

## SPECIAL GRAPHS

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*Dedicated to Dmitri V. Alekseevsky on the occasion of his sixty-fifth birthday*

ABSTRACT. A *special*  $p$ -form is a  $p$ -form which, in some orthonormal basis  $\{e_\mu\}$ , has components  $\varphi_{\mu_1 \dots \mu_p} = \varphi(e_{\mu_1}, \dots, e_{\mu_p})$  taking values in  $\{-1, 0, 1\}$ . We discuss graphs which characterise such forms.

### 1. CALIBRATIONS, SPECIAL FORMS AND GRAPHS

A constant  $p$ -form  $\varphi$  in a  $d$ -dimensional Euclidean space is a *calibration* if for any  $p$ -dimensional subspace spanned by a set of orthonormalised vectors  $e_1, \dots, e_p$ , the following condition holds:

$$(\varphi(e_1, \dots, e_p))^2 \leq 1, \quad (1)$$

with equality holding for at least one subspace. Let  $U$  be an oriented  $p$ -dimensional subspace of  $\mathbb{R}^d$  with oriented metric volume  $\text{vol}_U$ . The set of all such subspaces is the oriented Grassmannian  $\text{Gr}_p \mathbb{R}^d$ . A calibration  $\varphi \in \Lambda^p \mathbb{R}^d$  is thus a  $p$ -form with the property that the function  $\bar{\varphi} : \text{Gr}_p \mathbb{R}^d \rightarrow \mathbb{R}$  associated to  $\varphi$  and defined by  $U \mapsto \bar{\varphi}(U) := \langle \varphi, \text{vol}_U \rangle$  takes values in  $[-1, 1] \subset \mathbb{R}$  with at least one of the two extremal values  $\pm 1$  being achieved. The  $p$ -planes  $U$  for which  $\bar{\varphi}(U) = \pm 1$  are said to be calibrated by  $\varphi$ .

Almost all examples of calibrations known are invariant under a group  $G \subset \text{O}(\mathbb{R}^d)$  large enough so that it is relatively simple to check the calibration condition directly. Interestingly most of these examples, in particular the calibrations characterising special holonomy manifolds, for instance the  $G_2$ -invariant Cayley 3-form in seven dimensions, defined by the structure constants of the imaginary octonions, and the  $\text{Spin}(7)$ -invariant 4-forms in eight dimensions are special forms:

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**Definition 1.** A special  $p$ -form  $\varphi$  is a  $p$ -form  $\varphi \in \Lambda^p \mathbb{R}^d$  on  $d$ -dimensional Euclidian space  $\mathbb{R}^d$  in the orbit under the orthogonal group  $O(d, \mathbb{R})$  of

$$\varphi = \sum_{1 \leq \mu_1 < \dots < \mu_p \leq d} \varphi_{\mu_1 \dots \mu_p} e_{\mu_1} \wedge e_{\mu_2} \wedge \dots \wedge e_{\mu_p} \quad (2)$$

with  $\varphi_{\mu_1 \dots \mu_p} \in \{-1, 0, 1\}$  and  $(e_1, \dots, e_d)$  an orthonormal basis.

In other words, a  $p$ -form  $\varphi$  is special if there exist  $d$  orthonormal basis vectors  $e_\mu$ ,  $\mu = 1, \dots, d$ , such that for any subset of  $p$  basis vectors  $e_{\mu_1}, \dots, e_{\mu_p}$  we have

$$\varphi_{\mu_1 \dots \mu_p} := \varphi(e_{\mu_1}, \dots, e_{\mu_p}) \in \{-1, 0, 1\}. \quad (3)$$

Given a basis  $\{e_\mu\}$ , there are clearly only a finite number (obviously less than  $3^{\frac{d!}{p!(d-p)!}}$ ) of orbits of special  $p$ -forms under  $O(d, \mathbb{R})$  parametrised by the components  $\varphi_{\mu_1 \dots \mu_p} \in \{-1, 0, 1\}$ . Apparently different special  $p$ -forms may nevertheless be in the same orbit under  $O(d, \mathbb{R})$ , because the subgroup  $O(d, \mathbb{Z}) \subset O(d, \mathbb{R})$  of orthogonal matrices with integer coefficients maps the special form  $\varphi$  in equation (2) again into a special form with possibly different components. The group  $O(d, \mathbb{Z})$  is isomorphic to the semidirect product of the permutation group acting naturally on  $d$  copies of  $\mathbb{Z}_2$ . The action of  $(\sigma, \eta_1, \dots, \eta_d) \in S_d \times \mathbb{Z}_2^d \cong O(d, \mathbb{Z})$  on the antisymmetric tensor indices of  $\varphi$  is given by  $\varphi_{i_1 \dots i_p} \mapsto \eta_{i_1} \dots \eta_{i_p} \varphi_{\sigma(i_1) \dots \sigma(i_p)}$ , where  $\sigma \in S_d$  and  $\eta_i^2 = 1$ ,  $i = 1, \dots, d$ .

Let us now give an alternative description of special forms. An oriented  $p$ -subset of  $\{1, 2, \dots, d\}$  is given by the  $p$  elements  $s = \{\mu_1, \dots, \mu_p\}$  such that  $1 \leq \mu_1 < \mu_2 < \dots < \mu_p \leq d$ . The space of all such  $p$ -subsets is the *vertex space*  $\mathcal{P}^p(\{1, \dots, d\})$ . These oriented subsets of  $\{1, 2, \dots, d\}$  are in bijective correspondence to oriented coordinate subspaces  $\mathbb{R}^p \subset \mathbb{R}^d$  via  $s = \{\mu_1, \dots, \mu_p\} \mapsto e_{\mu_1} \wedge \dots \wedge e_{\mu_p}$ . A special  $p$ -form can be thought of as a function from  $\mathcal{P}^p(\{1, \dots, d\})$  to  $\varphi_{\mu_1 \dots \mu_p} \in \{-1, 0, 1\}$ . Consequently a special  $p$ -form is specified completely by the two sets  $\mathcal{J}^+$  and  $\mathcal{J}^-$  of oriented subsets  $\{\mu_1, \dots, \mu_p\} \subset \{1, \dots, d\}$  which have respectively  $\varphi_{\mu_1 \dots \mu_p} = +1$  and  $-1$ . We denote by  $\mu^{(a)}$ ,  $a = 1, \dots, |\varphi|$ , the elements in  $\mathcal{J} := \mathcal{J}^+ \cup \mathcal{J}^-$ , the support of  $\varphi$  and

$$\varphi = \sum_{a=1}^{|\varphi|} \varphi_{\mu_1^{(a)} \dots \mu_p^{(a)}} e_{\mu_1^{(a)}} \wedge \dots \wedge e_{\mu_p^{(a)}}, \quad (4)$$

where  $|\varphi| := |\mathcal{J}|$  is the weight of  $\varphi$ . For every permutation  $\sigma \in S_d$  of the basis vectors  $e_1, \dots, e_d$ , there exists a corresponding permutation of the oriented  $p$ -subsets.

Interestingly, we can define a metric on  $\mathcal{P}^p(\{1, \dots, d\})$  by setting the distance between two oriented  $p$ -subsets,  $s$  and  $\tilde{s}$ , to be  $(s, \tilde{s}) = p - \#(s \cap \tilde{s})$ , where  $\#(s \cap \tilde{s})$  is the number of elements in the intersection of the sets  $s$  and  $\tilde{s}$ . We can visualise

the restriction of this metric to the set  $\mathcal{J}$  by drawing a graph with labeled edges, the vertices corresponding to the elements of  $\mathcal{J}$  and the edges running between vertices labeled by a *distance* strictly less than  $p$ . Unfortunately the graph of a special  $p$ -form  $\varphi$  does not specify the components  $\varphi_{\mu_1 \dots \mu_p} \in \{-1, 0, 1\}$  completely up to the action of  $O(d, \mathbb{Z})$ ; we still need to specify some relative sign. Nevertheless the graph gives a very condensed way of encoding the characteristics of a special  $p$ -form.

Consider a graph  $\Gamma$  composed of a set of vertices  $V = \{v_i; i = 1, \dots, r\}$  connected by the maximum possible number of edges,  $r(r-1)/2$ , each labeled by a positive number  $d(v_i, v_j)$ , the *distance* between the vertices  $v_i$  and  $v_j$  at its ends. A graph is *admissible* if any triangle with edges labeled by distances  $d_i, d_j, d_k$  satisfies the triangle inequalities

$$1 \leq d_i \leq d_j + d_k \quad \text{and cyclic permutations.} \quad (5)$$

**Definition 2.** A realisation of a graph  $\Gamma$  is a map

$$\begin{aligned} \rho : \Gamma &\rightarrow \mathcal{P}^p(\{1, \dots, d\}) \\ v &\mapsto s_v \end{aligned} \quad (6)$$

which assigns to any vertex  $v$  an oriented  $p$ -subset  $s_v$  such that the distance between any two vertices  $d(v, w)$  is equal to the distance between the corresponding oriented  $p$ -subsets  $(s_v, s_w)$  and that

$$\# \left( \bigcup_{v \in \Gamma} s_v \right) = d \quad , \quad \# \left( \bigcap_{v \in \Gamma} s_v \right) = 0 . \quad (7)$$

Two realisations are equivalent if there exist a permutation  $\sigma \in S_d$  of the oriented  $p$ -subsets which maps one onto the other.

Consider the power set  $\mathcal{P}(V)$ , the set of all subsets  $S \subset V$  of vertices of the graph  $\Gamma$ ,

$$\mathcal{P}(V) = \{ \{\emptyset\}, \{v\}, \{w\}, \dots, \{v, w\}, \{x, y\}, \dots, \{v, w, x\}, \dots, \{V\} \} . \quad (8)$$

Clearly,  $\#\{\mathcal{P}(V)\} = 2^r$ . For every realisation, a graph function  $f$  associates a nonnegative integer to every  $S \in \mathcal{P}(V)$  as follows

$$\begin{aligned} F(S) &:= \left( \left( \bigcap_{v \in S} s_v \right) \cap \left( \bigcup_{v \notin S} s_v \right)^c \right) \geq 0 \\ f(S) &:= \#(F(S)) , \end{aligned} \quad (9)$$

where  $C$  denotes the complementary subset. Clearly, this function measures the number of indices which occur in every  $s_v$  for  $v \in S$  but do not occur in any  $s_v$  for  $v \notin S$ . Trivially, we have

$$\begin{aligned} f(\emptyset) &= 0 \\ f(V) &= 0 \\ \sum_{S \in \mathcal{P}(V)} f(S) &= d, \end{aligned} \tag{10}$$

because  $\bigcap_{v \in S} s_v$  is empty for the two first cases and  $\sum_{S \in \mathcal{P}(V)} f(S)$  contains all the indices  $\{1 \dots d\}$  and each index contributes to  $f$  for one and only one subset  $S$ .

**Theorem 1.** *To every graph function  $f$ , with non negative integer values, which satisfies*

$$d(v, \tilde{v}) = p - \sum_{\{S \in \mathcal{P}(V) | v, \tilde{v} \in S\}} f(S), \tag{11}$$

*there corresponds a class of equivalent realisations of the graph.*

In particular, if  $v = \tilde{v}$

$$d(v, v) = p - \sum_{\{S \in \mathcal{P}(V) | v \in S\}} f(S) = 0. \tag{12}$$

As a consequence, all realisations of a graph are simply obtained by finding all the non negative solutions to (11). Every realisation yields a simple and direct construction of a special form, up to choices of signs. Examples can easily be generated [DNW].

## 2. DEMOCRATIC GRAPHS

Consider a graph  $\Gamma$  with vertices  $v_i, i = 1, \dots, r$ , and the set of nonzero distances  $\{d(v_i, v_j) ; i < j\}$ . The  $r \times r$  *distance matrix*

$$M_{ij}^{[r]} = d(v_i, v_j) \tag{13}$$

is clearly symmetric, with diagonal elements equal to 0.

**Definition 3.** *A symmetry  $\sigma$  of a graph  $\Gamma$  is a permutation of the vertices  $v_i \mapsto \tilde{v}_i$  which leaves the distance matrix invariant, i.e.*

$$d(\tilde{v}_i, \tilde{v}_j) = d(v_i, v_j). \tag{14}$$

If there exists a realisation of a graph with symmetry  $\sigma$ , such that  $f(\sigma S) = f(S)$  for any  $S \in \mathcal{P}(V)$ , then the realisation has a permutation of the indices which induces  $\sigma$  on the monomials.

**Definition 4.** *A graph is democratic if, for every pair of vertices  $v_i, v_j$ , there exists some symmetry  $\sigma$  which maps  $v_i \mapsto \tilde{v}_i = v_j$ .*

Let  $\{d_a\}$  denote the set of unequal distances.

**Proposition 1.** *A necessary condition for a graph with  $r$  vertices to be democratic is that  $n_a^{(i)}$ , the number of vertices at distance  $d_a$  from vertex  $v_i$ , is independent of the choice of  $v_i$ . A graph satisfying this condition, i.e.  $n_a^{(i)} = n_a$ , will be called predemocratic.*

Note that  $\sum_a n_a^{(i)} = r - 1$  and that a predemocratic graph is not necessarily democratic.

First, consider graphs with an even number of vertices,  $r = 2n$ . Then, from every vertex there are  $r-1$  edges labeled by  $r-1$  distances. To have democracy, every vertex should have the same set of distances to its neighbouring vertices. Let  $d_i := d(v_1, v_{i+1})$ ,  $i = 1, \dots, r-1$ , be the distances between  $v_1$  and  $v_{i+1}$ . An example of a predemocratic graph with an even number of vertices  $r = 2n$  has distance matrix, up to relabeling, of the form

$$\begin{aligned} d(v_i, v_j) &= (1 - \delta_{ij}) d_{j+i-2 \pmod{r-1}} \\ d(v_i, v_r) &= (1 - \delta_{ir}) d_{2i-2 \pmod{r-1}} \end{aligned} \tag{15}$$

with  $d_0 \equiv d_{r-1}$ . For  $r = 4$ , the corresponding graph is the unique predemocratic graph and it is also democratic. For higher  $r$ 's, these graphs are in general not democratic.

Now, consider graphs with an odd number of vertices,  $r = 2n+1$ .

**Lemma 1.** *If the number of vertices  $r$  is odd, a predemocratic graph has all  $n_a$ 's even.*

*Proof:* For a predemocratic graph with  $r$  vertices, the total number of edges of distance  $d_a$  is clearly  $n_a r / 2$ .  $\square$

For  $r$  odd, if we set  $n_a = 2$ , for all  $a$ , there are  $(r-1)/2$  unequal distances  $d_a$ . Distance matrices with  $n_a = 4, 6, \dots$ , with all distances  $d_a$  different, can always be obtained from the distance matrices with  $n_a = 2$  by setting some  $d_a$ 's to be equal.

We now classify all distance matrices with  $n_a = 2$  for all  $a$  and all distances  $d_a$  different. Under these assumptions, the edges of length  $d_a$  for given  $a$  form a closed (possibly disconnected) curve  $\mathcal{C}_a$  containing every vertex once. Let us call the number of vertices in a connected piece of curve  $\mathcal{C}_a$  its pathlength, which is obviously between 3 and  $r$ .

**Lemma 2.** *A necessary condition for a predemocratic graph with an odd number of vertices  $r$  and  $n_a = 2$  to be democratic is that the curve  $\mathcal{C}_a$ , for every  $a$ , has connected pieces of equal pathlength  $3 \leq L_a \leq r$ . In other words,  $L_a$  is a divisor of  $r$  and  $\mathcal{C}_a$  consists of  $r/L_a$  disconnected pieces.*

*Proof:* The proof follows from the definition of a democratic graph: the curve  $\mathcal{C}_a$  as seen from any vertex has the same form, independent of the choice of the vertex.  $\square$

To give an example of a democratic graph with  $r = 2n + 1$  vertices and  $n_a = 2$  for all  $a$  we choose  $n$  distinct positive integers  $d_1, \dots, d_n$  and define the distance matrix  $M^{[r]}$  by

$$M_{ii}^{[r]} = 0 \quad , \quad M_{ij}^{[r]} = d_{\min\{|i-j|, 2n+1-|i-j|\}} \quad (16)$$

for  $0 \leq i, j \leq 2n$ . Evidently the matrix  $M^{[r]}$  has a cyclic isometry group  $\mathbb{Z}_r$  shifting the vertices  $v_i \mapsto v_{i+1 \pmod{r}}$ . Assuming that the distances  $d_1, \dots, d_n$  can be chosen in such a way that there exists a graph function  $f$  for the  $M^{[r]}$  invariant under  $\mathbb{Z}_r$  we get a realisation for  $M^{[r]}$ , which is a democratic graph with symmetry group containing  $\mathbb{Z}_r$ .

**Theorem 2.** *For an odd prime number  $r = 2n + 1$  every democratic graph with  $r$  vertices satisfying  $n_a = 2$  for all  $a$  (and  $n$  distinct distances  $d_a$ ) has a distance matrix of the form  $M_{ij}^{[r]}$  with a suitable choice of the positive integers  $d_1, \dots, d_n$ .*

*Proof:* Essentially we only need to show that the symmetry group of a democratic graph with a prime number  $r = 2n + 1$  of vertices and  $n_a = 2$  for all  $a$  must contain a cyclic subgroup of order  $r$  acting transitively on the vertices. Clearly all curves  $\mathcal{C}_a$  must be connected circles of length  $r$ , because  $r$  being prime has no proper divisors. We fix two vertices  $v_0$  and  $v_1$  and the curve  $\mathcal{C}_a$  containing the edge between them. By democracy there exists a symmetry  $\sigma$  sending  $v_0$  to  $\sigma(v_0) = v_1$  and mapping the curve  $\mathcal{C}_a$  to itself, because it is a symmetry and all  $d_a$  are distinct. Thus  $\sigma$  can only be the cyclic shift by one step along the curve  $\mathcal{C}_a$ , which clearly generates a cyclic group of symmetries of order  $r$  acting

transitively on the vertices. Setting  $v_i := \sigma^i(v_0)$  for  $0 \leq i \leq 2n$  we conclude that  $M_{ij}^{[r]} := d(v_i, v_j) = d(v_0, v_{j-i}) = d(v_{2n+1-j+i}, v_0)$  for all  $0 \leq i \leq j \leq 2n$ .  $\square$

Slightly more generally we can consider graphs with a group of symmetries isomorphic to  $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}$  acting transitively on the vertices. Of course if  $r_1$  and  $r_2$  are relatively prime, then the group of symmetries considered is isomorphic to  $\mathbb{Z}_{r_1 r_2}$ . Nevertheless we expect new features compared to the classification above, because  $r_1 r_2$  is no longer prime. With a group  $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}$  acting transitively on the vertices it is convenient to label the vertices by tuples  $(i_1, i_2) \in \{1, \dots, r_1\} \times \{1, \dots, r_2\}$ . Straightening this out by replacing  $(i_1, i_2)$  with  $i := i_2 + r_2(i_1 - 1)$  we get a distance matrix of the form

$$M^{[r_1][r_2]} = \begin{pmatrix} M^{[r_2]} & Q_1 & Q_2 & Q_3 & \dots & Q_2^t & Q_1^t \\ Q_1^t & M^{[r_2]} & Q_1 & Q_2 & \ddots & Q_3^t & Q_2^t \\ Q_2^t & Q_1^t & M^{[r_2]} & Q_1 & \ddots & Q_4^t & Q_3^t \\ Q_3^t & Q_2^t & Q_1^t & M^{[r_2]} & \ddots & Q_5^t & Q_4^t \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ Q_2 & Q_3 & Q_4 & Q_5 & \ddots & M^{[r_2]} & Q_1 \\ Q_1 & Q_2 & Q_3 & Q_4 & \ddots & Q_1^t & M^{[r_2]} \end{pmatrix}, \quad (17)$$

where  $M^{[r_2]}$  is the  $r_2 \times r_2$  distance matrix defined in (16) which depends on  $(r_2 - 1)/2$  arbitrary distances. The  $r_2 \times r_2$  matrices  $Q_i$ ,  $i = 1, \dots, (r_1 - 1)/2$ , depend on  $r_2$  arbitrary parameters. The first row of every  $Q_i$  is arbitrary and the  $r_2 - 1$  following rows are obtained by cyclically permuting the elements of the first row.

Clearly a graph function  $f$  for a matrix of the form (17), with the property that  $f(\sigma S) = f(S)$  for all sets of vertices  $S$  and all  $\sigma \in \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}$ , defines an equivalence class of democratic graphs. Conversely, for a democratic graph with  $r = r_1 r_2$  vertices,  $r_1, r_2$  prime and  $n_a = 2$  for all  $a$ , the distance matrix must be of the form  $M^{[r]}$  in (16) or  $M^{[r_1][r_2]}$  in (17).

In full generality, for every factorisation  $r = r_1 r_2 \cdots r_k$  with  $r_1 \geq r_2 \geq \cdots \geq r_k > 1$ , we can consider graphs having a group of symmetries isomorphic to  $\mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_k}$  acting transitively on the  $r$  vertices. A convenient labeling of the vertices is given by  $v^{i_1, \dots, i_k} := \sigma_1^{i_1} \sigma_2^{i_2} \cdots \sigma_k^{i_k}(v_0)$  where the tuples  $i_1, \dots, i_k \in \{1, \dots, r_1\} \times \cdots \times \{1, \dots, r_k\}$  and the  $\sigma_A$ ,  $A = 1, \dots, k$ , are generators of  $\mathbb{Z}_{r_A}$ , cyclic permutations of order  $r_A$ . Then the distance matrices have elements

$$d(v^{i_1, \dots, i_k}, v^{j_1, \dots, j_k}) = d(v_0, v^{j_1 - i_1, \dots, j_k - i_k}) = d_{j_1 - i_1, \dots, j_k - i_k}, \quad i_A < j_A, \quad A = 1, \dots, k. \quad (18)$$

If we have a graph function  $f$  for this matrix, with the property that  $f(\sigma S) = f(S)$  for all sets of vertices  $S$  and all  $\sigma \in \mathbb{Z}_{r_1} \times \cdots \times \mathbb{Z}_{r_k}$ , then the corresponding graph is democratic.

If we wish to represent the distance matrix  $M_{ij}^{[r]}$  in matrix form, corresponding to (16) or (17), then the labeling of the vertices by tuples turns out to be a nuisance, because we would have to straighten the indices as in the  $k = 2$  case. Instead, it is more natural to replace the vector space  $\mathbb{R}^r$  by a tensor product  $\mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_k}$ , with standard basis indexed by precisely the tuples above. The generators of the symmetry group then take the form

$$P^{[A]} = \mathbf{1}^{[1]} \otimes \mathbf{1}^{[2]} \otimes \cdots \otimes p^{[A]} \otimes \cdots \otimes \mathbf{1}^{[k]}. \quad (19)$$

where  $\mathbf{1}^{[B]}$  is the  $r_B \times r_B$  unit matrix and  $p^{[A]}$  are the permutation matrices of order  $r_A$ , which permute the indices  $i_A$  cyclically and thus induce an  $r \times r$  permutation on the indices  $i$ . As in the  $k = 2$  case above, there corresponds to every factorisation  $r = r_1 r_2 \cdots r_k$  a distance matrix invariant under all the permutations  $P^{[A]}$ , their powers and their products.

If  $r$  has  $m$  prime factors,  $r = s_1 \cdots s_m$ , with all the  $s_i$ 's different, the number of inequivalent democratic graphs with  $r$  vertices depends only on  $m$  and corresponds to the number of ways a set with  $n$  elements can be partitioned into disjoint, non-empty subsets. This is precisely the  $m$ -th Bell number  $B_m$ , which is given by the formula

$$B_{m+1} = \sum_{k=0}^m \binom{m}{k} B_k \quad , \quad B_0 = 1 . \quad (20)$$

If some of the prime factors  $s_i$  are equal, the partitions in different subsets leading to the same set of  $r_A$ 's yield equivalent graphs. We shall give further details and examples elsewhere [DNW].

## REFERENCES

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