{\varepsilon}) - U_m(K_{\varepsilon})||_2 < \frac{\varepsilon}{3}$ $\frac{\varepsilon}{3}$ for every $n, m \geq n_0$.

Then

$$
||U_n(K) - U_m(K)||
$$

\n
$$
\leq ||U_n(K) - U_n(K_{\varepsilon})|| + ||U_n(K_{\varepsilon}) - U_m(K_{\varepsilon})|| + ||U_m(K_{\varepsilon}) - U_m(K)||
$$

\n
$$
\leq ||K - K_{\varepsilon}|| + ||U_n(K_{\varepsilon}) - U_m(K_{\varepsilon})||_2 + ||K - K_{\varepsilon}|| < \varepsilon
$$

for every $n, m \geq n_0$.

Thus $(U_n(K))$ converges in norm. Let $D_1(K) = || \cdot || - \lim U_n(K)$.

Since $\phi(P_i)^{\perp}U_n(K)P_i = \phi(P_i)U_n(K)P_i^{\perp} = 0$ for every $i = 1, 2, ...n$, the limit $D_1(K)$ belongs to the diagonal $\Delta(\mathcal{U})$.

Since $||U_n(K)|| \le ||K||$ for all $n \in \mathbb{N}$, D_1 is a contraction.

We observe that if $K \in \Delta(U) \cap \mathcal{K}$ then $U_n(K) = K$ for all $n \in \mathbb{N}$ hence D_1 projects onto $\Delta(\mathcal{U}) \cap \mathcal{K}$.

Now $D_1|_{C_2}$ is the orthogonal projection onto $\Delta(\mathcal{U}) \cap C_2$, being the infimum of the sequence $(U_n|_{C_2})$.

We can also observe that $D|_{C_2}$ is an orthogonal projection in the Hilbert space C_2 .

If $T \in \Delta(\mathcal{U}) \cap C_2$ then by Proposition 6.3 $T = \sum_{n=1}^{\infty} E_n T F_n = D(T)$.

We conclude that $D|_{C_2}$ and $D_1|_{C_2}$ are both orthogonal projections onto $\Delta(\mathcal{U}) \cap C_2$, hence $D|_{C_2} = D_1|_{C_2}$.

Since C_2 is norm dense in K and $D|\kappa, D_1$ are norm continuous, $D|\kappa =$ $D_1.$ \Box

Proposition 6.11 *Suppose that* $\vee_n F_n = F$ *. Then the sequence* $(U_n(T)F)$ *converges strongly to the operator* $D(T)$ *for every* $T \in B(H_1, H_2)$.

Proof

First we observe that if $x \in F_m(H_1), m \in \mathbb{N}$, then the operator $x \otimes x^*$ is in $(\mathcal{S}_{1,\phi})'$.

Indeed, let $y \in E_m(H_2)$, then $R = y \otimes x^* \in \Delta(\mathcal{U})$. It follows that $R^*R = ||y||^2 x \otimes x^* \in \Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset (\mathcal{S}_{1,\phi})'.$

Let $T \in B(H_1, H_2)$ and $x \in F_m(H_1), m \in \mathbb{N}, ||x|| = 1$.

By Proposition 6.10

$$
U_i(Tx\otimes x^*)\stackrel{\|\cdot\|}{\to} D(Tx\otimes x^*), i\to\infty,
$$

hence

$$
U_i(Tx \otimes x^*)(x) \stackrel{\|\cdot\|}{\to} D(Tx \otimes x^*)(x), i \to \infty \tag{6.4}
$$

$$
D(Tx \otimes x^*)(x) = \sum_{n=1}^{\infty} E_n(Tx \otimes x^*) F_n(x) = E_m T(x) \tag{6.5}
$$

$$
D(T)(x) = \sum_{n=1}^{\infty} E_n T F_n(x) = E_m T(x)
$$
\n(6.6)

We have that

$$
V_i(Tx \otimes x^*) = \phi(P_i) (T x \otimes x^*) P_i + \phi(P_i)^{\perp} (T x \otimes x^*) P_i^{\perp}, i \in \mathbb{N}
$$

since $x \otimes x^* \in (\mathcal{S}_{1,\phi})'$,

$$
V_i(Tx \otimes x^*) = (\phi(P_i) T P_i) (x \otimes x^*) + (\phi(P_i)^{\perp} T P_i^{\perp}) (x \otimes x^*), i \in \mathbb{N},
$$

hence

$$
U_i(Tx \otimes x^*) = U_i(T)x \otimes x^* \Rightarrow U_i(Tx \otimes x^*)(x) = U_i(T)(x), i \in \mathbb{N} \qquad (6.7)
$$

Using (6.4), (6.5), (6.6), (6.7)

$$
U_i(T)(x) \stackrel{\|\cdot\|}{\to} D(T)(x), \ i \to \infty \text{ for all } x \in [\bigcup_{m=1}^{\infty} F_m(H_1)].
$$

Since the U_i are contractions $U_i(T)(x) \stackrel{\|\cdot\|}{\to} D(T)(x)$, for all $x \in F(H_1)$ Observe that $D(T)F = D(T)$. \Box

Remark 6.12 *The sequence* $(U_n(T))$ *has similar properties to the net of finite diagonal sums in the case of nest algebras.(Propositions 6.10,6.11 are analogous to Propositions 4.3,4.4 in [\[2\]](#page-33-0).)*

Theorem 6.13 *Let* $K \in \mathcal{U}$ *be compact. Then there exist unique compact operators* $K_1 \in \mathcal{U}_0, K_2 \in \Delta(\mathcal{U})$ *such that* $K = K_1 + K_2$ *. Moreover* $K_2 =$ $D(K).$

Proof

Let $K_2 = D(K)$ and $K_1 = K - K_2$ then $K_1 = \lim (K - U_n(K))$ (Proposition 6.10).

As in Theorem 3.1 $K - U_n(K) \in \mathcal{U}_0$ for all $n \in \mathbb{N}$. Hence $K_1 \in \mathcal{U}_0$.

The decomposition $K = K_1 + K_2$ in $\mathcal{U}_0 + \Delta(\mathcal{U}) \cap \mathcal{K}$ is unique because by Proposition 6.3, $\Delta(\mathcal{U}) \cap \mathcal{K} \subset \Delta(\mathcal{U})(I - Q_1)$, while by Theorem 3.3, $\mathcal{U} =$ $\mathcal{U}_0 \oplus \Delta(\mathcal{U})(I - Q_1).$ \Box

Corollary 6.14 *Let* $F \in \mathcal{U}$ *be a finite rank operator. Then there exist unique finite rank operators* $F_1 \in \mathcal{U}_0, F_2 \in \Delta(\mathcal{U})$ *such that* $F = F_1 + F_2$. Moreover $rankF_2 \leq rankF$ and $F_2 = D(F)$.

Proof

It can be shown that for each $n \in \mathbb{N}$ we have $rank(U_n(F)) \leq rank(F)$.

Therefore if $F_2 = || \cdot || \cdot || \cdot \lim U_n(F)$ then rank $(F_2) \le \text{rank}(F)$ and $F_2 =$ $D(F).$

Setting $F_1 = F - F_2$ we obtain the desired decomposition. \Box

Corollary 6.15 *Let* $K \in \mathcal{U} \cap C_p, 1 \leq p < \infty$. Then there exist unique *operators* $K_1 \in \mathcal{U}_0 \cap C_p, K_2 \in \Delta(\mathcal{U}) \cap C_p$ *such that* $K = K_1 + K_2$ *. Moreover* $||K_2||_p \leq ||K||_p.$

Proof

As in Theorem 6.13 $K = K_1 + D(K)$ where $K_1 \in \mathcal{U}_0$.

We observe that $D(K) \in C_p$ and $||D(K)||_p \leq ||K||_p$. \Box

7 Decomposition of a strongly reflexive masa bimodule

Let $\mathcal{U}, \mathcal{U}_0, \Delta(\mathcal{U}), \phi, \mathcal{D}_1, \mathcal{D}_2$ be as in section 3 and $\chi = \text{Map}(\Delta(\mathcal{U})).$

We now assume that U is a strongly reflexive masa bimodule.

Proposition 7.1 *The space* U_0 *is strongly reflexive.*

Proof

Let $T \in \mathcal{U}$, $P \in \mathcal{S}_{1,\phi}$. Since \mathcal{U} is a strongly reflexive masa bimodule there exists a net $(R_i) \subset [R_1(\mathcal{U})]$ such that $R_i \stackrel{wot}{\to} T$: Corollary 2.5 in [\[6\]](#page-33-7). So we have that $\phi(P)R_i P^{\perp} \stackrel{wot}{\rightarrow} \phi(P)TP^{\perp}$. Since $(\phi(P)R_i P^{\perp}) \subset [R_1(\mathcal{U}_0)]$ we conclude that $\phi(P)TP^{\perp} \in [R_1(\mathcal{U}_0)]^{-\text{wot}}$. We proved that $\phi(P)\mathcal{U}P^{\perp} \subset [R_1(\mathcal{U}_0)]^{-wd}$ for all $P \in \mathcal{S}_{1,\phi}$. Hence $\mathcal{U}_0 =$ $[R_1(\mathcal{U}_0)]^{-wot}.$ \Box

Remark 7.2 *The diagonal of a strongly reflexive masa bimodule is not nececcarily strongly reflexive. For example if* U *is a nonatomic nest algebra, then* ∆(U) *does not contain rank 1 operators.*

Proposition 7.3 i) $\mathcal{U}_0 = \mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0$. $ii)$ $U_0 \cap \Delta(U) = \Delta(U) \cap (R_1(\Delta(U))^*)^0$.

Proof

By Proposition 6.2 we have $\mathcal{U}_0 \subset (R_1(\Delta(\mathcal{U}))^*)^0$. It suffices to show that $\mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0 \subset \mathcal{U}_0$.

Since $\mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0$ is masa bimodule, as in Theorem 3.1 we can decompose it in the next sum:

$$
\mathcal{U} \cap (R_1(\Delta(\mathcal{U}))^*)^0 = \mathcal{U}_0 \cap (R_1(\Delta(\mathcal{U}))^*)^0 + \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0.
$$

Now we must prove that $\Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0 \subset \mathcal{U}_0$.

Using Theorem 2.2, there exist projections $P_1 \in \mathcal{D}_1, P_2 \in \mathcal{D}_2$ such that $[R_1(\Delta(\mathcal{U}))]^{-w^*} = P_2\Delta(\mathcal{U})P_1$ and $\Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0 = P_2^{\perp} \Delta(\mathcal{U})P_1^{\perp}$.

Let $T \in \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U}))^*)^0$.

Since U is a strongly reflexive masa bimodule there exists a net $(R_i) \subset$ $[R_1(\mathcal{U})]$ such that $R_i \stackrel{wot}{\rightarrow} T$ [\[6\]](#page-33-7).

By Proposition 6.1 there exist $M_i \in [R_1(\Delta(\mathcal{U}))], L_i \in \mathcal{U}_0$ such that $R_i =$ $M_i + L_i$.

Thus $M_i + L_i \stackrel{wot}{\rightarrow} T$ so $P_2^{\perp} M_i P_1^{\perp} + P_2^{\perp} L_i P_1^{\perp}$ $\stackrel{wot}{\rightarrow} P_2^{\perp}TP_1^{\perp}$ and thus $P_2^{\perp}L_iP_1^{\perp}$ $\overset{wot}{\rightarrow}$ T. It follows that $T \in \mathcal{U}_0$. \Box

Theorem 7.4 $\mathcal{U} = \mathcal{U}_0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}$.

Proof

By Theorem 2.2,

$$
\Delta(\mathcal{U}) = \Delta(\mathcal{U}) \cap (R_1(\Delta(\mathcal{U})^*)^0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}
$$

so by Proposition 7.3 $\Delta(\mathcal{U}) = \mathcal{U}_0 \cap \Delta(\mathcal{U}) + [R_1(\Delta(\mathcal{U}))]^{-w^*}.$

Since $\mathcal{U} = \mathcal{U}_0 + \Delta(\mathcal{U})$ we have that $\mathcal{U} = \mathcal{U}_0 + [R_1(\Delta(\mathcal{U}))]^{-w^*}$.

By Proposition 6.3 and Theorem 3.3 the previous sum is direct. \square

Propositions 3.6 and 3.7 have the following consequences:

Corollary 7.5 *i)The following are equivalent: a)* $R_1(\Delta(\mathcal{U})) = 0$. *b)* $\Delta(\mathcal{U})^* \Delta(\mathcal{U}) \subset \mathcal{L}_1 \cap \mathcal{A}_1$. *c*) $\Delta(\mathcal{U})\Delta(\mathcal{U})^*$ ⊂ $\mathcal{L}_2 \cap \mathcal{A}_2$. *ii)The following are equivalent: a)*∆(U) *is strongly reflexive. b)* $\Delta(\mathcal{U})$ ($\mathcal{L}_1 \cap \mathcal{A}_1$) = 0.

c) $(\mathcal{L}_2 \cap \mathcal{A}_2) \Delta(\mathcal{U}) = 0.$

Theorems 6.8, 7.4 and Corollary 5.3 give the following form of the decomposition of U when it is a strongly reflexive C.S.L. algebra.

Corollary 7.6 *If* S *is a completely distributive CSL in a Hilbert space* H *and* $\{A_n : n \in \mathbb{N}\} = \{A : A \text{ atom of } S\}$ then:

$$
Alg(\mathcal{S}) = Rad(Alg(\mathcal{S}))^{-w^*} \oplus \sum_{n=1}^{\infty} \oplus A_n B(H) A_n.
$$

Recall the notation $[R_1(\Delta(\mathcal{U}))]^{-w^*} = \sum_{n=1}^{\infty} \bigoplus \chi(I) \delta(F_n) B(H_1, H_2) F_n$, where $\{F_n : n \in \mathbb{N}\} = \{F : F \text{ atom of } U\}$ and

$$
D: B(H_1, H_2) \longrightarrow B(H_1, H_2): D(T) = \sum_{n=1}^{\infty} \chi(I) \delta(F_n) T F_n.
$$

Proposition 7.7 *Let* $\theta : \mathcal{U} \to \mathcal{U}$ *be the projection onto* $[R_1(\Delta(\mathcal{U}))]^{-w^*}$ *defined by the decomposition in Theorem 7.4. Then* $\theta = D|_{\mathcal{U}}$.

Proof

Since U decomposes as the direct sum of the masa bimodules U_0 and $[R_1(\Delta(\mathcal{U}))]^{-w^*}$, the map θ is a masa bimodule map:

$$
\theta(D_2TD_1) = D_2\theta(T)D_1
$$

for every $T \in \mathcal{U}, D_1 \in \mathcal{D}_1, D_2 \in \mathcal{D}_2$.

Hence if $T \in \mathcal{U}$:

$$
\theta(T) = \sum_{n=1}^{\infty} \chi(I)\delta(F_n)\theta(T)F_n = \sum_{n=1}^{\infty} \theta(\chi(I)\delta(F_n)TF_n)
$$

=
$$
\sum_{n=1}^{\infty} \chi(I)\delta(F_n)TF_n = D(T).
$$

Proposition 7.8 $\mathcal{U}_0 = \{T \in \mathcal{U} : \chi(I)\delta(F)TF = 0 \text{ for every atom } F \text{ of } \mathcal{U}\}.$

Proof

Let F be an atom of U . If $P \in \mathcal{S}_{1,\phi}$, as in Proposition 6.6 either $PF = F \Rightarrow P^{\perp}F = 0$ or $PF = 0 \Rightarrow$ $\chi(I)\delta(F)\phi(P)=0.$ So $\chi(I)\delta(F)\phi(P)TP^{\perp}F = 0$ for all $P \in \mathcal{S}_{1,\phi}$ and $T \in \mathcal{U}$, thus $\chi(I)\delta(F)\mathcal{U}_0F =$ 0 for every atom F . It follows that $\mathcal{U}_0 \subset \{T \in \mathcal{U} : \chi(I)\delta(F)TF = 0$, for every atom F in $\mathcal{U}\}$.

For the converse, let $T \in \mathcal{U} : \chi(I)\delta(F)TF = 0$ for every atom F in \mathcal{U} . By the previous proposition $D(T) = 0$, hence $T \in \mathcal{U}_0$. \Box

It is known that the linear span of the rank 1 operators in a strongly reflexive masa bimodule is wot dense in the module.

This is not true generally for the ultraweak topology [\[6\]](#page-33-7).

For the previous problem we have the next equivalence in proposition 7.10.

Firstly, we need the following lemma.

Lemma 7.9 If U is a reflexive masa bimodule (not necessarily strongly re*flexive) then:*

$$
[R_1(\mathcal{U})]^{-w^*}=[R_1(\mathcal{U}_0)]^{-w^*}\oplus[R_1(\Delta(\mathcal{U}))]^{-w^*}.
$$

Proof

Since $R_1(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})(I - Q_1)$ (Proposition 6.3), by Theorem 3.3 the previous sum is direct.

Clearly

$$
[R_1(\mathcal{U})]^{-w^*} \supset [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}.
$$

For the converse, let $T \in [R_1(\mathcal{U})]^{-w^*}$.

There is a net $(R_i) \subset [R_1(\mathcal{U})]$ with $R_i \xrightarrow{w^*} T$.

As in Proposition 6.1, we may decompose $R_i = L_i + M_i$ where $L_i \in [R_1(\mathcal{U}_0)]^{-\|\cdot\|_1}$ and $M_i \in [R_1(\Delta(\mathcal{U}))]$ for all i.

Since $M_i = D(R_i)$ (Corollary 6.14) and D is w^{*}-continuous, we have $M_i \stackrel{w^*}{\longrightarrow} M \in [R_1(\Delta(\mathcal{U}))]^{-w^*}.$

So
$$
L_i = R_i - M_i \xrightarrow{w^*} T - M = L \in [R_1(\mathcal{U}_0)]^{-w^*}
$$
.
Thus $T = L + M \in [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}$. \square

Proposition 7.10 *If* U *is a strongly reflexive masa bimodule, then:*

$$
\mathcal{U}=[R_1(\mathcal{U})]^{-w^*}\Leftrightarrow \mathcal{U}_0=[R_1(\mathcal{U}_0)]^{-w^*}.
$$

Proof

Suppose $\mathcal{U} = [R_1(\mathcal{U})]^{-w^*}$. Then by Theorem 7.4 we have $\mathcal{U} = \mathcal{U}_0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*}.$

It follows from the previous lemma that $\mathcal{U}_0 = [R_1(\mathcal{U}_0)]^{-w^*}$.

If conversely $\mathcal{U}_0 = [R_1(\mathcal{U}_0)]^{-w^*}$ then again by Theorem 7.4

$$
\mathcal{U} = \mathcal{U}_0 \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*} = [R_1(\mathcal{U}_0)]^{-w^*} \oplus [R_1(\Delta(\mathcal{U}))]^{-w^*} = [R_1(\mathcal{U})]^{-w^*}
$$

by Lemma 7.9. \Box

Acknowledgement: I would like to express appreciation to Prof. A. Katavolos for his helpful comments and suggestions during the preparation of this work, which is part of my doctoral thesis. I wish to thank him for the proof of Proposition 6.10.

I am also indebted to Dr. I.Todorov for helpful discussions and important suggestions that led to a substantial improvement of this paper.

This research was partly supported by Special Account Research Grant No. 70/3/7463 of the University of Athens.

References

- [1] W. B. Arverson, Operator algebras and invariant subspaces, *Ann. of Math.* 100 (2): 433-532, 1974.
- [2] Kenneth R. Davidson, *Nest algebras*, volume 191 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988. Triangular forms for operator algebras on Hilbert space.
- [3] K.R. Davidson and J.L. Orr, The Jacobson radical of a CSL algebra, *Trans. Amer. Math. Soc.* 344 (2): 925-947, 1994.
- [4] E. Effros, N. Ozawa and Z.J. Ruan, On injectivity and nuclearity for operator spaces, *Duke Math.* 110 (3): 489-521, 2001.
- [5] J.A. Erdos, Reflexivity for subspace maps and linear spaces of operators, *Proc. London Math. Soc.* 52 (3): 582-600, 1986.
- [6] J.A. Erdos, A. Katavolos and V.S. Shulman, Rank one subspaces of Bimodules over Maximal Abelian Selfadjoint Algebras, *J. Funct. Anal.* 157 (2): 554-587, 1998.
- [7] J.A. Erdos and S.C. Power, Weakly Closed Ideals of Nest Algebras, *J. Operator Theory* 7 : 219-235, 1982.
- [8] A. Hopenwasser, The radical of a reflexive operator algebra, *Pacific J. Math.* 65: 375-392, 1976.
- [9] A. Katavolos and I.G. Todorov, Normalizers of operator algebras and reflexivity, *Proc. London Math. Soc.* (3) 86: 463-484, 2003.
- [10] E.G. Katsoulis, Integration with respect to a commutative subspace lattice, *J. Operator Theory* 22 : 307-323, 1989.
- [11] D.R. Larson and B. Solel, Bimodules of nest subalgebras over von Neumann algebras, *Operator Theory: Advances and Applications,* Vol. 32: 159-180, 1988.
- [12] A.I. Loginov and V.S. Shulman, Hereditary and intermediate reflexivity of W*-algebras, (Russian) *Izv. Akad. Nauk. SSSR Ser. Mat.,* 39 1260 – 1273, 1975 (English translation : *Math. USSR-Izv.,* 9, 1189 – 1201, 1975.
- [13] I.G. Todorov, Modules over reflexive algebras, *Doctoral Thesis, University of Athens (in Greek),* 1999.
- [14] I.G. Todorov, Synthetic properties of ternary masa-bimodules, *Houston Journal of Math.* to appear.

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