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Degeneration and gluing of Kuranishi structures in Gromov-Witten theory and the degeneration/gluing axioms for open Gromov-Witten invariants under a symplectic cut

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(*In memory of Professor Raoul Bott.*)

Abstract

We construct a family Kuranishi structure in the Fukaya-Ono format on the moduli space $\overline{\mathcal{M}}_\bullet(W/B, L\,|\,\bullet)$ of open stable J-holomorphic maps to the fibers of an almost-complex degeneration family W/B that arises from a symplectic cut. The degenerate fiber of the family Kuranishi structure defines a Kuranishi structure on the moduli space of open stable maps to a singular symplectic space of the gluing form $Y_1 \cup_D Y_2$ from a symplectic cut, with a Lagrangian submanifold L contained in the smooth locus. The same discussion and construction apply also to relative open Gromov-Witten theory for a relative pair $(Z, L; D)$, where D is a codimension-2 symplectic submanifold of Z, disjoint from the Lagrangian submanifold L. We derive then the degeneration-gluing relations of these Kuranishi structures. The good flat behavior of the family Kuranishi structure on $\overline{\mathcal{M}}_\bullet(W/B, L\,|\,\bullet)$ motivates both a degeneration axiom and a gluing axiom for open Gromov-Witten invariants of a symplectic manifold X with a decorated Lagrangian submanifold L^α. When a symplectic cut at the boundary of a tubular neighborhood of L exists, the construction of open Gromov-Witten invariants of (X, L^α) can then be put in two steps: (1) use the degeneration axiom and the gluing axiom to fix the ambiguity in the choice of fundamental chain class; (2) intersection theory on the specific kind of singular Kuranishi space with the induced decoration on the moduli space of relative maps to the relative pairs from the degenerate target. Step (1) is analytical and is dealt with in this work. In the appendix we comment on the equivalence of Li-Ruan/Li's degeneration formula and Ionel-Parker's degeneration formula in closed Gromov-Witten theory.

Key words: symplectic cut, bordered Riemann surface, relative Maslov index, stable map, relative stable map, moduli space, Kuranishi structure modelled in a category, degeneration and gluing of Kuranishi structures, open Gromov-Witten invariant, relative open Gromov-Witten invariant, specialization, axiom, virtual fundamental chain, decorated Lagrangian submanifold, open/closed string duality.

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Professor Raoul Bott and mirror symmetry - a reminiscence of a curious mind, by C.-H.L..

In the fall 2000, after the semester-long lectures of Prof. Cumrun Vafa on string theory and stringy duality for both mathematicians and physicists in the spring that year, Prof. Raoul Bott got intrigued in mirror symmetry and had a conversation with Prof. Yau, the second author. It ended in a surprise e-mail from Prof. Yau to me one day with an assignment: to teach Prof. Bott what mirror symmetry is about. To "teach" a then-78-year-old legendary mathematician?! Unable to turn it away, I thus took the task as a new hand, expecting that my "student" would get bored very soon and I could resume the full focus on real projects. Amazingly, the outlining lecture for Prof. Bott was extended to weekly meetings for two intensive months in that semester.

Now, stringy duality, including mirror symmetry, is a very broad and technical subject. Its true/best explanation as yet remains largely physical, rather than mathematical. Its foundation lies on quantum field theory (QFT) and the rigidity of supersymmetric QFT's, together with numerous other mathematical and physical notions, objects, structures, and moduli problems that are incorporated into superstring theory along its continual fast-paced developments. With such an origin from physics, statements from stringy duality are unavoidably mysterious, shocking, and awe-inspiring to mathematicians. Anyone who ventures to lecture on such a subject before a mathematics master like Prof. Bott should expect to face many legitimate-yet-hard-to-give-a-round-off-answer questions, making him/her "hanging on the blackboard" forever. Although I focused only on the much limited topic on toric mirror symmetry, such embarrassing moments still happened no matter how complete-in-a-small-range I thought I had prepared. Yet, this is indeed how Prof. Bott in turn started to "teach" me what mirror symmetry is about! He rejected assumptions without sound reasons. He liked to see things derived from low/dirty scratches rather than from some high end. He constantly asked, "*WHY?*". With that energetic mind, he even attempted to provide his own pictures or explanations after listening to what I had presented. While each great mind has his/her own way of functioning, which can only inspire and is almost always unlearnable, it remains quite an experience to see how a great mind functions as he digests raw materials, thinks, polishes, and comments on them. His questions become a guide toward a deeper understanding.

Prof. Bott impressed me that he is not inclined to read a lot of literatures. This is very different from those from Yau's school. Once I brought for the lecture a pile of related papers marked with red under-lines and margin notes, he stared at them and asked me: "*How much time have you left for thinking?*" Actually, one reason string theory is demanding is that no matter how many notions/techniques one has finally brought to his/her mastery and employs them for fruitful results, there are always things that remain to be learned/understood when one attempts to reach a fuller/more-comprehensive picture. As so many intelligent people are devoted diligently and intensively to it, the growth and diversity and broadness of stringy literatures, including both mathematics and physics, can be terrifying. That particular question of his reminds me of the necessary balance between reading and independent thinking - a lesson I should keep in mind for good. He once said in a lecture at U.C. Berkeley: "*Doing mathematics should be like paddling a canoe downstream - natural and effortless.*" Most of us who study his works will never be able to reach such a Zen-like level of doing mathematics; yet perhaps this is part of what he meant to teach us through the insight, beauty, and elegance of his works. Among his far-reaching influences in mathematics, the orbifold/stack version of his joint work with Prof. Michael Atiyah that gives the *Atiyah-Bott Localization Formula* has been used again and again in the exact computations of Gromov-Witten invariants, a topic within mirror symmetry as well. The formula can be interpreted as a special mathematical version of Feynman's path-integral. Its format of localization can be generalized to other equivariant (co)homology theories, including equivariant K-theory, that can be used for gauge instanton counting for $d = 4$, $N = 2$ super Yang-Mills theory. Such theory (i.e. Seiberg-Witten theory) can be linked, too, to Gromov-Witten theory and mirror symmetry picture, e.g. with the mirror geometry encoded in the complex geometry of a family of Seiberg-Witten curves embeddable in a family of Calabi-Yau manifolds!

The news of Prof. Bott's passing away came in December 2005 while this work was being written with full vigor. These unforgettable hours with him on mirror symmetry are like a gift from him in his later years — completely unexpected, yet marking my mind deep. We thus dedicate this work to the memory of Prof. Bott, an inspiring and forever curious/learning mind.

0. Introduction and outline.

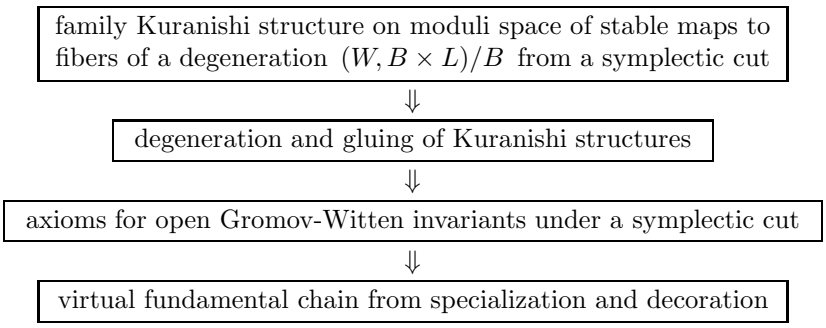
The moduli space of prestable labelled-bordered Riemann surfaces is an Artin stack locally modelled on a quotient of manifolds-with-corners. This leads to the singular real codimension-1 boundary in the Kuranishi structure $\mathcal{K}$ for the moduii space $\overline{\mathcal{M}}_\bullet(X, L\,|\,\bullet)$ of open stable maps to a symplectic manifold X with boundary confined in a Lagrangian submanifold L. Such boundary gives rise to an ambiguity in choosing the virtual fundamental chain on the Kuranishi structure $\mathcal{K}$ for defining open Gromov-Witten invariants of (X, L). To fix the ambiguity, an extra data (i.e. a "decoration") α on L has to be added to the problem and the induced effect of the decoration on L to the whole $\overline{\mathcal{M}}_\bullet(X, L\,|\,\bullet)$ and $\mathcal{K}$ has to be understood. Examples of such decoration α are a group action on L, a bundle map on the restriction $T_*X|_L$, or a diffeomorphism on a neighborhood of L in X that leaves L invariant. However, unless this decoration is extendable to the whole X, there is no obvious way to go from "α on L" to "an associated extra structure on $\overline{\mathcal{M}}_\bullet(X, L\,|\,\bullet)$ and $\mathcal{K}$" to help fix the choice of the virtual fundamental chain $[\overline{\mathcal{M}}_\bullet(X, L^\alpha)\,|\,\bullet]^{virt}$ on $\mathcal{K}$. The main goal of this work is to propose and explain a degeneration axiom and a gluing axiom under a symplectic cut for open Gromov-Witten invariants of a symplectic manifold with a decorated Lagrangian submanifold (X, L^α) to take care of the above technical issue for an important class of (X, L^α) that occurs in the compact version of conifold transitions of Calabi-Yau 3-folds in open/closed string duality in string theory ([Va1]).

Technically, we construct a Kuranishi structure for moduli spaces in

(1) a *family open Gromov-Witten theory* for a symplectic/almost-complex degeneration associated to a symplectic cut, and

(2) a *relative open Gromov-Witten theory* for a symplectic/almost-complex manifold X with a Lagrangian/totally-real submanifold L relative to a codimension-2 symplectic/almost-complex submanifold D that is disjoint from L.

Notions, constructions, and techniques developed by our predecessors in the various formats/ settings/categories are uniformized/merged into the present study. Such structure extends [F-O] and [Liu(C)] to a degeneration-family open Gromov-Witten theory and a relative open Gromov-Witten theory. In the case that L is empty, the study re-writes both the symplecto-analytic [L-R], [I-P1], [I-P2] and the algebro-geometric [Li1], [Li2] in the symplecto-analytic Fukaya-Ono format. For the technical step of constructing the transition data in the Kuranishi structure, we bring in also [Sie1]. How these Kuranishi structures are relevant to the construction of open Gromov-Witten invariants can be summarized by:



Compared with closed Gromov-Witten theory, it may look at first surprising that *in order to understand absolute open Gromov-Witten theory one has to understand both degeneration and relative open Gromov-Witten theory as well.* It could be true that this is not the only

way. However, from the viewpoint of algebraic geometry, the route we take in this project, of which the current work is a part, is an elaborate adoption of the *deformation-specialization technique* already long in use in enumerative (algebraic) geometry; (see, e.g. [Fu: Sec. 10.4] for an introduction). Furthermore, the conjectural *open/closed string duality* on Calabi-Yau threefolds that differ by an extremal transition ([Go-V], [O-V1], [O-V2], [Va1]) almost selects/specifies for us this route uniquely among other possible candidate constructions. Particularly for the motivation and the constant strong drive behind, we owe the credits of this work to enumerative algebraic geometers and string theorists – especially, our teachers Joe Harris and Cumrun Vafa and their respective school. The current work is a step toward a mathematical understanding of the compact version of the open/closed string duality for Calabi-Yau 3-folds in [Go-V], [O-V2], and [Va1] at the level of moduli spaces/stacks of stable maps, cf. the diagram in [L-Y2: Introduction]. (See also [D-F] for related discussions.)

Convention. Standard notations, terminology, operations, facts in (1) symplectic geometry; (2) algebraic geometry; (3) Sobolev theory; (4) topology can be found respectively in (1) [MD-S2], [G-S], [Woo]; (2) [Hart], [G-H]; (3) [MD-S3: Appendix. B], [Au: Chap. 2 - Chap. 3]; (4) [Sp].

- All *dimension*, *codimension*, *rank*, *index*, ..., etc. are with respect to $\mathbb{R}$ unless otherwise noted.
- $|\bullet|$ stands for the *cardinality* of $\bullet$ when $\bullet$ is a finite set or a finite group, for the *absolute value* or *norm* of $\bullet$ when $\bullet$ is a real or complex number or a vector, for the *sum* of the entries when $\bullet$ is a vector of integers referring to some combinatorial quantity (like number of marked points or contact order).
- The *complex projective space* of complex dimension n is denoted by $\mathbb{P}^n$.
- In denoting a stable map $f : \Sigma \to (X, L)$ to a symplectic space X with a Lagrangian submanifold L, it is assumed that $f(\partial\Sigma) \subset L$. Similarly, for a relative map $f : \Sigma \to (Z, L; D)$. When L is empty, so is $\partial\Sigma$.
- Properties of a map from a nodal (bordered) curve Σ to another is imposed on its normalization $\tilde{\Sigma}$; e.g. a C^∞ map f from Σ to $Y := Y_1 \cup_D Y_2$ is a continuous map $f : \Sigma \to Y$ such that its lift to $\tilde{\Sigma}_0$ is a C^∞ map to either Y_1 or Y_2, where Σ_0 runs over all irreducible components of Σ.
- Almost-complex (resp. complex) structures on different target (resp. domains) spaces are usually denoted by the same J (resp. j) unless the distinction is crucial to the discussion.
- The term "*orbifolds*" and "*sub-orbifolds*" are not restricted only to smooth ones.
- Omitted superscripts or subscripts are often denoted by $\dot{}$, $^\bullet$ or $\cdot$, $_\bullet$.
- Commonly used notations for different objects that have no chances of confusion:
  - $\mathbb{C}$ and $\mathbb{R}$: as the *complex plane* and the *real line* in differential geometry vs. as *ground fields* in algebraic geometry;
  - *curve class* β vs. *isomorphism* (α, β);
  - *isomorphism* α of curves or graphs vs. *decoration* α on a Lagrangian/almost-complex submanifold;
  - *universal curve* $\mathcal{C}$ over different bases vs. *category* $\mathcal{C}$;
  - *genus* g vs. *map* g;
  - *index set* I vs. *gluing map* $I_\bullet$ that *i*dentifies subsets.

Outline.

1. Symplectic cut and the direct system of expanded degenerations in the almost-complex category.
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4. The moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B} \,|\, [\beta], \vec{\gamma}, \mu)$ of stable $\check{W}^{1,p}$-maps.
   - 4.1 The moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ of stable $\check{W}^{1,p}$-maps to $(W[k], L[k])$, its relative tangent and relative obstruction bundles.
   - 4.2 The moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B} \,|\, [\beta], \vec{\gamma}, \mu)$ of stable $\check{W}^{1,p}$-maps to fibers of $(\widehat{W}, \widehat{L})/\widehat{B}$, the relative $\check{W}^{1,p}$-tangent-obstruction fibration complex.
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7. Degeneration and gluing of Kuranishi structures and axioms of open Gromov-Witten invariants under a symplectic cut.
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Appendix. The equivalence of Li-Ruan/Li's degeneration formula and Ionel-Parker's degeneration formula.

1 Symplectic cut and the direct system of expanded degenerations in the almost-complex category.

A direct system of expanded degenerations of almost-complex spaces that merges the symplectic construction (via multi- symplectic cut) in [I-P2: Sec. 2 and Sec. 12] and [L-R: Sec. 3] with the algebro-geometric construction (via blow-ups and blow-downs) in [Li: Sec. 1] without using the full language of stacks is given in this section. The fibers, up to a relative isomorphism, of the families in the system will occur as the targets of open stable maps in the problem. The same construction gives also a direct system of expanded relative pairs (cf. [I-P1: Remark 7.7], [L-R: Sec. 4]; [Gr-V: Sec. 2], [Li1: Sec. 4.1]) needed for relative open Gromov-Witten theory.

1.1 Symplectic cut and the associated expanded degenerations.

1.1.1 Expanded degenerations from a symplectic cut.

Symplectic cut and a compatible almost-complex degeneration.

Symplectic cut was introduced in [Le] and used in [I-P1], [I-P2], and [L-R]. We review it here to fix notations. Given a free Hamiltonian S^1 action on a connected open set U of a symplectic manifold X that separates X. Fix a Hamiltonian function $h : U \to (-l, l)$ of the S^1-action and let $X - h^{-1}(0) = X_+ \amalg X_-$. Then the manifold with boundary $\overline{X}_+ := X_+ \cup h^{-1}(0)$ (resp. $\overline{X}_- := X_- \cup h^{-1}(0)$) gives rise to a symplectic manifold Y_1 (resp. Y_2) by taking the quotient of the S^1-action on the boundary, and the boundary $h^{-1}(0)$ descends to a codimension-2 symplectic submanifold D in Y_1 (resp. Y_2). Let Y be the singular symplectic space from gluing Y_1 and Y_2 canonically along D. Then, there is a natural map $\xi : X \to Y$ that is modelled on a symplectic reduction (and hence an S^1-bundle) over the singular locus $D := Y_1 \cap Y_2$ on Y and is a symplectomorphism from $X - \xi^{-1}D$ to $Y - D$.

Definition 1.1.1.1 [symplectic cut]. With an abuse of language and a different naming than the original work [Le] of Lerman, we will call both the map $\xi : X \to Y = Y_1 \cup_D Y_2$ and the singular symplectic space Y a *symplectic cut* of X.

Given a symplectic cut $\xi : X \to Y = Y_1 \cup_D Y_2$, one can identify a small neighborhood of D in Y_1 (resp. Y_2) with a neighborhood of the zero-section of a complex line bundle $\mathbb{L}$ (resp. the dual complex line bundle $\mathbb{L}^*$) over D and construct a complex 1-parameter family $\pi : W \to B$ of symplectic spaces $W_\lambda := \pi^{-1}(\lambda)$, $\lambda \in B$, with a compatible almost-complex structure J_{W_λ} such that W_0 is symplecto-isomorphic to Y, with the restriction of J_{W_0} to a neighborhood of D almost-complex-isomorphic to the gluing of a neighborhood of the zero-section in $\mathbb{L}$ and a neighborhood of the zero-section in $\mathbb{L}^*$ along D, and W_λ, $\lambda \neq 0$, is symplecto-isomorphic to X. Here B is a small neighborhood of 0 in $\mathbb{C}$ and the total space of $\mathbb{L}$ and $\mathbb{L}^*$ are equipped with an $U(1)$-invariant almost-complex structure that combines an almost-complex structure J_D on D and the complex structure on fiber $\mathbb{C}$ via a $U(1)$-connection on $\mathbb{L}$ and $\mathbb{L}^*$. See [I-P2: Sec. 2] (and also [Go] and [MC-W]) for an explicit construction. The total space W is equipped with a symplectic structure ω_W and a compatible almost-complex structure J_W that gives $(\omega_{W_\lambda}, J_{W_\lambda})$ when restricted to W_λ. We will denote the family $\pi : W \to B$ also by W/B as in algebraic geometry and call W/B a *compatible almost-complex degeneration* associated to the symplectic cut $\xi : X \to Y$.

Fix and denote a local fiber complex coordinate of $\mathbb{L}$ (resp. $\mathbb{L}^*$) by w (resp. w') and treat both $\mathbb{L}$ and $\mathbb{L}^*$ as a $U(1)$-bundle. Let $0 < \varepsilon < 1$ be sufficiently small. Then, possibly after

shrinking, we may assume that

$$B \;=\; \{\lambda \in \mathbb{C} \,:\, |\lambda| < \varepsilon^2/2\}\,.$$

The following defines a subset of $\mathbb{L} \oplus \mathbb{L}^*$

$$(\mathbb{L} \oplus \mathbb{L}^*)_{\leq \varepsilon} \;=\; \{(\,\cdot\,, w, w') \,:\, |w| \leq \varepsilon\,,\, |w'| \leq \varepsilon\,,\, |ww'| \leq \varepsilon^2/2\}\,.$$

It admits a fibration $(\mathbb{L} \oplus \mathbb{L}^*)_{\leq \varepsilon} \to B$ defined by $(\,\cdot\,, w, w') \mapsto ww'$. With this fibration and an adjustment in the construction of W, there is a decomposition

$$W \;=\; (B \times \overline{U_1}) \cup (\mathbb{L} \oplus \mathbb{L}^*)_{\leq \varepsilon} \cup (B \times \overline{U_2})$$

over B, where B is taken to be $\{\lambda \in \mathbb{C} : |\lambda| < \varepsilon^2/2\}$, $U_1 = Y_1 - (\varepsilon$-neighborhood of the zero-section in $\mathbb{L})$, $U_2 = Y_2 - (\varepsilon$-neighborhood of the zero-section in $\mathbb{L}^*)$, and the gluing is along the related boundary circle-bundle over $B \times D$ in a way that respects the fibration of and the $U(1)$-action on these boundaries over $B \times D$. This decomposition allows us to construct an expanded almost-complex degeneration associated to W/B, which we explain in the next two themes.

Local expanded degenerations in the almost-complex category.

The expanded degeneration around D in the almost-complex category can be described by a finite collection of almost-complex manifolds together with a collection of gluing isomorphisms between open dense almost-complex submanifolds therein as follows.

Let $\mathbb{L}$ be a complex line bundle on the almost-complex manifold with the $\mathbb{C}^\times$-structure reduced to a $U(1)$-structure, and let α be a $U(1)$-connection on $\mathbb{L}$. This induces a unique $U(1)$-structure on the complex dual line bundle $\mathbb{L}^*$ to $\mathbb{L}$ on D with a $U(1)$-connection α^*. Denote the almost-complex structure on D be J_D; then the pairs (J_D, α) and (J_D, α^*), together with the fiberwise complex structures, determines almost-complex structures $J_{\mathbb{L}}$, $J_{\mathbb{L}^*}$ on the total space (still denoted by the same notation) of $\mathbb{L}$ and $\mathbb{L}^*$ respectively. $J_{\mathbb{L}}$ and $J_{\mathbb{L}^*}$ together induce an almost-complex structure $J_{\mathbb{L}\oplus\mathbb{L}^*}$ on (the total space of) $\mathbb{L} \oplus \mathbb{L}^*$.

Denote the zero-section of $\mathbb{L}$ or $\mathbb{L}^*$ by $\mathbf{0}$ and the projection map $\mathbb{L} \oplus \mathbb{L}^* \to \mathbb{L}$ (resp. $\mathbb{L} \oplus \mathbb{L}^* \to \mathbb{L}^*$) by pr (resp. pr'). Fix a system of local trivializations and $U(1)$-valued transition functions for $\mathbb{L}$. They induce a system of local trivializations and $U(1)$-valued transition functions on $\mathbb{L}^*$, and then a system of local trivializations and transition functions on $\mathbb{L} \oplus \mathbb{L}^*$. With respect to this, the map $\pi : \mathbb{L} \oplus \mathbb{L}^* \to \mathbb{C}$ given by $(x; w, w') \mapsto ww'$ is well-defined and compatible with $J_{\mathbb{L}\oplus\mathbb{L}^*}$, where (x, w) (resp. (x, w')) are local coordinates for $\mathbb{L}$ (resp. $\mathbb{L}^*$) in the specified local trivialization. Let $M_\lambda \subset \mathbb{L} \oplus \mathbb{L}^*$ be the preimage $\pi^{-1}(\lambda)$ of $\lambda \in \mathbb{C}$. For $\lambda \in \mathbb{C} - \{0\}$, M_λ is isomorphic to $\mathbb{L} - \mathbf{0}$ ($\simeq \mathbb{L}^* - \mathbf{0}$) as an almost-complex submanifold. For $\lambda = 0$, $M_0 = \mathbb{L} \vee \mathbb{L}^*$, the union of $\mathbb{L}$ and $\mathbb{L}^*$ with the zero-sections glued by the canonical isomorphism with D. Thus, the family $\pi : \mathbb{L} \oplus \mathbb{L}^* \to \mathbb{C}$ is a smoothing of M_0 over D in the almost-complex category.

Notation 1.1.1.2 [M_λ, $\lambda \neq 0$, *from gluing*]. Associated to the $U(1)$-structure on $\mathbb{L}$ and $\mathbb{L}^*$ is a well-defined norm function $|\,\cdot\,|$ on fibers of $\mathbb{L}$ and $\mathbb{L}^*$. Let $\mathbb{L}_{>\delta} = \{\,|w| > \delta\,\} \subset \mathbb{L}$, $\mathbb{L}^*_{>\delta} = \{\,|w'| > \delta\,\} \subset \mathbb{L}^*$, and, similarly, for $\mathbb{L}_{\leq\delta}$, $\mathbb{L}_{[\delta_1\,,\,\delta_2]}$, $\mathbb{L}^*_{\leq\delta}$ $\mathbb{L}^*_{[\delta_1\,,\,\delta_2]}$, $\cdots$, etc.. These bundles over D are equipped with the $U(1)$-connection (still denoted by α and α^*) from the restriction of that on $\mathbb{L}$ and $\mathbb{L}^*$ respectively. The local fiberwise maps $(x, w') \mapsto (x, w) = (x, \lambda/w')$, glue to a bundle isomorphism

$$\varphi_\lambda \;:\; \mathbb{L}^*_{[|\lambda|/\delta\,,\,\delta]} \;\xrightarrow{\sim}\; \mathbb{L}_{[|\lambda|/\delta\,,\,\delta]}\,, \qquad (x, w') \mapsto (x, w) = (x, \lambda/w')$$

for $0 \leq |\lambda| < \delta^2$ such that $\varphi_\lambda^* \alpha = -\alpha^*$. Thus, φ_λ is an isomorphism in the category of almost-complex manifolds as well. In terms of this, M_λ, $\lambda \neq 0$, is the almost-complex manifold obtained from gluing $\mathbb{L}_{>|\lambda|/\delta}$ and $\mathbb{L}^*_{>|\lambda|/\delta}$, with $|\lambda| < \delta^2$, by φ_λ. The maps

$$\begin{array}{rccccc} \theta_\lambda & : & \mathbb{L}_{>0} & \longrightarrow & M_\lambda \\ & & (\cdot\,, w) & \longmapsto & (\cdot\,, w, \frac{\lambda}{w}) \end{array} \qquad \text{and} \qquad \begin{array}{rccccc} \theta'_\lambda & : & \mathbb{L}^*_{>0} & \longrightarrow & M_\lambda \\ & & (\cdot\,, w') & \longmapsto & (\cdot\,, \frac{\lambda}{w'}, w') \end{array}$$

are almost-complex isomorphisms. We will denote the restriction of θ_λ (resp. θ'_λ, $\theta_\lambda \cup \theta'_\lambda$) to the subsets $\mathbb{L}_{[\delta_1,\delta_2]}$, ..., etc. of $\mathbb{L}_{>0}$ (resp. $\mathbb{L}^*_{[\delta'_1,\delta'_2]}$ ... of $\mathbb{L}^*_{>0}$, $\mathbb{L}_{[\delta_1,\delta_2]} \cup \mathbb{L}^*_{[\delta'_1,\delta'_2]}$ of $\mathbb{L}_{>0} \cup \mathbb{L}^*_{>0}$) by $\theta_{\lambda\,;\,[\delta_1,\delta_2]}$ (resp. $\theta'_{\lambda\,;\,[\delta'_1,\delta'_2]}$, $\theta_{\lambda\,;\,[\delta_1,\delta_2]} \cup \theta'_{\lambda\,;\,[\delta'_1,\delta'_2]}$).

For $k \in \mathbb{Z}_{\geq 0}$, let $B[k] = \mathbb{C}^{k+1}$, with coordinates $(\lambda_0, \ldots, \lambda_k)$, and $pr_i : B[k] \to \mathbb{C}$ be the i-th coordinate projection map. Let $(\mathbb{L} \oplus \mathbb{L}^*)_i = pr_i^*(\mathbb{L} \oplus \mathbb{L}^*)$, $i = 0, \ldots, k$, be the pulled-back of $\pi : \mathbb{L} \oplus \mathbb{L}^* \to \mathbb{C}$ to $B[k]$ via pr_i. The local coordinates of $(\mathbb{L} \oplus \mathbb{L}^*)_i$ will be denoted by $(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_k; x, w_i, w'_i)$ with $\lambda_i = w_i w'_i$. Let $(\mathbb{L} \oplus \mathbb{L}^*)_i^0 := (\mathbb{L} \oplus \mathbb{L}^*)_i - pr_i^*\mathbb{L}^*$, which is $\{w_i \neq 0\}$ in local coordinates, and $(\mathbb{L} \oplus \mathbb{L}^*)_i^\infty := (\mathbb{L} \oplus \mathbb{L}^*)_i - pr_i^*\mathbb{L}$, which is $\{w'_i \neq 0\}$ in local coordinates. We will use coordinates $(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_k; x, w_i, \lambda_i/w_i)$ and $(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_k; x, \lambda_i/w'_i, w'_i)$ respectively for these two open dense almost-complex submanifolds of $(\mathbb{L} \oplus \mathbb{L}^*)_i$. In terms of these, the following map

$$\begin{array}{ccc} (\mathbb{L} \oplus \mathbb{L}^*)_{i-1}^\infty & \overset{\varphi_{i-1,i}}{\longrightarrow} & (\mathbb{L} \oplus \mathbb{L}^*)_i^0 \\ (\lambda_0, \ldots, \lambda_{i-1}, \lambda_i, \ldots, \lambda_k; x, \frac{\lambda_{i-1}}{w'_{i-1}}, w'_{i-1}) & \longmapsto & (\lambda_0, \ldots, \lambda_{i-1}, \lambda_i, \ldots, \lambda_k; x, \frac{1}{w'_{i-1}}, \lambda_i w'_{i-1}) \end{array}$$

is an isomorphism in the almost-complex category for $i = 1, \ldots, k$. The system

$$\left(\{(\mathbb{L} \oplus \mathbb{L}^*)_i\}_{i=0}^k \,,\, \{\varphi_{\,i-1,\,i}\}_{i=1}^k \right)$$

of almost-complex manifolds and gluing data determines an almost-complex manifold $(\mathbb{L} \oplus \mathbb{L}^*)[k]$ that fibers over $B[k]$.

Definition 1.1.1.3 [expanded degeneration of $(\mathbb{L} \oplus \mathbb{L}^*)/\mathbb{C}$]. We will call the family of almost-complex spaces as constructed above, $\pi[k] : (\mathbb{L} \oplus \mathbb{L}^*)[k] \to B[k]$ (in short hand: $(\mathbb{L} \oplus \mathbb{L}^*)[k]/B[k]$), the *$k$-th expanded degeneration* of the degeneration $\pi : \mathbb{L} \oplus \mathbb{L}^* \to \mathbb{C}$.

We will use the above gluing construction of $(\mathbb{L} \oplus \mathbb{L}^*)[k]/B[k]$ as the foundation for the rest of the discussion on expanded degenerations.

The natural maps from pull-backs can be re-scaled to give maps $\tilde{\mathbf{p}}[k]_i : (\mathbb{L} \oplus \mathbb{L}^*)_i \to \mathbb{L} \oplus \mathbb{L}^*$ defined by

$$\begin{array}{l} (\lambda_0, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \ldots, \lambda_k; x, w_i, w'_i) \\ \qquad \longmapsto (\lambda_0 \cdots \lambda_k; x, (\lambda_0 \cdots \lambda_{i-1})\, w_i, (\lambda_{i+1} \cdots \lambda_k)\, w'_i) \end{array}$$

with $w_i w'_i = \lambda_i$, for $i = 0, \ldots, k$. These maps glue to a map $\tilde{\mathbf{p}}[k] : (\mathbb{L} \oplus \mathbb{L}^*)[k] \to \mathbb{L} \oplus \mathbb{L}^*$ over $\mathbf{p}[k] : B[k] \to \mathbb{C}$ in the almost-complex category.

All the gluing isomorphisms $\varphi_{\,i-1,\,i}$, $i = 1, \ldots, k$, are maps over $B[k]$. Thus, the fibers of $\pi[k] : (\mathbb{L} \oplus \mathbb{L}^*)[k] \to B[k]$ can be described by the corresponding gluing over a fixed values of $\vec{\lambda} = (\lambda_0, \ldots, \lambda_k)$, as follows. First, note that the φ_λ, $\lambda \in \mathbb{C}^\times$, defined in Notation 1.1.1.2 gives as well an isomorphism $\varphi_\lambda : \mathbb{L}^*_{>0} \to \mathbb{L}_{>0}$ in the almost-complex category. The gluing of $\mathbb{L}$ and $\mathbb{L}^*$ by φ_λ gives a ruled (i.e. $\mathbb{P}^1$-fibered) manifold Δ over D with a well-defined almost-complex structure that contains $\mathbb{L}$ and $\mathbb{L}^*$ as open almost-complex submanifolds. Different choices of λ

give rise to isomorphic almost-complex manifolds over D with a such isomorphism provided by the identity map on $\mathbb{L}$ (and hence on Δ). We will take Δ as from the gluing φ_1. Denote the zero-section $\mathbf{0}$ of $\mathbb{L}$ (resp. $\mathbb{L}^*$) by D_0 (resp. D_∞) in Δ. Let $(\mathbb{L} \vee \mathbb{L}^*)_i := \mathbb{L}_i \vee \mathbb{L}_i^*$, $i = 0, \ldots, k'$, be identical copies of $\mathbb{L} \vee \mathbb{L}^*$ and $(\mathbb{L} \vee \mathbb{L}^*)_{[k']}$ be the gluing of $(\mathbb{L} \vee \mathbb{L}^*)_i$, $i = 0, \ldots, k'$, by $\varphi_{i;1} : \mathbb{L}_{i-1}^* \to \mathbb{L}_i$, $(x, w'_{i-1}) \mapsto (x, w_i) = (x, 1/w'_{i-1})$. Then, as an almost-complex space,

$$(\mathbb{L} \vee \mathbb{L}^*)_{[k']} \;=\; \mathbb{L} \cup_{\mathbf{0}=D_{1,\infty}} \Delta_1 \cup_{D_{1,0}=D_{2,\infty}} \cdots \cup_{D_{k'-1,0}=D_{k',\infty}} \Delta_{k'} \cup_{D_{k',0}=\mathbf{0}} \mathbb{L}^*\,,$$

where $(\Delta_i; D_{i,0}, D_{i,\infty}) = (\Delta; D_0, D_\infty)$. There is a natural map $(\mathbb{L} \vee \mathbb{L}^*)_{[k']} \to \mathbb{L} \vee \mathbb{L}^*$ that restricts to the identity map on $\mathbb{L}$ and $\mathbb{L}^*$, and collapses all Δ_i to D. The natural $\mathbb{G}_m := \mathbb{C}^\times$-action on $\mathbb{L}$ extends to a $\mathbb{G}_m$-action on Δ as a group of automorphisms of Δ over D in the almost-complex category. For $\sigma \in \mathbb{G}_m$, the induced action coincides with the composition $\varphi_\sigma \circ {\varphi_1}^{-1}$ on $\Delta - D_0 \cup D_\infty$. It leaves $D_0 \cup D_\infty$ fixed. The relative automorphism group $Aut((\mathbb{L} \vee \mathbb{L}^*)_{[k']}/\mathbb{L} \vee \mathbb{L}^*)$ in the almost-complex category is the product $\prod_{i=1}^{k'} Aut(\Delta_i/D) = (\mathbb{C}^\times)^{k'}$. Now let $I = \{i_0, \ldots, i_{k'}\}$ be a subset of $\{0, \ldots, k\}$ and $\dot{H}_I$ be the locally closed submanifold of $B[k]$, whose points have coordinates $\lambda_i = 0$ exactly when $i \in I$. $B[k]$ is the disjoint union of $\dot{H}_I$, where I runs over all the subsets of $\{0, \ldots, k\}$. Let $\vec{\lambda} = (\lambda_0, \ldots, \lambda_k) \in \dot{H}_I$. Then, $\pi[k]^{-1}(\vec{\lambda})$ is the almost-complex space from the system

$$\left(\{M_{\lambda_i}\}_{i=0}^k\,,\, \{\varphi_{\,i-1,\,i\,;\,\vec{\lambda}}\}_{i=1}^k\right)\,,$$

where $M_{\lambda_i} = \{w_i w'_i = \lambda_i\} \subset \mathbb{L}_i \oplus \mathbb{L}_i^*$ and $\varphi_{\,i-1,\,i\,;\,\vec{\lambda}} : M_{\lambda_{i-1}} - \mathbb{L}_{i-1} \to M_{\lambda_i} - \mathbb{L}_i^*$ is given by $(\lambda_{i-1}/w_{i-1}, w'_{i-1}) \mapsto (w_i, \lambda_i/w_i) = (1/w'_{i-1}, \lambda_i w'_{i-1})$. This system can be reduced to the following system

$$\left(\{M_{\lambda_{i_j}}\}_{j=-1}^{k'+1}\,,\, \{\tilde{\varphi}_{\,j-1,\,j\,;\,\vec{\lambda}}\}_{j=0}^{k'+1}\right)\,,$$

where $M_{\lambda_{i_{-1}}} = \mathbb{L}_0 - \{\mathbf{0}\}$ and $M_{\lambda_{i_{k'+1}}} = \mathbb{L}_k^* - \{\mathbf{0}\}$ by convention, and $\tilde{\varphi}_{\,j-1,\,j\,;\,\vec{\lambda}} : M_{\lambda_{i_{j-1}}} - \mathbb{L}_{i_{j-1}} \to M_{\lambda_{i_j}} - \mathbb{L}_{i_j}^*$ is the composition $\varphi_{i_j-1\,,\,i_j} \circ \cdots \circ \varphi_{i_{j-1}+1\,,\,i_{j-1}+2} \circ \varphi_{i_{j-1}\,,\,i_{j-1}+1}$ with $\varphi_{i-1\,,0}$ and $\varphi_{k\,,\,k+1}$ being identity maps by convention.

In summary,

Lemma 1.1.1.4 [natural map and its fibers]. *Let $\mathbf{p}[k] : B[k] \to \mathbb{C}$ be the product map defined by $(\lambda_0, \ldots, \lambda_k) \mapsto \lambda_0 \cdots \lambda_k$. Then there is a natural map $\tilde{\mathbf{p}}[k] : (\mathbb{L} \oplus \mathbb{L}^*)[k] \to (\mathbb{L} \oplus \mathbb{L}^*)$ in the almost-complex category that covers $\mathbf{p}[k]$. The fiber of $\pi[k]$ at $\vec{\lambda} \in \dot{H}_I$ is isomorphic to $M_{\lambda_0 \cdots \lambda_k}$, if $I = \emptyset$, and to $(\mathbb{L} \vee \mathbb{L}^*)_{[k']}$, if $I = \{i_0, \ldots, i_{k'}\}$ is non-empty. In particular, $\tilde{\mathbf{p}}[k]$ is an isomorphism when restricted to the fiber over a point in the complement of complex codimension-2 coordinate subspaces.*

Expanded almost-complex degenerations associated to W/B.

Given a fibered almost-complex space W/B from a symplectic cut as constructed above, recall the decomposition over B

$$W \;=\; (B \times \overline{U_1}) \cup (\mathbb{L} \oplus \mathbb{L}^*)_{\le \varepsilon} \cup (B \times \overline{U_2})\,.$$

Let

$$\begin{aligned} \overline{U_1}[k] \;&:=\; pr_0^*((B \times \overline{U_1})/B) \;\simeq\; B[k] \times \overline{U_1}\,, \\ \overline{U_2}[k] \;&:=\; pr_k^*((B \times \overline{U_2})/B) \;\simeq\; B[k] \times \overline{U_2}\,, \end{aligned}$$

and define the *ε-truncation* $(\mathbb{L} \oplus \mathbb{L}^*)[k]_{\le \varepsilon}$ of $(\mathbb{L} \oplus \mathbb{L}^*)[k]$ as follows.

· First consider the preliminary truncation of $(\mathbb{L}\oplus\mathbb{L}^*)_0$ and $(\mathbb{L}\oplus\mathbb{L}^*)_k$ defined respectively by

$$\mathcal{U}_0' \;:=\; \{|w_0|\le\varepsilon\}\subset(\mathbb{L}\oplus\mathbb{L}^*)_0 \quad\text{and}\quad \mathcal{U}_k' \;:=\; \{|w_k'|\le\varepsilon\}\subset(\mathbb{L}\oplus\mathbb{L}^*)_k\,.$$

· Then the pair of gluing maps

$$(\,\varphi_{i-1,i}\circ\cdots\circ\varphi_{0,1}\,,\,\varphi_{i,i+1}\circ\circ\cdots\circ\varphi_{k-1,k}^{-1}\,)\,,$$

from $(\mathcal{U}_0',\mathcal{U}_k')$ to $\mathcal{U}_0'$, $(\mathbb{L}\oplus\mathbb{L}^*)_i$, $i=1,\ldots,k-1$, and $\mathcal{U}_k'$ respectively induces a truncation thereon defined by

$$\begin{aligned}
\mathcal{U}_0 &= \{\,(\lambda_0,\ldots,\lambda_k;x,\,w_0,\,w_0')\in(\mathbb{L}\oplus\mathbb{L}^*)_0\;:\;|w_0|\le\varepsilon\,,\;|w_0'|\le\tfrac{\varepsilon}{|\lambda_1\cdots\lambda_k|}\,\}\,,\\
\mathcal{U}_i &= \{\,(\lambda_0,\ldots,\lambda_k;x,\,w_i,\,w_i')\in(\mathbb{L}\oplus\mathbb{L}^*)_i\;:\;|w_i|\le\tfrac{\varepsilon}{|\lambda_0\cdots\lambda_{i-1}|}\,,\;|w_i'|\le\tfrac{\varepsilon}{|\lambda_{i+1}\cdots\lambda_k|}\,\}\,,\\
&\qquad\qquad i\;=\;1\,,\ldots,k-1\,,\\
\mathcal{U}_k &= \{\,(\lambda_0,\ldots,\lambda_k;x,\,w_k,\,w_k')\in(\mathbb{L}\oplus\mathbb{L}^*)_k\;:\;|w_k|\le\tfrac{\varepsilon}{|\lambda_0\cdots\lambda_{k-1}|}\,,\;|w_k'|\le\varepsilon\,\}\,.
\end{aligned}$$

· The gluing map $\varphi_{i-1,i}$ sends $\mathcal{U}_{i-1}\cap(\mathbb{L}\oplus\mathbb{L}^*)_i^\infty$ to $\mathcal{U}_i\cap(\mathbb{L}\oplus\mathbb{L}^*)_i^0$. Thus the collection $\{\varphi_{i-1,i}\}_{i=1}^k$ of gluing maps glue the collection $\{\mathcal{U}_i\}_{i=0}^k$ of almost-complex manifolds to an almost- complex manifold over $B[k]$, which will be denoted $(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\le\varepsilon}$ and called the *ε-truncation* of $(\mathbb{L}\oplus\mathbb{L}^*)[k]$.

Then the gluing $W \;=\; (B\times\overline{U_1})\cup(\mathbb{L}\oplus\mathbb{L}^*)_{\le\varepsilon}\cup(B\times\overline{U_2})$ induces via pr_1^* and pr_k^* a canonical gluing of

$$\overline{U_1}[k]\cup(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\le\varepsilon}\cup\overline{U_2}[k] \;=:\; W[k]$$

over $B[k]$, that goes with a map $\pi[k]:W[k]\to B[k]$. Here we shrink and re-define $B[k]$ to be

$$B[k] \;:=\; \{(\lambda_0,\ldots,\lambda_k)\;:\;|\lambda_i|<\varepsilon^2/2\,,\;i\;=\;0\,,\ldots,k\}\,.$$

The fiber of $\overline{U_1}[k]$, $(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\le\varepsilon}$, and $\overline{U_2}[k]$ over the same point in $B[k]$ glue to an almost-complex space.

Definition 1.1.1.5 [expanded degeneration of W/B]. The family $\pi[k]:W[k]\to B[k]$, in short $W[k]/B[k]$, of almost-complex spaces is called an *expanded almost-complex degeneration* of W/B.

Let $I\subset\{1,\ldots,n\}$ be non-empty. Then it follows from the construction that, for $\vec{\lambda}\in B[k]\cap\dot{H}_I$, the fiber almost-complex space $W[k]_{\vec{\lambda}}:=\pi[k]^{-1}(\vec{\lambda})$ is almost-complex-isomorphic to

$$Y_{[k']} \;:=\; Y_1\cup_{D=D_{1,\infty}}\Delta_1\cup_{D_{1,0}=D_{2,\infty}}\cdots\cup_{D_{k'-1,0}=D_{k',\infty}}\Delta_{k'}\cup_{D_{k',0}=D}Y_2$$

with $k'=|I|$. By construction, there is an almost-complex morphism

$$\tilde{\mathbf{p}}[k]\;:\;W[k]/B[k]\;\longrightarrow\;W/B\,,$$

cf. Lemma 1.1.1.4.

Neck-trunk decompositions of $W[k]/B[k]$ and re-forgings.

We introduce here neck-trunk decompositions of $W[k]/B[k]$ and re-forging morphisms for the discussion of rigidification in Sec. 4.2 and the gluing construction of a Kuranishi neighborhood in Sec. 5.3.

Recall $0 < \varepsilon < 1$ and consider the decomposition over B:

$$(\mathbb{L} \oplus \mathbb{L}^*)_{\leq 1/\varepsilon} \;=\; (\mathbb{L} \oplus \mathbb{L}^*)_{[\varepsilon,1/\varepsilon]\,;\,1} \cup (\mathbb{L} \oplus \mathbb{L}^*)_{\leq \varepsilon} \cup (\mathbb{L} \oplus \mathbb{L}^*)_{[\varepsilon,1/\varepsilon]\,;\,2}\,,$$

where

$$\begin{aligned}(\mathbb{L} \oplus \mathbb{L}^*)_{[\varepsilon,1/\varepsilon]\,;\,1} &= \{(\,\cdot\,, w, w') \,:\, \varepsilon \leq |w| \,\leq 1/\varepsilon\,,\, |ww'| \leq \varepsilon^2/2\}\,,\\ (\mathbb{L} \oplus \mathbb{L}^*)_{[\varepsilon,1/\varepsilon]\,;\,2} &= \{(\,\cdot\,, w, w') \,:\, \varepsilon \leq |w'| \leq 1/\varepsilon\,,\, |ww'| \leq \varepsilon^2/2\}\,,\end{aligned}$$

all three components fiber over B via $(\,\cdot\,, w, w') \mapsto ww'$, and the gluing is along their horizonal boundary over B. Let

$$\begin{aligned}\mathit{Neck}[k]_i &= \text{the image of } pr_i^*((\mathbb{L} \oplus \mathbb{L}^*)_{\leq \varepsilon}) \text{ in } W[k]\,, && i \;=\; 0\,,\,\ldots\,,\,k\,,\\ \mathit{Trunk}[k]_{i;1} &= \text{the image of } pr_i^*((\mathbb{L} \oplus \mathbb{L}^*)_{[\varepsilon,1/\varepsilon]\,;\,1}) \text{ in } W[k]\,, && i \;=\; 1\,,\,\ldots\,,\,k\,,\\ \mathit{Trunk}[k]_{i;2} &= \text{the image of } pr_i^*((\mathbb{L} \oplus \mathbb{L}^*)_{[\varepsilon,1/\varepsilon]\,;\,2}) \text{ in } W[k]\,, && i \;=\; 0\,,\,\ldots\,,\,k-1\,.\end{aligned}$$

Then all these spaces fiber over $B[k]$. Furthermore, since $0 < \varepsilon < 1$ and $|\lambda_i| < \varepsilon^2$ for all $\vec{\lambda} = (\lambda_0, \ldots, \lambda_k) \in B[k]$, one has

$$\mathit{Trunk}[k]_{i-1;2} \;=\; \mathit{Trunk}[k]_{i;1} \;=:\; \mathit{Trunk}[k]_i \qquad\qquad \text{and}$$

$$W[k]/B[k] \;=\; (\,\overline{U_1}[k] \,\cup\, \mathit{Neck}[k]_0 \,\cup\, \mathit{Trunk}[k]_1 \,\cup\, \mathit{Neck}[k]_1 \,\cup\, \cdots \,\cup\, \mathit{Trunk}[k]_k \,\cup\, \mathit{Neck}[k]_k \,\cup\, \overline{U_2}[k]\,)/B[k]\,,$$

where the gluings are along the horizontal boundary of each component over $B[k]$ and are all induced by the gluing maps $\varphi_{i-1,i}$'s. We shall call this a (ε-)*neck-trunk decomposition*, $\mathit{Neck}[k]_i$ a (ε-)*neck region*, and $\mathit{Trunk}[k]_i$ a (ε-)*trunk region* of $W[k]/B[k]$. When in need of expressing ε explicitly, we will denote a neck (resp. trunk) by $\mathit{Neck}_\varepsilon[k]_i$ (resp. $\mathit{Trunk}_\varepsilon[k]_i$).

Denote the fiber of $\mathit{Neck}[k]_i$, $\mathit{Trunk}[k]_j$, $i = 0\,,\,\ldots\,,\,k$, $j = 1\,,\,\ldots\,,\,k$, over $\vec{\lambda} \in B[k]$ by $\mathit{Neck}[k]_{i\,,\,\vec{\lambda}}$, $\mathit{Trunk}[k]_{j\,,\,\vec{\lambda}}$ respectively. Then $W[k]_{\vec{\lambda}}$ is divided to a gluing-along-boundary:

$$W[k]_{\vec{\lambda}} = \overline{U_1} \,\cup\, \mathit{Neck}[k]_{0\,,\,\vec{\lambda}} \,\cup\, \mathit{Trunk}[k]_{1\,,\,\lambda} \,\cup\, \mathit{Neck}[k]_{1\,,\,\lambda} \,\cup\, \cdots \,\cup\, \mathit{Trunk}[k]_{k\,,\,\lambda} \,\cup\, \mathit{Neck}[k]_{k\,,\,\lambda} \,\cup\, \overline{U_2}\,.$$

Recall Notation 1.1.1.2 and that $W[k]_{\vec{0}} = Y_{[k]}$ and denote $D_i = \Delta_i \cap \Delta_{i+1}$, $i = 0, \ldots, k$. Let $\vec{\lambda} \in B[k]$ and $0 \leq i_0 < \cdots < i_{k'} \leq k$ be the associated indices so that $\lambda_{i_j} = 0$. Then there are canonical almost-complex morphisms built-in to the construction:

$$\begin{aligned}&\mathit{Neck}[k]_{i\,,\,\vec{0}} \;=\; \mathit{Neck}[k]_{i\,,\,\vec{\lambda}}\,, \quad \mathit{Trunk}[k]_{j\,,\,\vec{0}} \;=\; \mathit{Trunk}[k]_{j\,,\,\vec{\lambda}}\,, && i\,,\,j \in \{i_0,\,\ldots\,,\,i_{k'}\}\,;\\ &pr_i^*(\theta_{\lambda_i\,;\,[\sqrt{|\lambda_i|},\varepsilon]} \cup \theta'_{\lambda_i\,,\,[\sqrt{|\lambda_i|},\varepsilon]}) : \mathit{Neck}[k]_{i\,,\,\vec{0}} - N_{\sqrt{|\lambda_i|}}(D_i) \to \mathit{Trunk}[k]_{i\,,\,\vec{\lambda}}\,, &&\\ &pr_i^*(\theta_{\lambda_i\,;\,[|\lambda_i|/\varepsilon,\varepsilon]} \cup \theta'_{\lambda_i\,,\,[|\lambda_i|/\varepsilon,\varepsilon]}) : \mathit{Neck}[k]_{i\,,\,\vec{0}} - N_{|\lambda_i|/\varepsilon}(D_i) \to \mathit{Trunk}[k]_{i\,,\,\vec{\lambda}}\,, && i \notin \{i_0,\,\ldots\,,\,i_{k'}\}\,;\\ &pr_j^*\theta_{\lambda_j\,;\,[\varepsilon,1/\varepsilon]} = pr_{j-1}^*\theta'_{\lambda_{j-1}\,;\,[\varepsilon,1/\varepsilon]} : \mathit{Trunk}[k]_{j\,,\,\vec{0}} \xrightarrow{\sim} \mathit{Trunk}[k]_{j\,,\,\vec{\lambda}}\,, && j \notin \{i_0,\,\ldots\,,\,i_{k'}\}\,.\end{aligned}$$

Here $N_{\cdot}(D_i)$ is the (open) tubular neighborhood of D_i in $Y_{[k]} = W[k]_{\vec{0}}$ of the specified radious from the norm on $\mathbb{L}$ and $\mathbb{L}^*$. The collection of these morphisms glue/descend to two almost-complex morphisms

$$\begin{aligned}I_{\vec{\lambda}} \;&:\; Y_{[k]} - \cup_{i=0}^k N_{\sqrt{|\lambda_i|}}(D_i) \;\longrightarrow\; W[k]_{\vec{\lambda}}\,,\\ I_{\vec{\lambda},\varepsilon} \;&:\; Y_{[k]} - \cup_{i=0}^k N_{|\lambda_i|/\varepsilon}(D_i) \;\longrightarrow\; W[k]_{\vec{\lambda}}\,,\end{aligned}$$

both of which shall be called a *re-forging morphism* from $W[k]_{\vec{0}}$ to $W[k]_{\vec{\lambda}}$. Note that $I_{\vec{\lambda}}$ glues along the paired boundary of the connected components of $Y_{[k]} - \cup_{i=0}^{k} N_{\sqrt{|\lambda_i|}}(D_i)$ while $I_{\vec{\lambda},\varepsilon}$ glues along the paired boundary of the connected components of $Y_{[k]} - \cup_{i=0}^{k} N_{|\lambda_i|/\varepsilon}(D_i)$ but along a collar of non-paired boundary associated to $i \notin \{i_0\,,\,\ldots\,,\,i_{k'}\}$.

Remark 1.1.1.6 [*trunk region*]. The discussion implies that $\mathit{Trunk}[k]_j \simeq B[k] \times \mathit{Trunk}[k]_{j\,,\vec{0}}$ canonically for $j = 1\,,\,\ldots\,,\,k$.

Remark 1.1.1.7 [*gluing map*]. With Notation 1.1.1.2,

$$I_{\vec{\lambda},\varepsilon} \circ pr_i^* \left(\varphi_{\lambda_i} \,:\, \mathbb{L}^*_{[|\lambda_i|/\varepsilon\,,\,\varepsilon]} \;\longrightarrow\; \mathbb{L}_{[|\lambda_i|/\varepsilon\,,\,\varepsilon]} \right) \;=\; Id_{\,\mathit{Neck}_\varepsilon[k]_{i\,,\,\vec{\lambda}}}\,,$$

where both $\mathbb{L}$ and $\mathbb{L}^*$ are regarded as canonically embedded in $\mathbb{L} \oplus \mathbb{L}^*$.

Remark 1.1.1.8 [*neck-trunk decomposition of* $W[k]_{\vec{\lambda}}$]. Let $0 \le i_0 < \cdots < i_{k'} \le k$ be the associated indices to a $\vec{\lambda} \in B[k]$ so that $\lambda_{i_j} = 0$. Then,

$$\begin{aligned}
&\left(\overline{U_1} \,\cup\, \mathit{Neck}[k]_{0\,,\,\vec{\lambda}} \,\cup\, \mathit{Trunk}[k]_{1\,,\,\vec{\lambda}} \,\cup\, \mathit{Neck}[k]_{1\,,\,\vec{\lambda}} \,\cup\, \cdots \,\cup\, \mathit{Trunk}[k]_{i_0\,,\,\vec{\lambda}}\right)\\
\bigcup \mathit{Neck}[k]_{i_0\,,\,\vec{\lambda}} \quad \bigcup \quad &\left(\mathit{Trunk}[k]_{i_0+1\,,\,\vec{\lambda}} \,\cup\, \mathit{Neck}[k]_{i_0+1\,,\,\vec{\lambda}} \,\cup\, \cdots \,\cup\, \mathit{Trunk}[k]_{i_1\,,\,\vec{\lambda}}\right)\\
\bigcup \mathit{Neck}[k]_{i_1\,,\,\vec{\lambda}} \quad \bigcup \quad \cdots \quad \bigcup \quad &\left(\mathit{Trunk}[k]_{i_{k'-1}+1\,,\,\vec{\lambda}} \,\cup\, \mathit{Neck}[k]_{i_{k'-1}+1\,,\,\vec{\lambda}} \,\cup\, \cdots \,\cup\, \mathit{Trunk}[k]_{i_{k'}\,,\,\vec{\lambda}}\right)\\
\bigcup \mathit{Neck}[k]_{i_{k'}\,,\,\vec{\lambda}} \quad \bigcup \quad &\left(\mathit{Trunk}[k]_{i_{k'}+1\,,\,\vec{\lambda}} \,\cup\, \mathit{Neck}[k]_{i_{k'}+1\,,\,\vec{\lambda}} \,\cup\, \cdots \,\cup\, \mathit{Trunk}[k]_{k\,,\,\vec{\lambda}} \,\cup\, \mathit{Neck}[k]_{k\,,\,\vec{\lambda}} \,\cup\, \overline{U_2}\right)
\end{aligned}$$

defines a *neck-trunk decomposition* of $W[k]_{\vec{\lambda}} \simeq Y_{[k']}$.

1.1.2 The pseudo-$\mathbb{G}_m[k]$-action on $W[k]/B[k]$ in almost-complex category.

Let $\mathbb{G}_m[k] := \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ (k times) with coordinates $(\sigma_1,\,\ldots\,,\,\sigma_k)$. It pseudo-acts[1] on $B[k]$ by

$$(\lambda_0,\,\ldots\,,\,\lambda_i,\,\ldots\,,\,\lambda_k) \longmapsto (\sigma_0\sigma_1^{-1}\lambda_0,\,\ldots\,,\,\sigma_i\sigma_{i+1}^{-1}\lambda_i,\,\ldots\,,\,\sigma_k\sigma_{k+1}^{-1}\lambda_k)\,,$$

where $\sigma_0 = \sigma_{k+1} = 1$ by convention. It admits a lifting to a pseudo-action on $W[k]/B[k]$ as follows.

Consider first the lifting of this pseudo-action to $(\mathbb{L} \oplus \mathbb{L}^*)_i$, $i = 0,\,\ldots\,,\,k$, over $B[k]$ by

$$\begin{aligned}
&(\lambda_0,\,\ldots\,,\,\lambda_i,\,\ldots\,,\,\lambda_k; x, w_i, w_i')\\
&\quad\longmapsto (\sigma_0\sigma_1^{-1}\lambda_0,\,\ldots\,,\,\sigma_i\sigma_{i+1}^{-1}\lambda_i,\,\ldots\,,\,\sigma_k\sigma_{k+1}^{-1}\lambda_k; x,\,\sigma_i w_i,\,\sigma_{i+1}^{-1}w_i')\,.
\end{aligned}$$

This is well-defined since $(\sigma_i w_i)(\sigma_{i+1}^{-1}w_i') = \sigma_i\sigma_{i+1}^{-1}\lambda_i$. This pseudo-action leaves both $(\mathbb{L} \oplus \mathbb{L}^*)_i^0$ and $(\mathbb{L} \oplus \mathbb{L}^*)_i^\infty$ invariant, and it follows from the explicit expression in Sec. 1.1.1 that the gluing map $\varphi_{i-1,i} : (\mathbb{L} \oplus \mathbb{L}^*)_{i-1}^\infty \to (\mathbb{L} \oplus \mathbb{L}^*)_i^0$ is $\mathbb{G}_m[k]$-equivariant, for $i = 1,\,\ldots\,,\,k$. Consequently, the pseudo-$\mathbb{G}_m[k]$-actions on $(\mathbb{L}\oplus\mathbb{L}^*)_i$, $i = 0,\,\ldots\,,\,k$, glue to a pseudo-$\mathbb{G}_m[k]$-action on $(\mathbb{L}\oplus\mathbb{L}^*)[k]$ that lifts the pseudo-$\mathbb{G}_m[k]$-action on $B[k]$. This pseudo-action embeds $\mathbb{G}_m[k]$ into $Aut((\mathbb{L}\oplus\mathbb{L}^*)[k])$ in the almost complex category; the isotropy group of $\vec{\lambda} \in B[k]$ under this pseudo-action coincides with $Aut(\pi[k]^{-1}(\vec{\lambda})/\mathbb{L} \vee \mathbb{L}^*)$.

[1]For non-algebraic-geometers: here $\mathbb{G}_m$ means the *multiplicative group* of the ground field (e.g., $\mathbb{C}^\times$ in $\mathbb{C}$ in our case) and is a standard notation from algebraic geometry. Also, given a group G with the identity e, a *pseudo-group action* of G on a space M is a map from a neighborhood of $e \times M$ in $G \times M$ to M that satisfies all the group-action axioms whenever items in the axioms are defined.

By construction, the pseudo-$\mathbb{G}_m[k]$-action on $(\mathbb{L}\oplus\mathbb{L}^*)[k]/B[k]$ descends to the trivial action on $(\mathbb{L}\oplus\mathbb{L}^*)/\mathbb{C}$ under $(\tilde{\mathbf{p}}[k], \mathbf{p}[k])$. It follows that $\mathbb{G}_m[k]$ leaves $(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\leq\varepsilon}$ invariant and its restriction to the horizontal boundary $\partial_{/B[k]}(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\leq\varepsilon} = B[k]\times(\partial\overline{U_1}\amalg\partial\overline{U_2})$ of $(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\leq\varepsilon}$ over $B[k]$ acts purely on the $B[k]$-factor. This together with the gluing form $W[k]/B[k] = (\overline{U_1}[k]\cup(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\leq\varepsilon}\cup\overline{U_2}[k])/B[k]$ of $W[k]/B[k]$ implies that the pseudo-$\mathbb{G}_m[k]$-action on $(\mathbb{L}\oplus\mathbb{L}^*)[k]_{\leq\varepsilon}$ extends to a pseudo-$\mathbb{G}_m[k]$-action on $W[k]/B[k]$ such that its restriction on $\overline{U_1}[k] = B[k]\times\overline{U_1}$ and $\overline{U_2}[k] = B[k]\times\overline{U_2}$ acts only on the $B[k]$-factor.

The following lemma follows immediately from the gluing construction of $W[k]/B[k]$ in Sec. 1.1.1.

Lemma 1.1.2.1 [$\mathbb{T}^k$-action on $W[k]/B[k]$]. *The restriction of the pseudo-$\mathbb{G}_m[k]$-action on $W[k]/B[k]$ to its maximal compact subgroup $\mathbb{T}^k := U(1)^k$ gives an honest $\mathbb{T}^k$-action on $W[k]/B[k]$. This $\mathbb{T}^k$-action leaves the neck-trunk decomposition of $W[k]/B[k]$ invariant; the two re-forging morphisms $I_{\vec{\lambda}}$ and $I_{\vec{\lambda},\varepsilon}$ are equivariant with respect to the stabilizer of the fiber $W[k]_{\vec{\lambda}}$ under the $\mathbb{T}^k$-action on $W[k]$.*

1.1.3 The topological quotient space $\widehat{W}/\widehat{B}$ associated to W/B.

We now construct a topological space $\widehat{W}/\widehat{B}$ with charts that accommodates all the fibers $\{W_\lambda\}_{\lambda\in B}\cup\{Y_{[k]}\}_{k\in\mathbb{Z}_{>0}}$ that occur in an expanded degeneration of W/B. For notation, given fibered spaces W' over B' and W'' over B'', a map $\varphi : W'/B' \to W''/B''$ means a map $\varphi : W' \to W''$ that is descendable to a map $\underline{\varphi} : B' \to B''$ on the base. Similarly, for a *pseudo-map*[2] $W'/B' \to W''/B''$.

Recall the base $B[k]$ with coordinates $(\lambda_0, \ldots, \lambda_k)$ from the product $\mathbb{C}^{k+1}$. To make the discussion more specific/concrete, for a subset $I = \{i_0, \ldots, i_{k'}\}$ of $\{0, \ldots, k\}$ let $B[k]_I^{\varepsilon^2/4}$ be the affine coordinate subspace of $B[k]$, whose points have coordinates $\lambda_i = \varepsilon^2/4$ for $i \notin I$ and denote $\pi[k]^{-1}(B[k]_I^{\varepsilon^2/4})$ by $W[k]_{B[k]_I^{\varepsilon^2/4}}$. Then one has a pseudo-embedding of almost-complex spaces via the composition

$$\varphi_{k',k;I} \,:\, W[k']/B[k'] \xrightarrow{\sim} W[k]_{B[k]_I^{\varepsilon^2/4}}/B[k]_I^{\varepsilon^2/4} \hookrightarrow W[k]/B[k]\,,$$

[2]Here a *pseudo-map* $f : A_1 \to A_2$ means a map f from a subset of A_1 to A_2. Similarly, a *pseudo-embedding* $f : A_1 \to A_2$ means a pseudo-map $f : A_1 \to A_2$ that is an embedding on where f is defined. For non-algebraic-geometers: the reason for introducing such notion here is as follows. In the full construction of a moduli stack via the *Isom*-functor, for two families of geometric objects in question (e.g. all the almost-complex isomorphism classes of fibers that occur in expanded degenerations of W/B) $\pi_1 : W_1/B_1$ and $\pi_2 : W_2 \to B_2$, one constucts/defines a universal "overlapping" family $\pi : W \to \mathbf{Isom}(\pi_1, \pi_2)$. Encoded into the construction of the family π are natural morphisms $p_1 : \mathbf{Isom}(\pi_1, \pi_2) \to B_1$ and $p_2 : \mathbf{Isom}(\pi_1, \pi_2) \to B_2$, and tautological isomorphisms $p_1^*W_1 \simeq W \simeq p_2^*W_2$ over $\mathbf{Isom}(\pi_1, \pi_2)$. In Grothendieck's picture, illuminated by Mumford, each of π_1 and π_2 gives a local chart of the "moduli space" behind, and the data from the *Isom*-construction gives the Grothendieck's generalized notion of "gluing" local charts B_1 and B_2 of the "moduli space". As we mean to avoid the distraction of such formality, in our case it happens that one may relate B_1 and B_2 instead by directly choosing (non-canonically and non-uniquely) a section to p_1, which is only defined on $Im p_1 \subset B_1$, and then post-compose it with p_2. This gives then a substitute "transition" map $\underline{\varphi} : Im p_1 \to B_2$. Furthermore, as long as the "quotient topology on the moduli space" is concerned, all that matters is that the domain $Im p_1$ of $\underline{\varphi}$ contains an *open* neighborhood of the point in B_1 over which the central fiber in question sits; the precise tracking of $Im p_1$ is irrelevant. Thus we directly re-denote $\underline{\varphi}$ as a *pseudo-map* $\underline{\varphi} : B_1 \to B_2$. Via the canonical isomorphism $p_1^*W_1 \simeq W \simeq p_2^*W_2$, accompanying the construction of $\underline{\phi}$ is also the pseudo-map $\varphi : W_1/B_1 \to W_2/B_2$ that covers $\underline{\varphi}$. See [L-MB] for details on stacks and [L-L-Y: Sec. 1] for a literature guide. Similar use of "*pseudo-*" applies to terms: *pseudo-embedding*, *pseudo-isomorphisms*, ..., etc., and their compositions.

where $W[k']/B[k'] \xrightarrow{\sim} W[k]_{B[k]_I^{\varepsilon^2/4}}/B[k]_I^{\varepsilon^2/4}$ is the almost-complex pseudo-isomorphism that lifts the pseudo-isomorphism $B[k'] \to B[k]_I^{\varepsilon^2/4}$ defined by $(\lambda'_0\,,\,\ldots\,,\,\lambda'_{k'}) \mapsto (\lambda_0\,,\,\ldots\,,\,\lambda_k)$ with $\lambda_i = (\frac{4}{\varepsilon^2})^{k-k'}\lambda'_j$, for $i = i_j \in I$, and $= \varepsilon^2/4$, for $i \notin I$. The defining domain of $\varphi_{k',k;I}$ contains an open neighborhood of the central fiber $\simeq Y_{[k']}$ of $W[k']/B[k']$. $\varphi_{k',k;I}$ is equivariant with respect to $\mathbb{G}_m[k'] \hookrightarrow \mathbb{G}_m[k]$ with

$$(\sigma'_1,\,\cdots,\,\sigma'_{k'}) \;\longmapsto\; (\underbrace{\sigma'_0,\,\cdots,\sigma'_0}_{i_0},\,\underbrace{\sigma'_1,\,\cdots,\,\sigma'_1}_{i_1-i_0},\,\cdots,\underbrace{\sigma'_{k'},\,\cdots,\,\sigma'_{k'}}_{i_{k'}-i_{k'-1}},\,\underbrace{\sigma'_{k'+1},\,\cdots,\,\sigma'_{k'+1}}_{k-i_{k'}})\,,$$

where $\sigma'_0 = \sigma'_{k'+1} = 1$ by convention and the multiplicity of each repeated entry is indicated. Let $W_{(k)}/B_{(k)}$ be the quotient space of $W[k]/B[k]$ by $\mathbb{G}_m[k]$ with the quotient topology. Then $(W - W_0)/(B - \{0\})$ embeds in $W_{(k)}/B_{(k)}$ canonically for all $k \in \mathbb{Z}_{\geq 0}$ and $\varphi_{k',k;I}$ induces an embedding

$$\varphi_{(k',k;I)} \;:\; W_{(k')}/B_{(k')} \;\hookrightarrow\; W_{(k)}/B_{(k)}$$

over W/B, for all $k' < k$, that restricts to the identity map on $(W - W_0)/(B - \{0\})$.

Let $\widehat{B} = B \cup \mathbb{Z}_{>0}$ with the topology generated by the open subsets of B and the subsets of $\widehat{B}$ of the form $U \cup \{1,\,\ldots\,,\,k\}$, where U is an open neighborhood of $0 \in B$ and $k \in \mathbb{Z}_{>0}$. Define the set

$$\widehat{W}/\widehat{B} \;:=\; \Big(\amalg_{k\in\mathbb{Z}_{\geq 0}} W_{(k)}/B_{(k)}\Big)\Big/\sim,$$

where $p \in W_{(k)}$ and $p' \in W_{(k')}$ with $k > k'$ are defined to be equivalent (in notation, $p \sim p'$) if p is the image of p' under some $\varphi_{(k',k;I)}$ (this defines $\widehat{W}$) and $p \in B[k]$ and $p' \in B[k']$ are equivalent if p is the image of p' under some $\underline{\varphi_{(k',k;I)}}$ (this reproduces $\widehat{B}$). As indicated, the fibrations $W_{(k)}/B_{(k)}$, $k \in \mathbb{Z}_{\geq 0}$, induce a fibration of $\widehat{W}$ over $\widehat{B}$. By construction, there are natural embeddings (of sets)

$$\varphi_{(k)} \;:\; W_{(k)}/B_{(k)} \;\hookrightarrow\; \widehat{W}/\widehat{B}\,,\quad k \;\in\; \mathbb{Z}_{\geq 0}\,.$$

Equip $\widehat{W}$ with the topology that specifies a subset $\widehat{U}$ of $\widehat{W}$ to be open if and only if $\widehat{U} = \cup_\alpha \widehat{U}_\alpha$ such that for each α there exists $k_\alpha \in \mathbb{Z}_{\geq 0}$ so that $\widehat{U}_\alpha = \varphi_{(k_\alpha)}(U_\alpha)$ for some open subset U_α of $W_{(k_\alpha)}$. We will call this topology the *quotient topology* on $\widehat{W}$. Note that this topology involves all $\varphi_{(k',k;I)}$ so that the information of how one $Y_{[k']}$ or W_λ degenerates to another $Y_{[k]}$ with $k > k'$ is all kept. By construction, both the natural map $\widehat{W} \to \widehat{B}$ and the defining maps

$$\varphi[k] \;:\; W[k]/B[k] \;\longrightarrow\; \widehat{W}/\widehat{B}$$

from the composition $W[k]/B[k] \to W_{(k)}/B_{(k)} \to \widehat{W}/\widehat{B}$ are continuous. $(W[k]/B[k], \varphi[k])$ is named a *standard local chart* on $\widehat{W}/\widehat{B}$ and the collection $\{(W[k]/B[k], \varphi[k]) : k \in \mathbb{Z}_{\geq 0}\}$ the *standard atlas* for $\widehat{W}/\widehat{B}$.

Finally, note that the collection of maps $\{\tilde{\mathbf{p}}[k] : W[k]/B[k] \to W/B\}_{k\in\mathbb{Z}_{\geq 0}}$ descends to a (continuous) tautological map

$$\widehat{\mathbf{p}} \;:\; \widehat{W}/\widehat{B} \;\longrightarrow\; W/B\,.$$

Remark 1.1.3.1 [*quotient topology versus stack*]. To identify consistently isomorphic fibers (as almost-complex spaces) in the collection $\{W[k]/B[k]\}_{k\in\mathbb{Z}_{\geq 0}}$ and make the final family universal, one has to employ Grothendieck's generalized notion in algebraic geometry of "*gluing*" via the Isom-functor construction, of a "*space*" as a collection of local charts together with a gluing

data in the generalized sense, and of a "*global structure*" as a descent datum. Following this, the collection $\{W[k]/B[k]\}_{k\in\mathbb{Z}_{\geq 0}}$ would be glued to an Artin stack $\mathcal{B}$, together with a universal expanded degeneration $\mathcal{W}$ over $\mathcal{B}$. The set of geometric points of $\mathcal{B}$ would be $B\cup\mathbb{Z}_{>0}$ with the corresponding set of isomorphism class of fibers of $\mathcal{W}/\mathcal{B}$ being $\{W_\lambda\}_{\lambda\in B}\cup\{Y_{[k]}\}_{k\in\mathbb{Z}_{>0}}$. (Cf. [Li1: Sec. 1]; see [L-L-Y: Sec. 1] for a brief tour on stacks). Since it is the stable maps, i.e. triples $(\Sigma, W[k]_\lambda, f:\Sigma\to W[k]_{\vec{\lambda}})$, that we want to study in this work, it turns out that what we finally need most essentially is a structure that describes the "nearness" between a W_λ or $Y_{[k]}$ and another $W_{\lambda'}$ or $Y_{[k']}$. For this reason, the space $\widehat{W}/\widehat{B}$ with the quotient topology and the standard atlas as constructed above that accommodates all $\{W_\lambda\}_{\lambda\in B}\cup\{Y_{[k]}\}_{k\in\mathbb{Z}_{>0}}$ suffices.

1.2 Symplectic/almost-complex relative pairs and their expansions.

A *symplectic* (resp. *almost-complex*) *relative pair* $(Z;D)$ is a symplectic (resp. almost-complex) manifold Z together with a real codimension-2 symplectic (resp. almost-complex) submanifold D. Given a symplectic relative pair $(Z;D)$ with a Hamiltonian $U(1)$-action on a (open) tubular neighborhood $N(D)$ of D in Z that fixes D, define $Z[1]$ to be the total space of a compatible almost-complex degeneration of a symplectic cut on Z associated to the given local $U(1)$-action around D. By construction, $Z[1]$ fibers over $A[1]:=B=\{\lambda\in\mathbb{C}:|\lambda|<\varepsilon^2/2\}$, $0<\varepsilon<1$, with the singular fiber $Z[1]_0=Z\cup_{D=D_{1,\infty}}\Delta_1$. Since the pinched locus of the symplectic cut is disjoint from D and it separates D with $Z-N_\varepsilon(D)$, $D[1]:=A[1]\times D$ embeds canonically in $Z[1]$ over $A[1]$ with $D[1]_0:=\{0\}\times D$ identical to $D_{1,0}$ in Δ_1.

The construction in Sec. 1.1.1 applied to the almost-complex degeneration $Z[1]/A[1]$ then gives rise to an *almost-complex expanded relative pair* $(Z[k];D[k])/A[k]$ with $A[k]=B[k-1]$ and $D[k]=A[k]\times D$, for $k\in\mathbb{Z}_{\geq 0}$. Its fiber, e.g., at $\vec{0}\in A[k]$ is the almost-complex relative pair

$$\begin{aligned}(Z[k];D[k])_{\vec{0}} &= (Z\cup_{D=D_{1,\infty}}\Delta_1\cup_{D_{1,0}=D_{2,\infty}}\cdots\cup_{D_{k-1,0}=D_{k,\infty}}\Delta_k\,;\,D_{k,0})\\ &=: (Z_{[k]};D_{[k]})\,.\end{aligned}$$

There is also the almost-complex morphism

$$\tilde{\mathbf{p}}[k]\;:\;(Z[k];D[k])/A[k]\;\longrightarrow\;(Z;D)/pt$$

from the construction.

Let $\overline{U}=Z-N_\varepsilon(D)$, where $N_\varepsilon(D)$ is the open ε-neighborhood of D in Z with respect to the norm on $\mathbb{L}$. Then $(Z[k];D[k])/A[k]$ admits a *neck-trunk decomposition*:

$$\begin{aligned}&Z[k]/A[k]\;=\\ &\quad(\overline{U}[k]\,\cup\,\mathit{Neck}[k]_0\,\cup\,\mathit{Trunk}[k]_1\,\cup\,\mathit{Neck}[k]_1\,\cup\cdots\cup\,\mathit{Trunk}[k]_k\,\cup\,N_\varepsilon(D)[k])/A[k]\,,\end{aligned}$$

where $\overline{U}[k]$, $\mathit{Neck}[k]_i$, $i=0,\ldots,k-1$, $\mathit{Trunk}[k]_j$, $j=1,\ldots,k$, here are similar to their counterpart: $\overline{U_1}[k-1]$, $\mathit{Neck}[k-1]_i$, and $\mathit{Trunk}[k-1]_j$, in Sec. 1.1.1 and $N_\varepsilon(D)[k]=A[k]\times N_\varepsilon(D)$, which contains $D[k]$. This induces a neck-trunk decomposition to the fiber $(Z[k];D[k])_{\vec{\lambda}}$ of $(Z[k];D[k])$ at $\vec{\lambda}\in A[k]$, cf. Remark 1.1.1.8. There are *re-forging morphisms* from $Z[k]_{\vec{0}}=Z_{[k]}$ to $Z[k]_{\vec{\lambda}}$ constructed in the same way as earlier:

$$\begin{aligned}I_{\vec{\lambda}} &: Z_{[k]}-\cup_{i=0}^{k-1}N_{\sqrt{|\lambda_i|}}(D_i)\;\longrightarrow\;Z[k]_{\vec{\lambda}}\,,\\ I_{\vec{\lambda},\varepsilon} &: Z_{[k]}-\cup_{i=0}^{k-1}N_{|\lambda_i|/\varepsilon}(D_i)\;\longrightarrow\;Z[k]_{\vec{\lambda}}\,,\qquad\vec{\lambda}\;\in\;A[k]\,.\end{aligned}$$

The group $\mathbb{G}_m[k]$ now pseudo-acts on $A[k]$ by

$$(\lambda_0,\ldots,\lambda_i,\ldots,\lambda_{k-1})\longmapsto(\sigma_0\sigma_1^{-1}\lambda_0,\ldots,\sigma_i\sigma_{i+1}^{-1}\lambda_i,\ldots,\sigma_{k-1}\sigma_k^{-1}\lambda_{k-1})\,,$$

where $\sigma_0 = 1$ by convention. Similar to Sec. 1.1.2, it lifts to a *pseudo-$\mathbb{G}_m[k]$-action* on $Z[k]$ that leaves $D[k]$ invariant in such a way that the pseudo-action on $D[k] = A[k] \times D$ acts only on the $A[k]$-factor. As a parallel to Lemma 1.1.2.1, the restriction of the pseudo-$\mathbb{G}_m[k]$-action on $(Z[k]; D[k])/A[k]$ to its maximal compact subgroup $\mathbb{T}^k$ gives an honest $\mathbb{T}^k$-action on $(Z[k]; D[k])/A[k]$. This $\mathbb{T}^k$-action leaves the neck-trunk decomposition of $(Z[k]; D[k])/A[k]$ invariant and the two re-forging morphisms $I_{\vec{\lambda}}$ and $I_{\vec{\lambda},\varepsilon}$ are equivariant with respect to the stabilizer of $Z[k]_{\vec{\lambda}}$ under the $\mathbb{T}^k$-action on $Z[k]$.

To connect the various expanded relative pairs, each $I' = \{ i_0, \ldots, i_{k'-1} \} \subset \{ 0, \ldots, k-1 \}$ is associated to a pseudo-embedding of almost-complex spaces

$$\varphi'_{k',k;I'} \,:\, (Z[k']; D[k'])/A[k'] \,\hookrightarrow (Z[k]; D[k])/A[k]\,,$$

which covers the pseudo-embedding $A[k'] \to A[k]$, defined by $(\lambda'_0\,,\,\ldots\,,\,\lambda'_{k'-1}) \mapsto (\lambda_0\,,\,\ldots\,,\,\lambda_{k-1})$ with $\lambda_i = (\frac{4}{\varepsilon^2})^{k-k'}\lambda'_j$, for $i = i_j \in I'$, and $= \varepsilon^2/4$, for $i \notin I'$. and is equivariant with respect to the group homomorphism $\mathbb{G}_m[k'] \hookrightarrow \mathbb{G}_m[k]$ defined by

$$(\sigma'_1, \cdots, \sigma'_{k'}) \,\longmapsto\, (\underbrace{1, \cdots, 1}_{i_0}, \underbrace{\sigma'_1, \cdots, \sigma'_1}_{i_1 - i_0}, \cdots, \underbrace{\sigma'_{k'-1}, \cdots, \sigma'_{k'-1}}_{i_{k'-1} - i_{k'-2}}, \underbrace{\sigma'_{k'}, \cdots, \sigma'_{k'}}_{k - i_{k'-1}})\,.$$

Let $(Z_{(k)}; D_{(k)})/A_{(k)}$ be the quotient space of $(Z[k]; D[k])/A[k]$ by $\mathbb{G}_m[k]$ with the quotient topology. Then $(Z; D)$ embeds in $(Z_{(k)}; D_{(k)})/A_{(k)}$ canonically for all $k \in \mathbb{Z}_{\geq 0}$ and $\varphi'_{k',k;I'}$ induces an embedding

$$\varphi'_{(k',k;I)} \,:\, (Z_{(k')}; D_{(k')})/A_{(k')} \,\hookrightarrow\, (Z_{(k)}; D_{(k)})/A_{(k)}\,,$$

for all $k' < k$, that restricts to the identity map on $(Z; D)$.

Let $\widehat{A} = \mathbb{Z}_{\geq 0}$ with the topology generated by the defining open subsets $\{i \in \mathbb{Z}_{\geq 0} \,:\, 0 \leq i \leq n\}$, $n \in \mathbb{Z}_{\geq 0}$. Then, the construction in Sec. 1.1.3 applied to $\{(Z[k]; D[k])/A[k]\}_{k \in \mathbb{Z}_{\geq 0}}$, where $(Z[0]; D[0])/A[0] = (Z; D)$ by convention, gives rise to a *topological relative pair* $(\widehat{Z}; \widehat{D})$ over $\widehat{A}$ with the *quotient topology*, the natural embeddings

$$\varphi_{(k)} \,:\, (Z_{(k)}; D_{(k)})/A_{(k)} \,\hookrightarrow\, (\widehat{Z}; \widehat{D})/\widehat{A}\,, \quad k \,\in\, \mathbb{Z}_{\geq 0}\,,$$

the *standard local charts*

$$\varphi[k] \,:\, (Z[k]; D[k])/A[k] \,\longrightarrow\, (\widehat{Z}; \widehat{D})/\widehat{A}\,, \quad k \,\in\, \mathbb{Z}_{\geq 0}\,,$$

and a (continuous) tautological map

$$\widehat{\mathbf{p}} \,:\, (\widehat{Z}; \widehat{D})/\widehat{A} \,\longrightarrow\, (Z; D)\,.$$

The topological relative pair $(\widehat{Z}; \widehat{D})/\widehat{A}$ equipped with the standard local charts substitutes the stack of expanded relative pairs obtained by gluing $(Z[k]; D[k])/A[k]$'s via the Isom-functor construction.

Readers are referred also to [I-P1: Sec. 3 and Sec. 6], [L-R: Sec. 3], and [Li1: Sec. 4] for related discussions.

2 Prestable labelled-bordered Riemann surfaces.

In this section we review/rephrase/modify definitions/facts of labelled-bordered Riemann surfaces with marked points to introduce and fix terminologies and notations that we will use. This is a classical topic with long history. Readers are referred to [Sie1: Sec. 2], [F-O: Sec. 9 and pp. 988 - 991], and [Liu(C): Sec. 2 - Sec. 4] for related discussions and guide to literatures. See also [Ab], [A-G], [D-M], [H-M], [I-S2], [Kn], [Ma], [Se], [Sil], and [Wol].

Prestable labelled-bordered Riemann surfaces with marked points.

Definition 2.1 [prestable labelled-bordered Riemann surface]. A *prestable labelled-bordered Riemann surface of* (*combinatorial*) *type* $((g,h),(n,\vec{m}))$ (*with labelled boundary and marked points*)[3], where $\vec{m}=(m_1\,,\,\ldots\,,\,m_h)$, consists of the following data:

- a *compact connected nodal bordered Riemann surface* Σ, whose points are locally modelled at 0 or $(0,0)$ in the following holomorphic models:

  (i) *interior point*:

  (i1) $\{z\in\mathbb{C}\,:\,|z|<1\}$ for a *smooth interior point*,

  (i2) $\{(z_1,z_2)\in\mathbb{C}^2\,:\,|z_1|<1\,,\,|z_2|<1\,,\,z_1z_2=0\}$ for an *interior node*;

  (b) *boundary point*:

  (b1) $\{z\in\mathbb{C}\,:\,|z|<1\,,\mathrm{Im}\,(z)\geq 0\}$ for a *smooth boundary point*,

  (b2) $\{(z_1,z_2)\in\mathbb{C}^2\,:\,|z_1|<1\,,\,|z_2|<1\,,\,z_1z_2=0\}/(z_1,z_2)\sim(\overline{z_2},\overline{z_1})$ for a *boundary node of type E*,

  (b3) $\{(z_1,z_2)\in\mathbb{C}^2\,:\,|z_1|<1\,,\,|z_2|<1\,,\,z_1z_2=0\}/(z_1,z_2)\sim(\overline{z_1},\overline{z_2})$ for a *boundary node of type H*;

  the number of interior (resp. boundary) node will be denoted $n_{\mathrm{i.n.}}$ (resp. $n_{\mathrm{b.n.}}$).

- *labelled boundary* and h: a *boundary component* of Σ is either the image of an *embedding* of S^1 in $\partial\Sigma$ or a boundary node of type E; Σ has h-many boundary components and they are labelled from 1 to h; the labelled boundary of Σ will be denoted by $\dot{\partial}\Sigma$ (or simply $\partial\Sigma$ when the labelling is understood); note that different boundary components of Σ may intersect at a boundary node of type H.

- *genus* g: each boundary component of Σ can be capped by a 2-disc; let $\hat{\Sigma}$ be the nodal Riemann surface without boundary obtained by capping all the boundary components of Σ by discs, then $\hat{\Sigma}$ has *arithmetic genus* g.

- *free marked points*: an n-tuple $\vec{p}=(p_1\,,\,\cdots\,,\,p_n)$ of *smooth interior points* or *double boundary points*[4], on Σ; the support of the latter free points is required to be smooth boundary points. The notation $n\doteq n'+n''$ means that there are n'-many interior marked points and n''-many free marked points supported in $\partial\Sigma$, when the distinction is needed.

[3]The definition here is based on [Liu(C): Definition 3.9]. We phrase it to make it manifest that an interior marked point on a nodal bordered Riemann surface is allowed to move and land on the boundary to become a *double* boundary point. This freedom is required to obtain a compact moduli space of stable bordered Riemann surfaces with marked points. We avoid the term *marked bordered Riemann surface* to reserve its more traditional meaning in the Teichmüller theory of Riemann surfaces.

[4]For non-algebraic-geometers: in the affine $\mathbb{R}$-scheme model a *smooth interior point* (resp. *smooth boundary point*) on Σ is modelled on a *complex closed point*, $(x^2-(c+\bar{c})x+c\bar{c})$, $c\in\mathbb{C}-\mathbb{R}$, (resp. *real closed point* $(x-a)$, $a\in\mathbb{R}$) in $Spec\,\mathbb{R}[x]$. A complex closed point in $Spec\,\mathbb{R}[x]$ can be deformed to a *double real point*, described by an ideal $(x-a)^2$ for some $a\in\mathbb{R}$, in $\mathbb{R}[x]$. While a real double point as above can be deformed to a complex closed point, a closed real point can only be deformed to another closed real point. In other words, an interior

· *boundary marked points*: an m_i-tuple of smooth boundary points $\vec{p}_i = (p_{i1}, \cdots, p_{i\,m_i})$ on the boundary component of Σ labelled by i for $i = 1, \ldots, h$; we require that the set of boundary marked points is disjoint from the support of free marked points that land on the boundary.

By definition, the set of nodes and the set of marked points on Σ are disjoint from each other. Any point in the union of the two is called a *special point* on Σ.

A *regular* or *smooth point* on Σ is either a smooth interior point or a smooth boundary point on Σ. The set of regular points on Σ with the induces topology and holomorphic/complex structure is denoted by $\Sigma_{\rm reg}$ and called the *regular* or *smooth locus* of Σ.

From the local model of points on Σ, one can define the *normalization* $\tilde{\Sigma}$ of Σ as in algebraic geometry. Topological, $\tilde{\Sigma}$ is obtained by first removing all the (interior as well as boundary) nodes on Σ and then filling all the resulting (interior as well as boundary) punctures by distinct points. $\tilde{\Sigma}$ is a possibly disconnected bordered Riemann surface (with neither interior nor boundary nodes). Let $\nu : \tilde{\Sigma} \to \Sigma$ be the normalization of Σ and $\tilde{\Sigma} = \amalg_i \tilde{\Sigma}_i$ be the disjoint union of connected components; then each $\nu(\tilde{\Sigma}_i)$ in Σ is called an *irreducible component* of Σ.

Let $\overline{\Sigma}$ be the nodal bordered Riemann surface with the same topology as Σ but with the complex-conjugated holomorphic structure from that of Σ. Then $\Sigma_{\mathbb{C}} := \Sigma \cup_{\partial\Sigma = \partial\overline{\Sigma}} \overline{\Sigma}$ has a canonically induced nodal Riemann surface structure without boundary. It is called the *Schottky/complex double* of Σ. By construction, there is an *involution* τ that acts on $\Sigma_{\mathbb{C}}$ by complex conjugation.

An *isomorphism* $h : (\Sigma, \dot{\partial}\Sigma, \vec{p}, \vec{p}_1, \cdots, \vec{p}_h) \to (\Sigma', \dot{\partial}\Sigma', \vec{p}', \vec{p}'_1, \cdots, \vec{p}'_h)$ from a labelled-bordered Riemann surface to another of the same type is a bi-holomorphic map $h : (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ that preserves the label of the boundary components and sends p_i to p'_i, q_{ij} to q'_{ij}. An *automorphism* of $(\Sigma, \dot{\partial}\Sigma, \vec{p}, \vec{p}_1, \cdots, \vec{p}_h)$ is an isomorphism from $(\Sigma, \dot{\partial}\Sigma, \vec{p}, \vec{p}_1, \cdots, \vec{p}_h)$ to itself. $(\Sigma, \dot{\partial}\Sigma, \vec{p}, \vec{p}_1, \cdots, \vec{p}_h)$ is called *stable* if its group $Aut(\Sigma, \dot{\partial}\Sigma, \vec{p}, \vec{p}_1, \cdots, \vec{p}_h)$ of automorphisms is finite.

We will denote the data $(\Sigma, \dot{\partial}\Sigma, \vec{p}, \vec{p}_1, \cdots, \vec{p}_h)$ also by $(\Sigma, \partial\Sigma)$ or Σ in short. The isomorphism class of labelled-bordered Riemann surfaces isomorphic to Σ will be denoted $[\Sigma]$. When there is no chance of confusion, we will call Σ also a *curve* and denote it by C, as a 1-dimensional scheme over $Spec\,\mathbb{C}$ or $Spec\,\mathbb{R}$ in algebraic geometry with labelled irreducible components of $\mathbb{R}$-locus and marked points. The *moduli space* of isomorphism classes of *stable* (resp. *prestable*) labelled-bordered Riemann surfaces of type $((g,h),(n,\vec{m}))$ will be denoted $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$ (resp. $\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$).

Theorem 2.2 $[\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}]$**.** *The moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$ of stable labelled-bordered Riemann surfaces with marked points of type $((g,h),(n,\vec{m}))$, with its topology defined via the dilatation of quasi-conformal maps and their composition with circle/arc-with-ends-in-boundary pinching maps or via the local Fenchel-Nielsen coordinates associated to pants-decompositions, is a compact, Hausdorff, orientable orbifold-with-corners.*

See [Liu(C): Theorem 4.9, Theorem 4.14] and the quoted references there.

The universal deformation $\mathcal{C}/Def(\Sigma)$ of Σ, canonically acted upon by $Aut(\Sigma)$, provides a local orbifold-chart $\psi_{[\Sigma]} : Def(\Sigma) \to \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$ around $[\Sigma]$ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$. Topologically this

marked point in Σ can be deformed to a double boundary point on $\partial\Sigma$ and vice versa. Together, we name them *free* marked points on Σ. Thus, a free marked point, whether in the interior or on boundary, always have real 2-dimensional family of deformations. In particular, fixing a complex point always contributes *two* real constraints whether that point is in the interior or on the boundary. In contrast, a boundary marked point on Σ can move around only in the boundary $\partial\Sigma$ and contributes only one real condition.

is a quotient of a neighborhood of the origin in the manifold-with-corners

$$Ext^1_{\Sigma_{\mathbb{C}}}\left(\Omega_{\Sigma_{\mathbb{C}}}(\sum_{i=1}^n(p_i+\overline{p_i})\,+\,\sum_{j=1}^h\sum_{k=1}^{m_j}p_{jk})\,,\,\mathcal{O}_{\Sigma_{\mathbb{C}}}\right)^{\tau}$$
$$\simeq\ \mathbb{C}^{3g-3+h+n'}\times\overline{\mathbb{H}}^{n''}\times\mathbb{R}^{h-n_{bn}+m_1+\cdots+m_h}\times(\mathbb{R}_{\geq 0})^{n_{bn}}$$

by $Aut(\Sigma)$, where $\bullet^{\tau}$ is the fixed-point locus of the induced action of τ on $\bullet$, $\overline{\mathbb{H}}$ = the closed upper half-plane $\{z\in\mathbb{C}:\mathrm{Im}\,(z)\geq 0\}$, and $n\doteq n'+n''$. As an orbifold, $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$ goes with a *universal family*, denotes also by $\mathcal{C}/\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$. We will call this $\mathcal{C}$ the *universal curve* of type $((g,h),(n,\vec{m}))$. $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$ is *naturally stratified* by a finite collection of locally closed sub-orbifolds-with-corners. The stratification is governed by the *topological type* (i.e. equivalence up to homeomorphisms of the underlying topology of punctured bordered Riemann surfaces) and the *degeneration patterns* of a labelled-bordered Riemann surface with marked points. See, e.g., [Liu(C): Figures 1, 2, 3, 9, 10, 11] for illustrations of such stratifications.

Local chart on $\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$.

There are 11 types of *unstable* (*irreducible*) *components* that can happen for a prestable labelled-bordered Riemann surface: (1) *closed* component: $(g=0)$ $\mathbb{P}^1$ with 0, 1, or 2 special points; $(g=1)$ torus without special points or nodal torus with one node and without marked points; (2) *bordered* component: (all with $g=0$) 2-disc D^2 with 0 or 1 free marked point; or D^2 with 1 or 2 boundary marked points; annulus without special point or nodal annulus with one node and without marked points. These components contribute positive-dimensional subgroups to $Aut(\Sigma)$. The following discussion is an immediate generalization of [F-O: pp. 989 - 990] and [Sie1: Sec. 2.2] to the case of labelled-bordered Riemann surfaces. The moduli space $\widetilde{M}_{(g,h),(n,\vec{m})}$ of isomorphism classes of *prestable* labelled-bordered Riemann surfaces of type $((g,h),(n,\vec{m}))$ can be associated to an Artin stack. The discussion below gives a substitute quotient topology structure.

A *semi-universal deformation* $\mathcal{C}/Def(\Sigma)$ of Σ, together with a specification of an *approximate pseudo-$Aut(\Sigma)$-action* on $\mathcal{C}/Def(\Sigma)$, defines a local chart

$$\psi_{[\Sigma]}\ :\ Def(\Sigma)\ \longrightarrow\ \widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$$

of $[\Sigma]\in\widetilde{M}_{(g,h),(n,\vec{m})}$. Such a pair of data can be constructed as follows:

(1) *the defining family $\mathcal{C}/Def(\Sigma)$ of the chart*: Let $\Sigma'=(\Sigma,(p'_{\cdot})_{\cdot})$, where $(p'_{\cdot})_{\cdot}$ is a minimal tuple of rigidifying additional marked points on Σ that are disjoint from all the existing special points of Σ. Take $\mathcal{C}/Def(\Sigma)$ to be the universal deformation $\mathcal{C}'/Def(\Sigma')$ of Σ' with the sections $s'_{\cdot}$ associated to $p'_{\cdot}$'s removed.

(2) *the approximate pseudo-$Aut(\Sigma)$-action*: Let e be the identity element of $Aut(\Sigma)$ and recall that the central fiber $\mathcal{C}'_0$ of $\mathcal{C}'/Def(\Sigma')$ is Σ'. Consider the product family $(Aut(\Sigma)\times\mathcal{C}')/(Aut(\Sigma)\times Def(\Sigma'))$. First, extend the section $s_{\cdot}$ over $\{e\}\times Def(\Sigma')$ to over $Aut(\Sigma)\times\{0\}$ by setting $s'_{\cdot}(\sigma,0)=\sigma\cdot p'_{\cdot}$. Then, further extend them to a collection of sections $s'_{\cdot}$ over a neighborhood (still denoted by $Aut(\Sigma)\times Def(\Sigma')$, though in general it may not be a product) of $Aut(\Sigma)\times\{0\}\subset Aut(\Sigma)\times Def(\Sigma')$ whose image in a fiber are disjoint from each other and from the special points and the image of the existing sections associated Σ on that fiber. This can always be done but is non-canonical/non-unique. Denote the resulting family by $((Aut(\Sigma)\times\mathcal{C})/(Aut(\Sigma)Def(\Sigma))\,,\,(s'_{\cdot})_{\cdot})$ and the restriction of $s'_{\cdot}$ to over $\{\sigma\}\times Def(\Sigma)$ by $s'_{\cdot,\sigma}$.

· From the universal property of the family $\mathcal{C}'/Def(\Sigma')$ the unique isomorphism from the central fiber $(\Sigma, (\sigma\cdot p'_{\cdot})_{\cdot})$ of the family $((\{\sigma\}\times\mathcal{C})/(\{\sigma\}\times Def(\Sigma))\,,\,(s_{\cdot,\sigma})_{\cdot})$ to the central fiber Σ' of $\mathcal{C}'/Def(\Sigma')$ extends to a unique isomorphism

$$\Phi'_\sigma\ :\ (\{\sigma\}\times\mathcal{C})/(\{\sigma\}\times Def(\Sigma))\ \longrightarrow\ \mathcal{C}'/Def(\Sigma')\,,$$

assuming that the neighborhood of $Aut(\Sigma)\times\{0\}$ in $Aut(\Sigma)\times Def(\Sigma)$ we chose is small enough.

· Let

$$F_\sigma\ :\ ((\{\sigma\}\times\mathcal{C})/(\{\sigma\}\times Def(\Sigma))\,,\,(s'_{\cdot,\sigma})_{\cdot})\ \longrightarrow\ \mathcal{C}/Def(\Sigma)\,,$$

be the forgetful isomorphism that forgets the tuple $(s'_{\cdot,\sigma})_{\cdot}$ of rigidifying section. The morphism

$$\begin{array}{rcccc}\Phi_{[\Sigma]} & : & (Aut(\Sigma)\times\mathcal{C})/(Aut(\Sigma)\times Def(\Sigma)) & \longrightarrow & \mathcal{C}/Def(\Sigma)\\ & & (\sigma,x) & \longmapsto & \sigma\cdot x\ :=\ (F_\sigma\circ\Phi'^{-1}_\sigma\circ F_e^{-1})(x)\end{array}$$

defines then an *approximate*[5] pseudo-$Aut(\Sigma)$-action on $\mathcal{C}/Def(\Sigma)$.

(3) *The coordinate map* $\psi_{[\Sigma]}$: The family $\mathcal{C}/Def(\Sigma)$ specifies a map $\psi_{[\Sigma]}: Def(\Sigma)\to\widetilde{M}_{(g,h),(n,\vec{m})}$ by sending $b\in Def(\Sigma)$ to the isomorphism class $[C_b]\in\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$ of the fiber C_b of $\mathcal{C}$ over b.

Topologically, $\psi_{[\Sigma]}$ is a quotient of a neighborhood of the origin of the manifold-with-corners

$$\begin{aligned}&Ext^1_{\Sigma_\mathbb{C}}\left(\Omega_{\Sigma_\mathbb{C}}(\textstyle\sum_{i=1}^n(p_i+\overline{p_i})\ +\ \sum_{j=1}^h\sum_{k=1}^{m_j}p_{jk}+D_{\text{rigidifying}})\,,\,\mathcal{O}_{\Sigma_\mathbb{C}}\right)^\tau\\ &\quad\simeq\ \mathbb{C}^{3g-3+h+n'+d_c}\times\overline{\mathbb{H}}^{n''}\times\mathbb{R}^{h-n_{bn}+m_1+\cdots+m_h+d_b}\times(\mathbb{R}_{\geq 0})^{n_{bn}}\end{aligned}$$

by the induced $Aut(\Sigma)$-action, where $D_{\text{rigidifying}}$ is a minimal τ-invariant rigidifying divisor on $\Sigma_\mathbb{C}$ whose support is disjoint from the existing special points on Σ_C, $n=\doteq n'+n''$, and d_c (resp. d_b) is the complex (resp. real) dimension of the product of the automorphism group of the closed (resp. bordered) unstable components of Σ. The stacky (real) dimension of these charts, i.e. $dim\,Def(\Sigma)-dim\,Aut(\Sigma)$, remains $6g-6+3h+2n+m_1+,\cdots+m_h$.

Definition 2.3 [standard local chart of $\widetilde{M}_{(g,h),(n,\vec{m})}$]. We will call the tuple $(Def(\Sigma),\Phi_{[\Sigma]},\psi_{[\Sigma]})$, in short $Def(\Sigma)$, a *standard local chart* of $[\Sigma]\in\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$ and the $\mathcal{C}$ that accompanies $Def(\Sigma)$ in the construction and is equipped with the approximate pseudo-$Aut(\Sigma)$-action the *universal curve* over the chart $Def(\Sigma)$.

Resemblance of the approximate pseudo-action with a pseudo-action.

$\Phi_{[\Sigma]}$ defines a relation $\sim$ on $Def(\Sigma)$ generated by $b_1\sim b_2$ if there exists a $\sigma\in Aut(\Sigma)$ such that $b_2=\sigma\cdot b_1$. As the major step of the construction is a morphism to the universal deformation space of Σ with added rigidifying marked points, it remains true that two fibers C_{b_1} and C_{b_2} of $\mathcal{C}/Def(\Sigma)$ are isomorphic if and only if $b_1\sim b_2$; and, in this case, an isomorphism $C_{b_2}\simeq C_{b_1}$ can be given by the composition $\sigma_1\cdot\ldots\cdot\sigma_k\cdot$ for some $\sigma_1\,,\,\ldots\,,\,\sigma_k\in Aut(\Sigma)$. Furthermore, as long

[5]Here, the term "*approximate*" is referring to the fact that the composition law $\Phi_{[\Sigma]}(\sigma_1,\Phi_{[\Sigma]}(\sigma_2,x))=\Phi_{[\Sigma]}(\sigma_1\sigma_2,x)$ may not hold but, for $Def(\Sigma)$ small enough, $\Phi_{[\Sigma]}(\sigma_1,\Phi_{[\Sigma]}(\sigma_2,x))$ is always in a small neighborhood of $\Phi_{[\Sigma]}(\sigma_1\sigma_2,x)$.

as $Def(\Sigma)$ in the construction is small enough, the map $\sigma : \mathcal{C}/Def(\Sigma) \to \mathcal{C}/Def(\Sigma)$ is bijective on the domain it is defined. These two properties make the approximate pseudo-$Aut(\Sigma)$-action on $\mathcal{C}/Def(\Sigma)$ equally good as a genuine one.

Definition 2.4 [$Aut(\Sigma)$**-orbit].** An equivalence class of $\sim$ in $Def(\Sigma)$ is called an $Aut(\Sigma)$-*orbit* on $Def(\Sigma)$. Similarly for the approximate pseudo-$Aut(\Sigma)$-action on $\mathcal{C}$.

$Def(\Sigma)$ admits a stratification by locally closed subsets such that points in the same stratum have the corresponding fibers in $\mathcal{C}$ of the same topological type. It follows that the approximate pseudo-$Aut(\Sigma)$-action leaves each stratum invariant and points of $Def(\Sigma)$ in the same fiber have their $Aut(\Sigma)$-orbits of the same dimension. When not of the finitely many exceptional types, a general point $b \in Def(\Sigma)$ has the $Aut(\Sigma)$-orbit $Aut(\Sigma) \cdot b$ of the same dimension as $Aut(\Sigma)$, while $0 \in Def(\Sigma)$, which corresponds to the fiber Σ, is always a fixed point of $Aut(\Sigma)$.

Remark 2.5 [*abelian* $Aut(\Sigma)$]. When $Aut(\Sigma)$ is abelian, a similar construction as in Sec. 1.1.2 shows that $Aut(\Sigma)$ does pseudo-acts on $\mathcal{C}/Def(\Sigma)$ in this case.

Lemma 2.6 [pseudo-$\Gamma\cdot Aut_e(\Sigma)^\circ$**-action].** *Let* Γ *be a finite subgroup of* $Aut(\Sigma)$, $Aut_e(\Sigma)^\circ$ *be a small enough neighborhood of the identity element* e *of* $Aut(\Sigma)$, *and* $\Gamma\cdot Aut_e(\Sigma) = \cup_{\sigma\in\Gamma}\, \sigma\cdot Aut_e(\Sigma)^\circ$. *Then, possibly after shrinking* $Def(\Sigma)$, *the defining* $\Gamma\cdot Aut_e(\Sigma)^\circ$*-action on the center fiber* Σ *of* $\mathcal{C}/Def(\Sigma)$ *extends to a pseudo-action on* $\mathcal{C}/Def(\Sigma)$ *by isomorphisms. This pseudo-*$\Gamma\cdot Aut_e(\Sigma)^\circ$*-action extends to an approximate pseudo-*$Aut(\Sigma)$*-action on* $\mathcal{C}/Def(\Sigma)$ *by isomorphisms.*

Proof. Fix a rigidifying devisor $\sum_\cdot p'_\cdot$ on Σ away from the nodes and let $\Sigma = (\cup_{q_i} N_i)\cup(\cup_j V_j)$ be a neck-trunk decomposition of Σ (cf. the thick-thin decomposition of Σ when Σ is of hyperbolic type), where N_i is a neck on Σ in a small neighborhood of node q_i with q_i running over the set of nodes of Σ, and V_j be a connected component of $\Sigma - \cup_{q_i} N_i$, such that $\Gamma\cdot Aut_e(\Sigma)^\circ(\cup_i \partial N_i)$ remains in a tubular neighborhood of $\cup_i \partial N_i$ in Σ and the $\Gamma\cdot Aut_e(\Sigma)^\circ$-orbits of all marked points, including the added regidifying ones $p'_\cdot$, are away from this tubular neighborhood. As Γ sends nodes to nodes, this can be realized as long as $Aut_e(\Sigma)^\circ$ is small enough. Extend this neck-trunk decomposition of Σ to a neck-trunk decomposition

$$\mathcal{C}/Def(\Sigma) \;=\; \big(\cup_{q_i}\, Neck(q_i)\big) \bigcup \big(\cup_j\, Trunk_j\big)$$

of $\mathcal{C}/Def(\Sigma)$, where $\{q_i\}_i$ is the set of nodes of Σ; $Neck(q_i)$ is a neck region in $\mathcal{C}$ associated to q_i; and $\{Trunk_j/Def(\Sigma)\}_j$ is the set of connected components of $\mathcal{C}/Def(\Sigma) - Neck(q_i)$, equipped with a fixed product decomposition $Trunk_j = Def(\Sigma)\times V_j$. This can be realized as long as $Def(\Sigma)$ is small enough. Denote the section of $\mathcal{C}/Def(\Sigma)$ associated to $p'_\cdot$ by $s'_\cdot$. The specification of a neck-trunk decomposition of $\mathcal{C}/Def(\Sigma)$ specifies simultaneously how each fiber of $\mathcal{C}/Def(\Sigma)$ is obtained from a cut-and-paste of Σ, (cf. the re-forging morphisms in Sec. 1.1.1). This then induces a pseudo-$\Gamma\cdot Aut_e(\Sigma)^\circ$-action

$$\Phi^\circ_{[\Sigma]} \;:\; (\Gamma\cdot Aut_e(\Sigma)^\circ)\times(\mathcal{C}/Def(\Sigma)) \;\longrightarrow\; \mathcal{C}/Def(\Sigma)$$

on $\mathcal{C}/Def(\Sigma)$ as the cut-and-paste region remain near the neck region of Σ under the smallness assumption of $Aut_e(\Sigma)^\circ$. This proves the first statement of the lemma.

To extend this to an approximate pseudo-$Aut(\Sigma)$-action on $\mathcal{C}/Def(\Sigma)$, consider the product family $(Aut(\Sigma)\times\mathcal{C})/(Aut(\Sigma)\times Def(\Sigma))$. Recall $s'_\cdot$ the sections of $\mathcal{C}/Def(\Sigma)$ that correspond to the added rigidifying points $p'_\cdot$ on Σ. Their image lies in the trunk region of $\mathcal{C}/Def(\Sigma)$. Extend these sections first to over $\Gamma\cdot Aut_e(\Sigma)^\circ \times Def(\Sigma)$ by setting $s'_{\cdot,\sigma} = \sigma\cdot s'_\sigma$ over $\{\sigma\}\times Def(\Sigma)$,

where $(\{\sigma\}\times\mathcal{C})/(\{\sigma\}\times Def(\Sigma))$ is canonically identified with $\mathcal{C}/Def(\Sigma)$. These sections again have their image in the trunk region of $(\{\sigma\}\times\mathcal{C})/(\{\sigma\}\times Def(\Sigma))$. Extend these sections next to over $Aut(\Sigma)\times\{0\}$ as well by the $Aut(\Sigma)$-action on Σ. Finally extend the resulting sections to over $Aut(\Sigma)\times Def(\Sigma)$. This then defines an approximate pseudo-$Aut(\Sigma)$-action on $\mathcal{C}/Def(\Sigma)$ by isomorphisms that extends the pseudo-$\Gamma\cdot Aut_e(\Sigma)^\circ$-action constructed. This concludes the proof.

□

The same argument gives also:

Lemma 2.7 [finite group]. *Any finite group action on Σ by automorphisms extends to an action on $\mathcal{C}/Def(\Sigma)$ by isomorphisms. This action extends to a pseudo- $\Gamma\cdot Aut_e(\Sigma)^\circ$-action on $\mathcal{C}/Def(\Sigma)$ and then to an approximate pseudo-$Aut(\Sigma)$ on $\mathcal{C}/Def(\Sigma)$, both by isomorphisms.*

The quotient topology on $\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$ and the stabilization morphism.

The *quotient topology* on $\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$ is defined by setting a subset $U\subset\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$ to be *open* if $U=\cup_\alpha U_\alpha$ such that there exist a collection of standard local charts $(V_\alpha,\Phi_\alpha,\psi_\alpha)$ of $\widetilde{M}_{(g,h),(n,\vec{m})}$ such that $U_\alpha\subset\psi_\alpha(V_\alpha)$ and that $\psi_\alpha^{-1}(U_\alpha)$ is open in V_α. This is similar to the construction in Sec. 1.1.3 and Sec. 1.2 for the quotient topology on $\widehat{B}$ and $\widehat{A}$.

For $((g,h),(n,\vec{m}))$ with $2(2g+h+n)+m_1+\cdots+m_h\geq 5$, *stabilization* of prestable labelled-bordered Riemann surfaces by contracting the unstable components gives rise to a *flat local complete intersection morphism*[6] $st:\mathcal{C}/Def(\Sigma)\to\mathcal{C}_{\rm st}/Def(\Sigma_{\rm st})$, together with a group homomorphism $Aut(\Sigma)\to Aut(\Sigma_{\rm st})$ that makes st equivariant, for each $[\Sigma]\in\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}$. The collection of these pairs of morphisms on local charts-with-structure-group descend to the *stabilization morphism* $\widetilde{st}:\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}\to\overline{M}_{(g,h),(n,\vec{m})}$. We say that *$\widetilde{st}$ is a local complete intersection morphism in the stacky sense.* It is continuous with respect to the quotient topology on $\widetilde{M}_{(g,h),(n,\vec{m})}$. The inclusion $\overline{M}_{(g,h),(n,\vec{m})}\hookrightarrow\widetilde{M}_{(g,h,(n,\vec{m})}$ is a section to $\widetilde{st}$ with open-dense image.

Remark 2.8 [local factorization of st]. Assume that $2(2g+h+n)+m_1+\cdots+m_h\geq 5$. Let $\Sigma=\Sigma^s\cup\Sigma^u$, where the subcurve Σ^u consists of all the unstable irreducible components of Σ and Σ^s is the union of the remaining irreducible components. Then a connected component of Σ^u may intersect Σ^s at either 1 or 2 nodes of Σ; it is called a *tree* in the formal case and a *chain* in the latter case, in which it can only be either a chain of $\mathbb{P}^1$ of the form $\mathbb{P}^1_{(1)}\cup\cdots\cup\mathbb{P}^1_{(k)}$ with 0 of $\mathbb{P}^1_{(i)}$ glued to ∞ of $\mathbb{P}^1_{(i+1)}$, or a chain of discs $D^2=\{z\in\mathbb{C}:|z|\leq 1\}$ of the form $D^2_{(1)}\cup\cdots\cup D^2_{(k)}$ with $-\sqrt{-1}$ of $D^2_{(i)}$ glued to $\sqrt{-1}$ of $D^2_{(i+1)}$. These are reflected to the stabilization map: locally st can be factorized to a composition of a projection map of a product space, for no collapsing or collapsing a tree of unstable components; a map of the form $\pi[k]:B[k]\to B$ in Lemma 1.1.1.4, for collapsing a chain of unstable $\mathbb{P}^1$ components; and a map of the form

$$(\mathbb{R}_{\geq 0})^{k+1}\longrightarrow\mathbb{R}_{\geq 0}\,,\qquad(t_0,\ldots,t_k)\longmapsto t_0\cdots t_k\,,$$

for collapsing a chain of unstable discs. Cf. [Sie1: end of Sec. 2.2].

[6] See [Fu] for a general definition of *local complete intersection morphism*. Such a morphism has a well-defined Gysin map, and hence push-pull, on cycles.

3 The moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)$ of stable maps.

In the previous two sections, we discuss respectively the targets and the domains of the maps we want to study. However, as a lesson from the various standard moduli problems in algebraic geometry, which can almost always be traced back to the complicated problem of Hilbert-schemes, to render a reasonable moduli space of maps from bordered Riemann surfaces to fibers of $\widehat{W}/\widehat{B}$, we need to fix some combinatorial quantities of such maps that are constant for a continuous/flat family. The closed Gromov-Witten theory indicates a partial set of such data: the *combinatorial type of domain curves*, the image curve class $\beta \in H_2(X, L; \mathbb{Z})$, and *boundary loop class* $\vec{\gamma}$ from $H_1(L; \mathbb{Z})$. The study of [Liu(C)] implies that for open Gromov-Witten theory the boundary effect is reflected also in the *Maslov index* $\mu \in \mathbb{Z}$, which is not fixed by β in general. This quantity thus has to be generalized to our case and be included in the combinatorial data. This is done in Sec. 3.1 and the generalized Maslov index does enter the operator index in Sec. 5.3.1. However, this addition of data is not enough. While it turns out that the datum $\vec{\gamma}$ from $H_1(L; \mathbb{Z})$ is not influenced, the datum $\beta \in H_2(X, L; \mathbb{Z})$ is not the correct choice of the image curve class datum in our case since in general it is not well-defined to all fibers in the family $W[k]/B[k]$, which contains X as a fiber, due to the monodromy effect. It thus has to be enlarged to and replace by the *minimal common monodromy-invariant curve-class subset* $[\beta] \subset H_2(X, L; \mathbb{Z})$, generated by β under the monodromy of $W[k]/B[k]$, for all $k \in \mathbb{Z}_{\geq 0}$. This is done in Sec. 3.2. Once these combinatorial data are identified, one can then define the related moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)$ of maps accordingly. This is done in Sec. 3.3.

3.1 Maslov index of a map to a singular space or a relative pair.

A generalization of the notion of Maslov index to a map from a bordered Riemann surface to a relative pair or a singular space from a symplectic cut is given in this subsection. This quantity is needed to select a reasonable (union of) component(s) of the moduli space of stable maps in question.

Given a C^∞ map $f : (\Sigma, \partial\Sigma) \to (X, L)$ from a prestable bordered Riemann surface Σ to a smooth symplectic manifold X. Endow X with a compatible almost-complex structure J that renders T_*X a complex vector bundle with $T_*L \hookrightarrow (T_*X)|_L$ as a totally real subbundle. Then $E := f^*(\det(T_*X))$ is a complex line bundle on Σ whose restriction to $\partial\Sigma$ contains a real line subbundle $E_{\mathbb{R}}(L)$ associated to $f^*(T_*L)$. The Maslov index of f in this case (cf. [K-L: Definition 3.7.2]) is defined by:

Definition 3.1.1 [Maslov index - smooth target]. The *Maslov index* $\mu(f)$ of the C^∞ map f above is twice the index of a general extension of $E_{\mathbb{R}}(L) \subset E|_{\partial\Sigma}$ to a real-line subbundle with isolated singularities in E, still denoted by $E_{\mathbb{R}}(L)$, over the whole Σ. For convenience, we set $\mu(f) = 2\, deg\,(f^* \det(T_*X))$ if either L or $\partial\Sigma$ is empty.

Note that this definition is more in the almost-complex category than in the symplectic category. However, $\mu(f)$ thus defined is independent of the choice of ω-compatible J on X and the general extension $E_{\mathbb{R}}$ on Σ. To turn the real line field language to the more convenient real vector field language, one considers the complex line bundle $E^{\otimes 2}$ and rephrases $\mu(f)$ as the index of a general global section s of $E^{\otimes 2}$ that extends the section s_L in $E^{\otimes 2}|_{\partial\Sigma}$ determined by $f^*(T_*L)$.

In the complex Kähler category, the key object in the above description of $\mu(f)$, namely the (complex) determinant line bundle $K := \det \Omega_X = (\det T_*X)^{-1}$, can be defined for a *singular* Y from a symplectic cut. Once having this, the Maslov index of a C^∞ map $f : \Sigma \to (Y, L)$, with L disjoint from the singular locus Y_{sing} of Y, can be defined in exactly the same way as above: the index of a global section s in $f^*(K^{\otimes(-2)})$ that extends a global section s_L in $(f^*(K^{\otimes(-2)}))|_{\partial\Sigma}$

determined by $f^*(T_*L)$. Taking det of a coherent sheaf in algebraic geometry brings in a twisting effect from a divisor whose support is contained in the non-locally-free locus of the coherent sheaf (cf. [Kn-M]). For Ω_Y in Kähler category, such locus coincides with the singular locus of Y. One can compute such effect explicitly and compare them with the contribution to $\mu(f)$ from each individual smooth irreducible component of Y. The result can be stated in both the symplectic and the almost-complex category. This gives rise to the following definitions.

Definition 3.1.2 [Maslov index - relative pair and symplectic gluing]. The Maslov index of a C^∞ map from a bordered Riemann surface Σ to a symplectic pair or a symplectic space from a symplectic cut is defined as follows:

(1) Let $(Z, L; D)$ be a smooth symplectic pair $(Z; D)$ with a Lagrangian submanifold L disjoint from D and $f : (\Sigma, \partial\Sigma) \to (Z, L)$ be a C^∞ map. Then, define the *Maslov index* of f *relative to* D to be

$$\mu^{rel}(f) \;=\; \mu(f) \;-\; 2\, f_*[\Sigma] \cdot D\,,$$

where $\mu(f)$ is the usual Maslov index of f as defined in Definition 3.1.1. (If L is empty, then set $\mu(f) = \deg f^*(K_Z^{\otimes(-2)}) = -2\, f_*[\Sigma] \cdot K_Z$. Note that both L and D in the definition can be disconnected.)

(2) Let $(Y, L) = (Y_1, L_1) \cup_{D_1 \simeq D_2} (Y_2, L_2)$ be the singular symplectic space from gluing of two Lagrangian-decorated relative pairs $(Y_1, L_1; D_1)$ and $(Y_2, L_2; D_2)$ and $f = f_1 \sqcup f_2 : \Sigma := \Sigma_1 \cup \Sigma_2 \to (Y_1, L_1) \cup_D (Y_2, L_2)$ be a C^∞ map to (Y, L). Then, define the *Maslov index* of f to be

$$\mu(f) \;=\; \mu^{rel}(f_1) + \mu^{rel}(f_2) \;=\; (\mu(f_1) - 2f_{1\,*}[\Sigma_1] \cdot D_1) \;+\; (\mu(f_2) - 2f_{2\,*}[\Sigma_2] \cdot D_2)\,.$$

(3) For a C^∞ map f to a symplectic space from gluing a finite collection of Lagrangian-decorated symplectic pairs, apply Item (1) and Item (2) above inductively to define the Maslov index $\mu(f)$ or $\mu^{rel}(f)$.

The same definitions hold in the almost-complex category with L replaced by a totally real submanifold and D replaced by a real-codimension-2 almost-complex submanifold.

Example 3.1.3 [relative Maslov index]. (Cf. Sec. 1.2.) Given $(Z, L; D)$, let $(Z_{[k]}, L_{[k]}; D_{[k]})$ be the central fiber of its k-th expanded relative pairs. For an open relative stable map $f : \Sigma \to (Z_{[k]}, L_{[k]}; D_{[k]})$ with the corresponding decomposition $f = f_0 \sqcup f_1 \sqcup \cdots \sqcup f_k$, where $f_0 : \Sigma_0 \to Y$ and $f_i : \Sigma_i \to \Delta_i$, $i = 1, \ldots, k$, the Maslov index of f as a relative map is then

$$\mu^{rel}(f) \;=\; (\mu(f_0) - 2f_{0\,*}[\Sigma_0] \cdot D_0) \;-\; 2 \sum_{i=1}^{k} f_{i\,*}[\Sigma_i] \cdot (K_{\Delta_i} + D_{i,0} + D_{i,\infty})\,,$$

where $\mu(f_0)$ is defined as in Definition 3.1.1 for smooth target.

We list as lemmas the basic invariance properties of the Maslov index of C^∞ maps, as defined above, that are part of the foundations of later discussions. The proof of these lemmas are straightforward and hence omitted.

Lemma 3.1.4 [invariance under homotopy and deformation]. (1) *Let Z be a smooth manifold of even dimension, L be a smooth submanifold of Z of the middle dimension, and D be a smooth codimension-2 submanifold of Z disjoint from L. Let $f_t : \Sigma \to (Z, \omega_t)$, $t \in [0, 1]$, be a homotopy class of C^∞ maps from a prestable bordered Riemann surface Σ to $(Z; D)$ with*

$f_t(\partial\Sigma) \subset L$ and ω_t is a 1-parameter family of symplectic structures (say, of class C^2) on Z keeping L a Lagrangian submanifold and D a symplectic submanifold. Then $\mu^{rel}(f_0) = \mu^{rel}(f_1)$. (2) Let $Y = Y_1 \cup_D Y_2$ be a space from gluing smooth even-dimensional (manifold, codimension-2 submanifold)-pairs and L be a smooth submanifold of Y of the middle dimension disjoint from D. Let $f_t : \Sigma \to (Y, \omega_t)$, $t \in [0,1]$, be a homotopy class of C^∞ maps from a prestable bordered Riemann surface Σ to Y with $f_t(\partial\Sigma) \subset L$ and ω_t is a 1-parameter family of symplectic structures (say, of class C^2) on Y keeping L a Lagrangian submanifold and D a symplectic submanifold. Then $\mu(f_0) = \mu(f_1)$.

Lemma 3.1.5 [invariance under domain degeneration]. *Let (X, L) be either a smooth symplectic manifold or a singular symplectic space from symplectic cut, with a Lagrangian submanifold L disjoint from X_{sing}. Let $p : \Sigma \to \underline{\Sigma}$ be a pinching map that arise from a degeneration of Σ that pinches a finite disjoint union of simple loops on Σ. Given a C^∞ map $f : \Sigma \to (X, L)$ and a family of deformations of f to a $g : \underline{\Sigma} \to (X, L)$, Then $\mu(f) = \mu(g)$. Similarly for C^∞ maps into $(Z, L; D)$.*

Lemma 3.1.6 [invariance under symplectic cut on target]. *Let $\xi : (X, L) \to Y := (Y_1, L_1) \cup_D (Y_2, L_2)$ be a symplectic cut with L_1 and L_2 disjoint form D. (1) Let $f : \Sigma \to (X, L)$ be a C^∞ map that intersects $\xi^{-1}(D)$ at a finite union of S^1-orbits and $g : \underline{\Sigma} \to Y$ be the C^∞ map descended from f, where $\underline{\Sigma}$ is obtained from Σ by pinching each connected component of $f^{-1}(\xi^{-1}(D))$ to a nodal point. Then $\mu(g) = \mu(f)$. (2) Conversely, let $g : \underline{\Sigma} \to \underline{X}$ be a pre-deformable C^∞ map (cf. Definition 3.3.1) and $f : \Sigma \to X$ be a lifting of g, where Σ is a deformation of $\underline{\Sigma}$ that smoothes exactly the nodes $g^{-1}(D)$ in $\underline{\Sigma}$. Then $\mu(f) = \mu(g)$.*

We remark that, if one associates the symplectic cut ξ to a symplectic deformation family as constructed in [Go], [MC-W], and [I-P2], then Lemma 3.1.6 is a corollary of [I-P2: Lemma 2.2]. The same statements of these lemmas, with L replaced by a totally real submanifold and D replaced by a real-codimension-2 almost-complex submanifold, in the almost-complex category hold as well.

Remark 3.1.7 [homotopy vs. homology]. As in the absolute case in [Liu(C)], the Maslov index of an open relative stable map $f : \Sigma \to (Z, L; D)$ or the singular (Y, L) influences the deformation properties of f. Though a homotopy invariant, it is not determined by the image class $f_*[\Sigma]$ of f in $H_2(Z, L; \mathbb{Z})$ or $H_2(Y, L; \mathbb{Z})$, cf. [K-L: Remark 4.2.2].

3.2 Monodromy effect and the choice of curve class data in H_2.

Recall the symplectic cut $\xi : X \to Y = Y_1 \cup_D Y_2$ and the associated almost-complex degeneration W/B. Let L be an Lagrangian submanifold disjoint from the cutting locus $\xi^{-1}(D)$ then it gives rise to $(W, B \times L)/B$, where L is totally real in each fiber of W/B; and the construction in Sec. 1.1 extends immediately to give expanded degenerations $(W[k], L[k])/B[k]$ with the equivariant pseudo-$\mathbb{G}_m[k]$-action, the topological space $(\widehat{W}, \widehat{B} \times L)/\widehat{B}$, the standard local charts $\varphi[k] : (W[k], L[k])/B[k] \to (\widehat{W}, \widehat{L})/\widehat{B}$ of $(\widehat{W}, \widehat{L})/\widehat{B}$ with the product-induced map $\tilde{\mathbf{p}}[k] : (W[k], L[k])/B[k] \to (W, L)/B$. Note that $\widehat{L} = \widehat{B} \times L$. We remark that $L[k] \simeq B[k] \times L$ is a coisotropic submanifold in $W[k]$ and is fiberwise Lagrangian/totally-real over $B[k]$. We can assume that $L[k]$ is contained in the trunk region $\overline{U_1}[k] \cup \overline{U_2}[k]$ of $W[k]/B[k]$. $\mathbf{p}[k]$ sends the discriminant locus $\{\lambda_0 \cdots \lambda_k = 0\} \subset B[k]$ of $W[k]/B[k]$ to the discriminant locus $\{0\} \subset B$ of W/B and the complement $B[k]_{\text{reg}} := B[k] - \{\lambda_0 \cdots \lambda_k = 0\}$ to the complement $B_{\text{reg}} := B - \{0\}$.

Note that $\pi_1(B[k]_{\mathrm{reg}}) \simeq \mathbb{Z}^{\oplus(k+1)}$ is generated by the canonically-oriented meridian S^1 of the $(k+1)$-many coordinate hyperplanes of $B[k]$. Fix topological trivializations

$$W[k]_{\mathbb{R}_{\geq 0}\cdot(\varepsilon^2/4\,,\cdots,\varepsilon^2/4)} \;\simeq\; (\mathbb{R}_{\geq 0}\cdot(\varepsilon^2/4\,,\,\cdots\,,\,\varepsilon^2/4))\times X$$

along the diagonal ray of $B[k]$'s. This fixes an isomorphism

$$H_2(W[k]_\bullet, L[k]_\bullet;\mathbb{Z}) \;\simeq\; H_2(X,L;\mathbb{Z})\,, \qquad \text{for } \bullet \in \mathbb{R}_{\geq 0}\cdot(\varepsilon^2/4\,,\,\cdots\,,\,\varepsilon^2/4)\,.$$

Via these identifications, $\pi_1(B[k]_{\mathrm{reg}})$ acts on $H_2(X,L;\mathbb{Z})$ by monodromy. Furthermore, since L is contained in the truck region of X, one has:

Lemma 3.2.1 [trivial monodromy on $H_1(L;\mathbb{Z})$]. *As a fiber of $(W[k],L[k])/B[k]$, the monodromy $\pi_1(B[k]_{\mathrm{reg}})$-action on $H_1(L;\mathbb{Z})$ is well-defined and is trivial; and the connecting homomorphism $\partial : H_2(X,L;\mathbb{Z}) \to H_1(L;\mathbb{Z})$ is equivariant with respect to the $\pi_1(B[k]_{\mathrm{reg}})$-action.*

Lemma/Definition 3.2.2 [$(\widehat{W},\widehat{L})/\widehat{B}$-monodromy orbit]. *For each $\beta \in H_2(X,L;\mathbb{Z})$, all the monodromy-orbits $\pi_1(B[k]_{\mathrm{reg}})\cdot\beta$, $k\in\mathbb{Z}_{\geq 0}$, coincide.* We will name it the *$(\widehat{W},\widehat{L})/\widehat{B}$-monodromy orbit* of β and denote it by $[\beta]$.

Proof. Observe that the following diagram commutes

$$\begin{array}{ccc} H_2(X,L;\mathbb{Z}) & \stackrel{\tilde{\mathbf{p}}[k]_*}{\longrightarrow} & H_2(X,L;\mathbb{Z}) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \mathbf{p}[k]_*(a)} \\ H_2(X,L;\mathbb{Z}) & \stackrel{\tilde{\mathbf{p}}[k]_*}{\longrightarrow} & H_2(X,L;\mathbb{Z}) \end{array}$$

for all $a \in \pi_1(B[k]_{\mathrm{reg}})$; i.e. $\tilde{\mathbf{p}}[k]_*$ is equivariant with respect to the monodromy actions. As $\tilde{\mathbf{p}}[k]_*$ is the identity map under our identification and $\mathbf{p}[k]_* : \pi_1(B[k]_{\mathrm{reg}}) \to \pi_1(B_{\mathrm{reg}})$ is surjective, the lemma follows immediately.

□

Since the difference of two elements in a same $[\beta]$ lies in the kernel of the map

$$\xi_* \;:\; H_2(X,L;\mathbb{Z}) \;\longrightarrow\; H_2(Y,L;\mathbb{Z})\,,$$

each $[\beta]$ determines a class, denoted by $\xi_*[\beta]$, in $H_2(Y,L;\mathbb{Z})$. For simplicity of notation, we will denote $\xi_*[\beta]$ also by $[\beta]$.

Comparison 3.2.3 [*Li-Ruan and Ionel-Parker*]. Though in different format, it should be noted that $(\widehat{W},\widehat{L})/\widehat{B}$-monodromy orbits in $H_2(X,L;\mathbb{Z})$ coincides with $\xi_*^{-1}(0)$-cosets, where $\xi_* : H_2(X,L;\mathbb{Z}) \to H_2(Y,L;\mathbb{Z})$ for the moment. Thus, the curve class considered here is of the same kind as [L-R: Sec. 5] when L is empty. Furthermore, $\xi_*^{-1}(0)$ is generated precisely by the "*rim tori*" of [I-P1: Sec. 5] since the monodromy of all $W[k]/B[k]$ are generated exactly by uniform simultaneous Dehn twists over D. As remarked in ibidem it is with respect to such a collection in $H_2(X;\mathbb{Z})$ that one expects to have a degeneration-formula/gluing-theorem of Gromov-Witten invariants. Thus the combinatorial data we use to restrict the moduli problem of maps from bordered Riemann surfaces to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$ is the same, when L is empty, as those in [L-R], [I-P1], and [I-P2]. See Appendix for a further comparison of [L-R] versus [I-P1], [I-P2].

Comparison 3.2.4 [*refinement of* [Li1] *and* [Li2]]. In the algebro-geometric setting ([Li1], [Li2]) without L, one assumes the existence of a relative ample line bundle H on W/B and considers

a fixed H-degree curve class, which in general corresponds to a collection of curve classes in $H_2(X;\mathbb{Z})$ (or $A_1(X)$). Note that, since $H|_{W_b}$, $b \in B$, form a flat family of line bundles with base B, the first Chern class of $H|_X$, and hence the fixed H-degree class, must be monodromy invariant. As the moduli space of maps to fibers of $\widehat{W}/\widehat{B}$ associated to different monodromy orbits must be disjoint from each other, Jun Li's degeneration formula in [Li1] and [Li2] indeed always splits into a disjoint/independent collection[7] of degeneration formulas, one for each monodromy orbit in the fixed H-degree curve class. Since the discussion in this subsection produces the same monodromy on $H_2(X;\mathbb{Z})$ (or $A_1(X)$) as the one associated to the Artin stack $\mathfrak{W}/\mathfrak{B}$ of expanded degenerations associated to W/B, constructed in [Li1], the $\widehat{W}/\widehat{B}$-monodromy-orbit refinement of [Li2] is the finest refinement of Jun Li's formula (and is indeed implicitly already in [Li2], had a discussion of monodromy at the level of the stack $\mathfrak{W}/\mathfrak{B}$ been made. Further, it has to be so for [Li1], [Li2] to be consistent with [L-R], [I-P1], [I-P2]. So this is also a consistency check statement. See Comparison 3.2.3 above and Appendix). The examples studied in [L-Y1] are [L-Y2] are both special cases of such refinement: there the $\mathfrak{W}/\mathfrak{B}$-monodromy on $H_2(X;\mathbb{Z})$ (or $A_1(X)$) is trivial and hence the degeneration formula of Jun Li refines to one associated to each fixed curve class in $H_2(X;\mathbb{Z})$ (or $A_1(X)$).

3.3 The moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$.

We now define the moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$ and highlight its basic properties.

Moduli space of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$: its topology.

Definition 3.3.1 [stable map to fibers of $(W[k],L[k])/B[k]$]. Let $[\beta]$ be the $(\widehat{W},\widehat{L})/\widehat{B}$-monodromy orbit of $\beta \in H_2(X,L;\mathbb{Z})$, $\vec{\gamma} = (\gamma_1\,,\,\ldots\,,\gamma_h) \in H_1(L;\mathbb{Z})^{\oplus h}$ such that $\partial\beta = \gamma_1 + \cdots + \gamma_h$, and $\mu \in \mathbb{Z}$. A map $f : (\Sigma,\partial\Sigma)/pt \to (W[k],L[k])/B[k]$ from a bordered Riemann surface Σ to a fiber[8] of $(W[k],L[k])/B[k]$ is called *prestable* of (*combinatorial*) *type* $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ if the following conditions are satisfied:

- Σ is a *prestable labelled-bordered Riemann surface* of type $((g,h),(n,\vec{m}))$;
- f is *continuous* and $\tilde{f} := \nu\circ f$ is *J-holomorphic*: $J\circ d\tilde{f} = d\tilde{f}\circ j$, where $\nu : \tilde{\Sigma}\to\Sigma$ is the normalization of Σ;
- $\tilde{\mathbf{p}}[k]_*(f_*[\Sigma,\partial\Sigma]) \in [\beta]$; $\tilde{\mathbf{p}}[k]_*(f_*[\dot{\partial}\Sigma]) = \vec{\gamma}$; $\mu(f) = \mu$;
- the automorphism group $Aut^{\,\mathrm{rigid}}(f)$ of f as a map to (the rigid) $W[k]$ is finite.

An *isomorphism* between two prestable maps $f_1 : \Sigma_1/pt \to W[k]/B[k]$, $f_2 : \Sigma_2/pt \to W[k]/B[k]$ of the same type is a pair (α,β)[9], where $\alpha : \Sigma_1 \to \Sigma_2$ is an isomorphism of prestable

[7] The moduli stacks involved for different monodromy orbits are disjoint from each other. They are substacks, consisting of disjoint collections of connected components, of the moduli stack constructed in [Li1] and are equipped with the tangent-obstruction complex and the virtual fundamental class from the restriction of those constructed in [Li2] to related connected components. See [L-Y1] for an explicit example and discussion.

[8] When the fiber in question is almost-complex isomorphic to a W_λ with $\lambda \neq 0$, the existing definitions from Gromov-Witten theory for smooth targets apply. Thus, all our focus here is on maps with singular targets. Such focus of discussions to singular targets prevails the whole manuscript.

[9] The use of notation (α,β) here is so compelling. There should be no confusion of this β with the curve class β. Similarly, for the occasional use of a *map* g, versus the genus g.

labelled-bordered Riemann surfaces with marked points and $\beta \in \mathbb{G}_m[k]$ acts on $W[k]/B[k]$ as in Sec. 1.1.2 such that $f_1 \circ \beta = f_2 \circ \alpha$. The isomorphism class associated to a prestable map f will be denoted by $[f]$. The *group of automorphisms* $Aut(f)$ of a prestable $f : \Sigma/pt \to W[k]/B[k]$ consists then of elements $(\alpha, \beta) \in Aut(\Sigma) \times \mathbb{G}_m[k]$ such that $\beta \circ f = f \circ \alpha$.

A prestable map $f : \Sigma/pt \to (W[k], L[k])/B[k]$, with image in fiber $(W[k]_{\vec{\lambda}}, L[k]_{\vec{\lambda}})$, is called *non-degenerate* if no irreducible components of Σ are mapped into the singular locus $W[k]_{\vec{\lambda},\, sing}$ of $W[k]_{\vec{\lambda}}$. For f non-degenerate, $\Lambda := f^{-1}(W[k]_{\vec{\lambda},\, sing})$ consists of interior nodes on Σ. A node $q \in \Lambda$ is called a *distinguished node* on Σ *under* f.

Assume that the target fiber $W[k]_{\vec{\lambda}} \simeq Y_{[k']}$ for some k'. Decompose a non-degenerate prestable f by

$$f \,=\, \cup_{i=0}^{k'} f_i \,:\, \Sigma \,=\, \cup_{i=0}^{k'} \Sigma_{(i)} \;\longrightarrow\; Y_{[k']} \,=\, \cup_{i=0}^{k'} \Delta_i$$

with $f_{(i)} = f_{\Sigma_{(i)}} : \Sigma_{(i)} \to \Delta_i$. Recall $D_i := \Delta_i \cap \Delta_{i+1}$. Let $\Lambda_i := f^{-1}(D_i)$ and called it the *i-th subset of distinguished nodes*. Associated to $q_{\cdot} \in \Lambda_i$ are unique $q_{\cdot,1}$ on $\Sigma_{(i)}$ and $q_{\cdot,2}$ on $\Sigma_{(i+1)}$. From the normal form of J-holomorphic map at a point ([Ye: Theorem 3.1] and [I-P1: Lemma 3.4]), $f_i^{-1}(D_i)$ is a divisor of the form $\sum_{q_{ij} \in \Lambda_i} s_{ij,1}\, q_{ij,1}$ on $\Sigma_{(i)}$ and $f_{i+1}^{-1}(D_i)$ is a divisor of the form $\sum_{q_{ij} \in \Lambda_i} s_{ij,2}\, q_{ij,2}$ on $\Sigma_{(i+1)}$. A prestable f is called *pre-deformable* if it is non-degenerate and $s_{ij,1} = s_{ij,2}$ $(=: s_{ij})$ for all $q_{ij} \in \Lambda_i$, $i = 0, \ldots, k$. We call s_{ij} the *contact order* of f at q_{ij} along D_i. Both the non-degeneracy condition and the pre-deformability condition are preserved under isomorphisms between prestable maps.

Finally, a prestable $f : \Sigma/pt \to (W[k].L[k])/B[k]$ is called *stable* if f is pre-deformable and its group $Aut(f)$ of automorphisms is finite. The moduli space of isomorphism classes of stable maps to fibers of $(W[k], L[k])/B[k]$ of type $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ is denoted by $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$.

We have assumed that the almost-complex structure on $W[k]$ is C^∞; thus, all maps parameterized by $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ are C^∞ as well when restricted/lifted to the connected components of the normalization of the domains.

For $[f : \Sigma/pt \to W[k]/B[k]] \in \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$, fix a Hermitian metric[10] on $\mathcal{C}/Def(\Sigma)$ and on $W[k]$. Define the *energy*[11] of $f : \Sigma/pt \to W[k]/B[k]$ to be

$$E(f) \;=\; \frac{1}{2}\int_\Sigma |df|^2\, d\mu\,,$$

where $|df|^2$ is the norm-squared of df with respect to the metric on $W[k]$ and on Σ, and $d\mu$ is the area-form on Σ with respect to the metric on Σ. Then one can define a topology on $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ similar to [Pa: Sec. 2.1] and [Ye: Definition 0.2]; see also [Gr2], [P-W], [R-T1], [Sie1]; [Liu(C)]; [I-P1], [L-R]. A point $[f']$ in $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ is said to in the $(\varepsilon_1,\varepsilon_2)$-neighborhood $U_{\varepsilon_1,\varepsilon_2}([f])$ of $[f]$ if they have representatives $f : \Sigma/pt \to W[k]/B[k]$ and $f' : \Sigma'/pt \to W[k]/B[k]$ so that

(1) there exists a surjective *collapsing/pinching map* $c : \Sigma' \to \Sigma$ that is a diffeomorphism from the complement of a collection of simple loops and simple arc with ends on $\partial\Sigma'$ on Σ' to the complement of the set of nodes on Σ', and collapses/pinches each simple loop (resp. arc) in the collection to an interior (resp. boundary) node of Σ such that

[10] A *Hermitian metric* on an almost-complex space is a (Riemannian) metric so that the almost-complex structure is an isometry. Different choices of such auxiliary metrics on domains and targets define the same topology. Here for $W[k]$ which is equipped with a compatible pair (J,ω) the metric is chosen to be the one associated to the pair (J,ω).

[11] Note that $E(f)$ is conformally invariant with respect to the metric on Σ. For J ω-tame and f J-holomorphic, $E(f)$ coincides with the symplectic area, and is determined by $[\beta] \subset H_2(X, L; \mathbb{Z})$.

·	(nearness of domain)	Σ' is isomorphic to a fiber of $\mathcal{C}/Def(\Sigma)$ with $\|j - c_*j\|_{C^\infty} < \varepsilon_2$ on $\Sigma - U_{\varepsilon_1}$ and $c(p'_\cdot)$ in the ε_2-neighborhood of $p_\cdot$, where $p_\cdot$, $p'_\cdot$ are marked points on Σ, Σ' that are paired by their label;
·	(nearness of target and map)	$\|f - f' \circ c^{-1}\|_{C^\infty} < \varepsilon_2$ on $\Sigma - U_{\varepsilon_1}$, as maps to $W[k]$;
(2)	(nearness of energy)[12]	$\|E(f) - E(f')\| < \varepsilon_2$.

Here, U_{ε_1} is the ε_1-neighborhood of the set of nodes of Σ that is small enough so that it contains no marked points. The system $\{\, U_{\varepsilon_1,\varepsilon_2}([f])\,\}_{f;\,\varepsilon_1,\,\varepsilon_2}$ of subsets generates the C^∞-*topology*[13] on $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$.

The pseudo-embedding $\varphi_{k',k;I} : (W[k'],L[k])/B[k'] \hookrightarrow (W[k],L[k])/B[k]$, $k' < k$, $I \subset \{0\,,\,\ldots\,,\,k\}$, from Sec. 1.1.3 induces a *pseudo-embedding*

$$\begin{aligned}\varphi_{k',k;I} \,:\, &\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k'],L[k'])/B[k']\,|\,[\beta],\vec{\gamma},\mu)\\ &\hookrightarrow\, \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)\,.\end{aligned}$$

Define the set of isomorphism classes of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$:

$$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu) \;:=\; \Big(\amalg_{k=0}^{\infty}\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)\Big)\Big/\sim,$$

where the equivalence relation $\sim$ is generated by $[f] \sim \varphi_{k',k;I}([f'])$ for $[f] \in \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}($ $(W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ and $[f'] \in$ the defining domain of $\varphi_{k',k;I}$ on $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k'],$ $L[k'])/B[k']\,|\,[\beta],\vec{\gamma},\mu)$. By construction, there are embeddings of sets

$$\varphi_{(k)} \,:\, \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu) \,\hookrightarrow\, \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)\,,\quad k \in \mathbb{Z}_{\geq 0}\,.$$

A subset U of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$ is said to be *open* if $U = \cup_\alpha U_\alpha$ such that U_α is contained in the image of some $\varphi_{(k)}$ and $\varphi_{(k)}^{-1}(U_\alpha)$ is open in $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],$ $\vec{\gamma},\mu)$. This defines the C^∞-*topology* on the moduli space[14] $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$ of stable maps to fibers of $\widehat{W}/\widehat{B}$. By construction, $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$. fibers naturally over B; in notation $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$.

Definition 3.3.2 [tautological cover]. By construction,

$$\Big\{\,\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)\,\Big\}_{k\in\mathbb{Z}_{\geq 0}}$$

[12]This condition is redundant here as $E(f) = E(f')$ currently. We reserve it here to stress its importance to Compactness Theorem.

[13]Let $\mathcal{M}$ be the moduli space of pre-deformable stable maps to fibers of $((W[k],L[k])/B[k])^{\text{rigid}}$ with the C^∞-topology from [Ye: Definition 0.2]. Then $\mathbb{G}_m[k]$ acts on $\mathcal{M}$, and our moduli space $\mathcal{M}_\cdot^{\text{non-rigid}}((W[k],L[k])/B[k]\,|\,\cdots)$ is contained in $\mathcal{M}/\mathbb{G}_m[k]$ with the quotient topology. The induced subset-topology on $\mathcal{M}_\cdot^{\text{non-rigid}}((W[k],L[k])/B[k]\,|\,\cdots)$ coincides with its C^∞-topology.

[14]We could have used the notation $\mathcal{M}_{(h,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B})\,|\,[\beta],\vec{\gamma},\mu)$ for the moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$. Our choice of the latter reflects the intention to keep in mind that maps to singular fibers are meant to be limited to those that are approachable from maps to smooth fibers, (reflected, e.g. by the pre-deformability condition). As we will show that this is indeed so at the level of Kuranishi/virtual neighborhoods on the moduli space.

is an open cover of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$. We will call it the *tautological cover* of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$.

Indeed, there exists k_0 depending $(W/B, L)$ and $((g,h),(n,\vec{m})|[\beta],\vec{\gamma},\mu)$ such that

$$\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k_0], L[k_0])/B[k_0]\,|\,[\beta],\vec{\gamma},\mu) \supset \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k_0+1], L[k_0+1])/B[k_0+1]\,|\,[\beta],\vec{\gamma},\mu)$$
$$\supset \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k_0+2], L[k_0+2])/B[k_0+2]\,|\,[\beta],\vec{\gamma},\mu) \supset \cdots .$$

Thus, the tautological cover of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ is finite in effect, cf. Theorem 3.3.8. The universal maps on the universal curve over each $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ are glued to give the universal map (between spaces with charts)

$$F \,:\, \mathcal{C}/\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu) \longrightarrow (\widehat{W},\widehat{L})/\widehat{B}\,.$$

Remark 3.3.3 [*on Definition 3.3.1*]. For the meaning/reason of the various conditions in Definition 3.3.1: [Liu (C): Lemma 6.13], which is generalized to Lemma 5.3.1.1 in Sec. 5.3.1, explains the role of *Maslov index* μ on infinitesimal deformations of an open stable map; [L-R: Lemma 3.11 (3)], [I-P1: Lemma 3.3], and [Li1: Proposition 2.2] give the reason to the important *pre-deformability condition*, as we want to single out maps that contribute to the degeneration formula; [I-P1: Sec. 6, Step 3] explains why morphisms of maps in question are defined so that the singular targets become *non-rigid* on the ruled-manifold-components from expansion, as it has to so that the choice of complex-scaling renormalizations in "stretching/pulling out" a degenerate component that falls into $W[k]_{\lambda\,,\,sing}$ becomes irrelevant. Furthermore, we will see in Sec. 5.3.5 that it is the *combination of all three* that renders the moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ "*virtually flat*" over B. Only so can one hope for a degeneration formula.

We now highlight three basic properties of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ in parallel to [L-R: Sec.3.3] and [I-P1: Theorem 7.4] (and to the existing literature quoted earlier on non-family case as well).

Hausdorffness.

Let $\mathcal{M}^{\text{rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ be the moduli space of isomorphism classes of stable maps to fibers of *rigid* $(W[k], L[k])/B[k]$ of type $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$. This is defined the same as in Definition 3.3.1 except that a morphism between $f_1 : \Sigma_1/pt \to W[k]/B[k]$ and $f_2 : \Sigma_2/pt \to W[k]/B[k]$ is taken to be an isomorphism $\alpha : \Sigma_1 \to \Sigma_2$ such that $f_1 = f_2 \circ \alpha$, and the stability condition for f to the rigid $W[k]/B[k]$ is that $Aut^{\,\text{rigid}}(f)$ is finite. Then [Sie1: proof of Proposition 3.8] (see also [F-O: Lemma 10.4]) can be applied to show that $\mathcal{M}^{\text{rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ is Hausdorff. This space is indeed a singular subspace of a manifold and hence is metrizable. The moduli space $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ is the quotient space of $\mathcal{M}^{\text{rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ by the $\mathbb{G}_m[k]$-action. Due to the stability condition, all the $\mathbb{G}_m[k]$-orbits on $\mathcal{M}^{\text{rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ have the same (real) dimension $2k$. This implies that $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$ is also Hausdorff, for $k \in \mathbb{Z}_{\geq 0}$.

Given now $[f]$, $[f'] \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$, assume, without loss of generality, that the image fiber of f (resp. f') is $Y_{[k]}$ or W_λ, $\lambda \neq 0$ (resp. $Y_{[k']}$ or $W_{\lambda'}$) with $k \geq k'$. Then $[f]$,

$[f'] \in \mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k] \,|\, [\beta], \vec{\gamma}, \mu)$. As $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((W[k], L[k])/B[k] \,|\, [\beta], \vec{\gamma}, \mu)$ embeds in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)/B$, this implies, by the way we define the C^∞-topology on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)/B$, that there are disjoint open subsets in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)/B$ that separate $[f]$ and $[f']$. It follows that:

Proposition 3.3.4 [Hausdorffness]. *$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)/B$ with the C^∞-topology is Hausdorff.*

This proposition can be regarded as a corollary of the stability condition on maps, in much the same reason as in geometric-invariant-theory quotients in algebraic geometry.

Finite stratification.

We first generalize a simplified version of the constructions/operations of [B-M: Sec. 1] to incorporate both the boundary of bordered Riemann surfaces and the consideration in [I-P1: Sec. 7]. This defines a category $\mathfrak{G}$ of graphs[15] whose objects label the topological types of stable maps to fibers of $(\widehat{W}, \widehat{L})/\widehat{B}$.

Definition 3.3.5 [weighted layered $(A_2 \to A_1)$-graph]. Let $A_2 \to A_1$ be a pair of abelian groups with a morphism. A *weighted layered $(A_2 \to A_1)$-graph* τ consists of the following data:

(1) (*graph with hands, bridges, legs, and fingers*) a *graph* τ, whose set of vertices, edges, legs, hands, bridges, and fingers are denoted by $V(\tau)$, $E(\tau)$, $L(\tau)$, $H(\tau)$, $B(\tau)$, and $F(\tau)$ respectively; among them the sets $H(\tau)$, $F(\tau)$, $L(\tau)$ are ordered, with $L(\tau)$ also bi-colored by $(\text{blue}, \text{red})$;

· the gluing of hands to vertices (resp. edges to vertices, bridges to hands, legs to vertices, fingers to hands) defines the *attaching map* $H(\tau) \to V(\tau)$ (resp. $E(\tau) \to \mathit{Sym}^2(V(\tau))$, $B(\tau) \to \mathit{Sym}^2(H(\tau))$, $L(\tau) \to V(\tau)$, $F(\tau) \to H(\tau)$, where $\mathit{Sym}^2(\,\cdot\,)$ is the symmetric product of $\cdot$); the attaching map $F(\tau) \to H(\tau)$, together with the ordering on the sets $H(\tau)$ and $F(\tau)$, groups elements of $F(\tau)$ into a tuple of tuples;

(2) (*layer structure*) a map $\mathit{layer} : V(\tau) \to \{\,0\,, \cdots, k+1\,\}$, for a $k \in \mathbb{Z}_{\geq 0}$, such that $\mathit{Im}\,(\mathit{layer})$ is either $\{0\}$, $\{k+1\}$, or the whole $\{\,0\,, \cdots, k+1\,\}$ and that, if v_1, $v_2 \in V(\tau)$ is connected by an edge $e \in E(\tau)$, then either $\mathit{layer}(v_1) = \mathit{layer}(v_2)$, in which case we call e an *ordinary edge*, or $|\mathit{layer}(v_1) - \mathit{layer}(v_2)| = 1$, in which case we call e a *distinguished edge*; the set of ordinary (resp. distinguished) edges is denoted by $E^o(\tau)$ (resp. $E^\dagger(\tau)$); by definition, $E(\tau) = E^o(\tau) \amalg E^\dagger(\tau)$;

· we require that a hand can be attached only to a vertex in $\mathit{layer}^{-1}(\{0\,, k+1\})$ and a red leg can be attached only to a vertex to which there is a hand attached;

[15] In this simplified presentation, we directly identify a (plain) graph with its geometric realization, i.e. a simplicial 1-complex consisting of a finite collection of points (i.e. *vertices*); a finite collection of (un-oriented) line segments (i.e. *edges*), with both ends attached to vertices; a finite collection of (un-oriented) line segments (i.e. *legs*, *hands* or *roots*) with only one end attached to vertices; a finite collection of (un-oriented) line segments (i.e. *bridges*) with both ends attached to free ends of hands, and a finite collection of (un-oriented) line segments (i.e. *fingers*) with only one end attached to free end of hands. We will denote a graph by τ (not to be confused with the involution τ in Definition 2.1). The full formal language in [B-M: Sec. 1] can be recovered whenever needed.

(3) (*weight functions*)

$$\begin{aligned} & g \,:\, V(\tau) \,\longrightarrow\, \mathbb{Z}_{\geq 0}\,; \\ & b : V(\tau) \to A_2\,,\ \gamma : H(\tau) \to A_1 \quad \text{such that the morphism } A_2 \to A_1 \\ & \qquad\qquad\qquad\qquad\qquad\quad \text{takes } \textstyle\sum_{v\in V(\tau)} b(v) \text{ to } \sum_{h\in H(\tau)} \gamma(h)\,; \\ & \mathit{ord} : E^\dagger(\tau) \to \mathbb{Z}_{\geq 1}\,; \end{aligned}$$

(4) an assignment $\tau \mapsto \mu(\tau) \in \mathbb{Z}$, called the *index* of τ.

An *isomorphism* $\alpha : \tau_1 \to \tau_2$ between two weighted layered $(A_2 \to A_1)$-graphs is an isotopy class of isomorphisms $\tau_1 \to \tau_2$ as a simplicial complex that induces isomorphisms of sets, ordered sets, or bi-colored ordered sets whichever applicable: $V(\tau_1) \xrightarrow{\sim} V(\tau_2)$, $H(\tau_1) \xrightarrow{\sim} H(\tau_2)$, $E(\tau_1) \xrightarrow{\sim} E(\tau_2)$, $B(\tau_1) \xrightarrow{\sim} B(\tau_2)$, $L(\tau_1) \xrightarrow{\sim} L(\tau_2)$, $F(\tau_1) \xrightarrow{\sim} F(\tau_2)$ and that preserves the layer $\mathit{layer}(\,\cdot\,)$, weights $g(\,\cdot\,)$, $b(\,\cdot\,)$, $\gamma(\,\cdot\,)$, $\mathit{ord}(\,\cdot\,)$, and the index $\mu(\,\cdot\,)$.

Denote by $\mathfrak{G}(A_2 \to A_1)$ (or simply $\mathfrak{G}$ when $A_2 \to A_1$ is understood) the category whose objects are weighted layered $(A_2 \to A_1)$-graphs and whose morphisms are given by isomorphisms.

Define the *core* τ^0 of a weighted layered $(A_2 \to A_1)$-graph to be the (weighted layered) subgraph of τ by removing the hands, bridges, legs, and fingers from τ. For a connected weighted layered $(A_2 \to A_1)$-graph τ, define the *genus* of τ to be

$$g(\tau) \;=\; 1 - \chi(\tau^0) + \sum_{v\in V(\tau)} g(v)$$

and the *b-weight* $b(\tau)$ of τ to be

$$b(\tau) \;=\; \sum_{v\in V(\tau)} b(v)\,.$$

For general τ, define its genus and b-weight by summing genus and b-weight over its connected components. Let τ_1, τ_2 be connected weighted layered $(A_2 \to A_1)$-graphs of the same index. A *contraction* from τ_1 to τ_2 is a homotopy class of surjective simplicial pseudo-maps $c : \tau_1 \to \tau_2$ such that

- the defining domain of c contains $\tau_1 - B(\tau_1)$;
- let $\mathit{layer} : V(\tau_1) \to \{0\,,\,\ldots\,,\,k_1+1\}$ and $\mathit{layer} : V(\tau_2) \to \{0\,,\,\ldots\,,\,k_2+1\}$ be the layer structure of τ_1 and τ_2 respectively; then $k_1 \geq k_2$ and there exists a non-decreasing map $I : \{0\,,\,\ldots\,,\,k_1{+}1\} \to \{0\,,\,\ldots\,,\,k_2{+}1\}$ such that $I \circ \mathit{layer}(v) = \mathit{layer}(c(v))$ for all $v \in V(\tau_1)$;
- c is a deformation retract on its defining domain; the induced maps from $H(\tau_1)$ to $H(\tau_2)$, $L(\tau_1)$ to $L(\tau_2)$, and $F(\tau_1)$ to $F(\tau_2)$ are bijective;
- let $v \in V(\tau_2)$, then $c^{-1}(v)$ is connected and $g(v) = g(c^{-1}(v))$, $b(v) = b(c^{-1}(v))$;
- if $e \in E^\dagger(\tau_1)$ is not mapped to a vertex of τ_2 then $c(e) \in E^\dagger(\tau_2)$ and $\mathit{ord}(e) = \mathit{ord}(c(e))$.

A (*red-to-blue*) *color change* $rb : \tau_1 \to \tau_2$ is a change of the color of some red legs to blue, leaving everything else the same. Both contractions and color-changes preserve g and b-weight of weighted layered $(A_2 \to A_1)$-graphs.

Associated to a point $[f : \Sigma/pt \to (\widehat{W}, \widehat{L})/\widehat{B}] \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta], \vec{\gamma}, \mu)$, with target isomorphic to $(Y_{[k]}, L)$, is a weighted layered graph $\tau_{[f]}$ via the following correspondence

$f:\Sigma\to(Y_{[k]},L)$	$(H_2(Y,L;\mathbb{Z})\stackrel{\partial}{\to}H_1(L;\mathbb{Z}))$-graph τ
irreducible component Σ_v of Σ	*vertex* $v\in V(\tau)$
labelled boundary component $(\partial\Sigma_v)_h$ of Σ_v (including *boundary node* q_h of *type E*)	*hand* $h\in H(\tau)$ attached to v
ordinary interior node q connecting Σ_{v_1}, Σ_{v_2}	*ordinary edge* e_q with ends attached to (v_1,v_2)
distinguished node q connecting Σ_{v_1}, Σ_{v_2}	*distinguished edge* e_q with ends attached to (v_1,v_2)
boundary node q of *type H* connecting $(\partial\Sigma)_{h_1}$ and $(\partial\Sigma)_{h_2}$	*bridge* b_q attached to the free ends of (h_1,h_2)
free marked point p on Σ_v	*leg* l_p attached to vertex v
interior marked point	*blue* leg
boundary free marked point	*red* leg
boundary marked point $p\in(\partial\Sigma)_h$	*finger* f_p attached to the free end of hand h
Σ_v such that $f(\Sigma_v)\subset\Delta_i$	*layer*$(v)=i$, $v\in V(\tau)$
$g(\Sigma_v)$	$g(v)$, $v\in V(\tau)$
$f_*[\Sigma_v]$	$b(v)$, $v\in V(\tau)$
$f_*[(\partial\Sigma_v)_h]$ if $\partial\Sigma_v\neq\emptyset$	$\gamma(h)$, $h\in H(\tau)$
Maslov index $\mu(f)$	$\mu(\tau)$
distinguished node q of *contact order* s	*ord*$(e_q)=s$, $e_q\in E^\dagger(\tau)$

where it is understood that, when $\Sigma_{v_1}=\Sigma_{v_2}$, $v_1=v_2$. It is clear that τ is defined to the isomorphism class $[f]$ of f. We call $\tau_{[f]}$ the *dual* (weighted layered) *graph* of f or $[f]$. Two stable maps $f_1:\Sigma_1/pt\to W[k_1]/B[k_1]$, $f_2:\Sigma_2/pt\to W[k_2]/B[k_2]$ are said to be *of the same topological type* if $\tau_{[f_1]}$ is isomorphic to $\tau_{[f_2]}$ in the category $\mathfrak{G}$. Degenerations of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$ are reflected contravariantly by compositions of contractions and color-changes of their dual graphs.

The following fundamental lemma on J-holomorphic maps to fibers of $(W[k],L[k])/B[k]$ is a consequence of [Ye: Lemma 4.1, Lemma 4.3, Lemma 4.5] and [I-P2: the explicit construction in Sec. 2], (see also [Gr2]; [MD-S1: Lemma 4.5.2], [F-O: Lemma 8.1], [Pa: Proposition 3.1.3], [P-W]; and [I-P1: Lemma 1.5], [L-R: Lemma 3.8 and Lemma 3.9]):

Lemma 3.3.6 [energy lower bound]. *One can fix Hermitian metrics on $W[k]$, $k\in\mathbb{Z}_{\geq 0}$, so that there exists a $\delta_0>0$ that depends only on (X,J,ω) such that, for all $k\in\mathbb{Z}_{\geq 0}$,*

- *any non-constant J-holomorphic map $f:\Sigma/pt\to(W[k],L[k])/B[k]$ has $E(f)\geq\delta_0$;*
- *for any sequence $f_i:\Sigma/pt\to(W[k],L[k])/B[k]$ of J-holomorphic maps on Σ and any blow-up point*[16] *$z\in\Sigma$,*

$$\lim_{r\to 0}\limsup_{i\to\infty}E(f|_{B_r(z)})\geq\delta_0\,.$$

The following lemma is parallel to [L-R: Lemma 3.15]. It follows from Lemma 3.3.6.

Lemma 3.3.7 [finite stratification]. *The classification of stable maps by their topological types gives rise to a finite stratification of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$, with each stratum S_τ labelled by a weighted layered $(H_2(Y,L;\mathbb{Z}),H_1(L;\mathbb{Z}))$-graph $\tau\in\mathfrak{G}$.*

Compactness.

[16] This is a point on Σ to which a positive energy of f_i condenses/accumulates in the limit. This is where a bubbling occurs. See, e.g. [MD-S1: Lemma 4.5.5], [P-W], [Ye: Sec. 4], and [L-R: Sec. 3.2].

The following fundamental compactness result of Gromov-Witten theory in the current contents is closely related to [L-R: Theorem 3.16, Corollary 3.17, Theorem 3.20, Theorem 3.21], and [I-P1: Theorem 7.4]. It follows from Lemma 3.3.6, Lemma 3.3.7, the compactness technique/results in [Ye], the compactness techniques/results in [L-R: Sec. 3.2], and the compactness technique/results in [I-P1, particularly Sec. 6, Step 3] and [I-P2], as the effect around the boundary of domains that is mapped to L is taken care of in [Ye], the effect for degeneration of domains due around the degeneration of the neck regions of targets is taken care of in [I-P1], and these two regions are disjoint from each other in our situation. See also [Gr2], [F-O], [I-S1], [Pa], [P-W], and [R-T1] for the non-family case.

Theorem 3.3.8 [compactness/B]. *The moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$ of the specified type, with the C^∞-topology, is compact over a compact subset of B.*

Remark 3.3.9 [*finiteness of curve classes in* $[\beta]$]. It should be noted that, while $[\beta]$ corresponds to an element in $H_2(Y, L:\mathbb{Z})$ under the map $\xi_*: H_2(X,L;\mathbb{Z}) \to H_2(Y,L;\mathbb{Z})$ from the symplectic cut $\xi: X \to Y$, the $(W, B\times L)/B$-monodromy orbit $[\beta]$ could be an infinite subset in $H_2(X,L;\mathbb{Z})$. However, only finitely many $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(X, L\,|\,\beta',\vec{\gamma},\mu)$, $\beta' \in [\beta]$, can be non-empty since all the related stable maps to X of the specified type have the same energy and one has the compactness result of [Ye] in this case.

Before leaving this section, we should mention that the moduli problem of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$ has non-trivial obstructions. The space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ is very singular in general. The construction of a family Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,$ $L\,|\,[\beta],\vec{\gamma},\mu)/B$, to be done in Sec. 5.3 and Sec. 5.4, is meant to accommodate such singularities due to obstructions.

4 The moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ of stable $\check{W}^{1,p}$-maps.

In this section we introduce the moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ of $\check{W}^{1,p}$-maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$. This space fibers over B and is locally embeddable into a Banach orbifold-with-corners; it contains $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ as a finite dimensional, compact-over-B, singular sub-orbifold-with-corners. Members of the system of Kuranishi neighborhoods for $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$, to be constructed in Sec. 5.3, are embedded in the local singular-orbifold-charts of $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)/B$ as finite dimensional, locally closed, algebraic-type subsets that are flat over B. The related Banach relative tangent-space fibration $T^1_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/(\widetilde{\mathcal{M}}_\bullet\times\widehat{B})}$ and the related Banach relative obstruction-space fibration $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/(\widetilde{\mathcal{M}}_\bullet\times\widehat{B})}$ on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ over $\widetilde{\mathcal{M}}_{(g,h),(n,\vec{m})}\times\widehat{B}$ and their flattening stratification are also given.

The foundation (Sec. 4.1) of the construction of these moduli spaces and fibrations are in [Sie1: Sec. 4 - Sec. 6]; see also [Ru] and [L-R]. These spaces will be used to study the gluability and the gluing of family Kuranishi neighborhoods in Sec. 5.4.

We assume throughout the work that $2 < p < \infty$ to ensure the continuity of $W^{1,p}$- and $\check{W}^{1,p}$-maps (to be defined below) on a bordered Riemann surface.

4.1 The moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ of stable $\check{W}^{1,p}$-maps to $(W[k],L[k])$, its relative tangent and relative obstruction bundles.

For a labelled-bordered Riemann surface Σ with marked points with a Kähler metric so that all the boundary components are geodesics, let U_ε be the ε-neighborhood of the set of nodes on Σ with respect to the metric. Consider a measure μ on Σ defined as follows.

- on $\Sigma - U_\varepsilon$, μ coincides with the area-form associated to the metric;
- around the $\varepsilon/2$-neighborhood of a node with local polar coordinates (r,θ), $\mu = drd\theta$, where r is the distance function to the node and θ parameterizes the angular direction; θ runs over $[0,2\pi]$ for an interior node or a boundary node of type E and over the disjoint union of two finite closed intervals for a boundary node of type H;
- on $U_\varepsilon - U_{\varepsilon/2}$, μ is realized as a non-degenerate 2-form that interpolates smoothly the above two 2-forms.

Define the $\check{L}^p$- (resp. $W^{k,p}$)-norm for a function f on Σ to be the L^p- (resp. $W^{k,p}$-)norm of f with respect to μ:

$$\|f\|_{\check{L}} \;=\; \left(\int_\Sigma |f|^p\,\mu\right)^{1/p}, \qquad \|f\|_{\check{W}^{k,p}} \;=\; \left(\int_\Sigma \sum_{|\mu|\le k} |\partial^\nu f|^p\,\mu\right)^{1/p}.$$

The completion of $C^\infty(\Sigma)$ with respect to the norm $\|\cdot\|_{\check{L}^p}$ and $\|\cdot\|_{\check{W}^{k,p}}$ is denoted by $\check{L}(\Sigma)$ and $\check{W}^{k,p}(\Sigma)$ respectively. For $2<p<\infty$,

$$C^\infty(\Sigma)\subset \check{W}^{1,p}(\Sigma)\subset W^{1,p}(\Sigma)\subset C^0(\Sigma)\,.$$

The notion of bounded $\check{L}$- or $\check{W}^{k,p}$-norm depends only on the complex structure on Σ, not the Kähler metric, ε, or the smooth interpolation on $U_\varepsilon - U_{\varepsilon/2}$. In particular, though such measure μ on Σ is not invariant under $Aut(\Sigma)$ in general, the notion of functions of bounded $\check{L}^p$- or $\check{W}^{1,p}$-norm on Σ is invariant $Aut(\Sigma)$. The notion generalizes to maps to manifolds or sections of a bundle. The choice of such Sobolev sections makes the local trivialization over a base S of the space of Sobolev sections for vector bundles of a family $\mathcal{C}_S/S$ of prestable labelled-bordered Riemann surfaces with marked points over S that occurs in our problem possible; see [Sie1: Sec. 4] for the technical details, which can be generalized to our case.

The symplectic cut $\xi: X\to Y$ extends to a strong deformation retract $r: W/B\to Y/\{0\}$ such that the restriction $r_\lambda: W_\lambda\to Y$ is also a symplectic cut. The post-composition of r with $\tilde{\mathbf{p}}[k]: W[k]/B[k]\to W/B$ defines a map $\mathbf{r}[k]: W[k]/B[k]\to Y/pt$.

Definition 4.1.1 [stable $\check{W}^{1,p}$-map to $(W[k],L[k])$]. A $\check{W}^{1,p}$-map $h:(\Sigma,\partial\Sigma)\to(W[k],L[k])$ is said to be of (*combinatorial*) *type* $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ if Σ is a labelled-bordered Riemann surface of type $((g,h),(n,\vec{m}))$, $\mathbf{r}[k]_*h_*([\Sigma]) = \xi_*([\beta])$, $\mathbf{r}[k]_*(h_*[\dot{\partial}\Sigma]) = \vec{\gamma}$, and the relative homotopy class of $\mathbf{r}[k]\circ h$ contains a map of Maslov index μ. h is called *stable* if the restriction of h to each unstable component of Σ is non-constant.

An *isomorphism* from $h_1:\Sigma_1\to W[k]$ to $h_2:\Sigma_2\to W[k]$ is an isomorphism $\alpha:\Sigma_1\to\Sigma_2$ such that $h_1 = h_2\circ\alpha$. The isomorphism class of h is denoted by $[h]$. The moduli space of isomorphisms classes of stable $\check{W}^{1,p}$-maps to $(W[k],L[k])$ of type $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ is denoted by $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$.

Note that the stability condition implies that $Aut(h)$ is finite for a stable $\check{W}^{1,p}$-map h.

As $2 < p < \infty$, a $\check{W}^{1,p}$-map is continuous and one can define the C^0-*topology* on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ by defining the $(\varepsilon_1, \varepsilon_2)$-neighborhood $U_{\varepsilon_1,\varepsilon_2}([h])$ of $[h]$ to consist of all $[h' : (\Sigma', \partial\Sigma') \to (W[k], L[k])] \in \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ such that there exists a surjective *collapsing/pinching map* $c : \Sigma' \to \Sigma$ that is a diffeomorphism from the complement of a collection of simple loops and simple arc with ends on $\partial\Sigma'$ on Σ' to the complement of the set of nodes on Σ', and collapses/pinches each simple loop (resp. arc) in the collection to an interior (resp. boundary) node of Σ so that

· (nearness of domain)	Σ' is isomorphic to a fiber of $\mathcal{C}/Def(\Sigma)$ with $\|j - c_* j\|_{C^\infty} < \varepsilon_2$ on $\Sigma - U_{\varepsilon_1}$ and $c(p'_\cdot)$ in the ε_2-neighborhood of $p_\cdot$, where $p_\cdot$, $p'_\cdot$ are marked points on Σ, Σ' that are paired by their label;
· (nearness of map)	$\|h - h' \circ c^{-1}\|_{C^0} < \varepsilon_2$ on $\Sigma_{\rm reg}$.

Here, U_{ε_1} is the ε_1-neighborhood of the set of nodes of Σ that is small enough so that it contains no marked points. This topology is equivalent to the L^∞-topology, [Sie1: Proposition 5.3].

A *Banach space-with-corners* is the direct product of a Banach space and a polyhedral cone at the origin in a finite-dimensional (real) vector space. A *Banach orbifold-with-corners* is an orbifold locally modelled on a finite quotient of a neighborhood of the origin of a Banach space-with-corners. The same techniques for the proof of [Sie1: Proposition 3.8 and Theorem 5.1] can be applied to prove the following theorem:

Theorem 4.1.2 $[\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)]$**.** *The C^0-topology on the moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ is Hausdorff. There exists a refinement of the C^0-topology on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ so that it becomes a (Hausdorff) Banach orbifold-with-corners.*

We shall call the refined topology in the above theorem the $\check{W}^{1,p}$-*topology* on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$. With this topology, a local orbifold-chart of $[h]$ is modelled on the quotient of the Banach space-with-corners from a rigidifying slice $V''_{[h]}$ to the approximate pseudo-$Aut(\Sigma)$-action on $Def(\Sigma) \times W^{1,p}(\Sigma, \partial\Sigma; h^*T_*W[k], (h|_\partial)^*T_*L[k])$ by $Aut(h)$. Here $W^{1,p}(\Sigma, \partial\Sigma; h^*T_*W[k], (h|_{\partial\Sigma})^*T_*L[k])$ is the Banach space of $\check{W}^{1,p}$-sections s of the vector bundle $h^*T_*W[k]$ on Σ with $s|_{\partial\Sigma}$ taking values in $(h|_{\partial\Sigma})^*T_*L[k]$. We call $(V''_{[h]}, \Gamma_{[h]} := Aut([h]))$, a *Banach orbifold-with-corners chart* of $[h]$ in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$. (We will call $[h]$ the center of the chart for convenience.)

The system of the equivariant relative tangent bundle of the local Banach orbifold-with-corners charts over the deformation space of the domain curve in the center glue to a Banach orbi-bundle $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$, whose fiber at $[h : (\Sigma, \partial\Sigma) \to (W[k], L[k])]$ is given by the Banach $Aut(h)$-space $\check{W}^{1,p}(\Sigma, \partial\Sigma; h^*T_*W[k], (h|_{\partial\Sigma})^*T_*(L[k]))$. We call this orbi-bundle the *relative tangent bundle* of $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ over $\widetilde{M}_{(g,h),(n,\vec{m})}$.

The same construction in [Sie1: Sec. 6.1] gives a Banach orbi-bundle $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$, whose fiber at $[h : (\Sigma, \partial\Sigma) \to (W[k], L[k])]$ is given by the Banach $Aut(h)$-space $\check{L}^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J h^*T_*W[k])$ of $\check{L}^p$-sections of $\Lambda^{0,1}\Sigma \otimes_J h^*T_*W[k]$. We call this orbi-bundle the *relative obstruction bundle* of $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu)$ over $\widetilde{M}_{(g,h),(n,\vec{m})}$.

The nonlinear Cauchy-Riemann operator $h \mapsto \bar{\partial}_J h := \frac{1}{2}(dh + J \circ dh \circ j)$ defines a section (in the sense of orbi-bundle)

$$s_{\bar{\partial}_J} \;:\; \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k], L[k] \,|\, [\beta], \vec{\gamma}, \mu) \;\longrightarrow\; T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$$

of the relative obstruction bundle $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$.

A connection ∇ on $T_*W[k]$ induces an *partial connection* on the orbi-bundle $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$, using the parallel transport on $T_*W[k]$ associated to ∇. Denote this ∇-induced partial connection on $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ also by ∇; then its associated horizontal distribution H^∇ at a point $([h],\eta)$ over $[h]$ projects isomorphically to the relative tangent space $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet\,,\,[h]}$ of $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ over $\widetilde{M}_{(g,h),(n,\vec{m})}$ at $[h]$. One thus has a well-defined *vertical projection* π^v to the tangent space $T_{([h],\eta)}\,T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet\,,\,[h]}$ for a tangent vector at $([h],\eta)$ that projects into $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet\,,\,[h]}$. Together with the vector space translations on fibers of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$, the composition $\pi^v\circ d\,s_{\bar\partial_J}$ defines an orbi-bundle homomorphism

$$D\bar\partial_J\ :\ T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\longrightarrow\ T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,.$$

We shall call $D\bar\partial_J$ the ∇-*induced linearization* of the nonlinear Cauchy-Riemann operator $\bar\partial_J$. The expression for $D\bar\partial_J$ can be computed explicitly. See, e.g., [MD-S1: Eq. (3.2), Remark 3.3.1], [Liu(C): Proposition 6.12], and [Sie1: Sec. 6.3]. Note that, by definition, the J-holomorphy locus in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ is sent by $s_{\bar\partial_J}$ to the image of the zero-section of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$; the linearization $D_h\bar\partial_J$ for h J-holomorphic is thus independent of ∇.

Finally, we remark that, as $L[k]$ is a coisotropic submanifold that contains properly a symplectic submanifold (e.g. $Re\,(B[k])\times L$) in $W[k]$, the restriction of the orbi-bundle homomorphism $D\bar\partial_J$ to each fiber is not Fredholm. Instead, $D\bar\partial_J$, when restricted to fibers, has a finite-dimensional cokernel but an infinite-dimensional kernel in general.

4.2 The moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ of stable $\check{W}^{1,p}$-maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$, the relative $\check{W}^{1,p}$-tangent-obstruction fibration complex.

Definition 4.2.1 [stable $\check{W}^{1,p}$-map to $(W[k],L[k])/B[k]$]. A $\check{W}^{1,p}$-map $h:(\Sigma,\partial\Sigma)/pt\to(W[k],L[k])/B[k]$ from a bordered Riemann surface Σ to a fiber of $(W[k],L[k])/B[k]$ is called *prestable* of (*combinatorial*) *type* $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ if h is a stable $\check{W}^{1,p}$-map from $(\Sigma,\partial\Sigma)$ to $(W[k],L[k])$ of type $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ such that the image of h lies in a fiber of $(W[k],L[k])/B[k]$. An *isomorphism* between two prestable $\check{W}^{1,p}$-maps $h_1:\Sigma_1/pt\to W[k]/B[k]$, $h_2:\Sigma_2/pt\to W[k]/B[k]$ of the same type is a pair (α,β) where $\alpha:\Sigma_1\to\Sigma_2$ is an isomorphism of prestable labelled-bordered Riemann surfaces with marked points and $\beta\in\mathbb{G}_m[k]$ acts on $W[k]/B[k]$ as in Sec. 1.1.3 such that $f_1\circ\beta=f_2\circ\alpha$. The isomorphism class associated to a prestable $\check{W}^{1,p}$-map h will be denoted by $[h]$. The *group of automorphisms* $Aut\,(h)$ of a prestable $h:\Sigma/pt\to W[k]/B[k]$ consists of elements $(\alpha,\beta)\in Aut(\Sigma)\times\mathbb{G}_m[k]$ such that $\beta\circ h=h\circ\alpha$.

A prestable $\check{W}^{1,p}$-map $h:\Sigma/pt\to(W[k].L[k])/B[k]$ is called *stable* if $Aut(h)$ is finite. The moduli space of isomorphism classes of stable $\check{W}^{1,p}$-maps to fibers of $(W[k],L[k])/B[k]$ of type $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ is denoted by $\mathcal{W}^{1,p}_{(g,h),(n,\vec{m})}((W[k],L[k])/B[k]\,|\,[\beta],\vec{\gamma},\mu)$.

Once having the notion of stable $\check{W}^{1,p}$-maps to the fibers of $(W[k],L[k])/B[k]$, one can apply the same procedure/routine of gluings as in Sec. 3.3 to define/obtain the moduli space $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ of (isomorphism classes of) stable $\check{W}^{1,p}$-maps to the fibers of $(\widehat{W},\widehat{L})/\widehat{B}$.

The $\check{W}^{1,p}$-topology and the singular orbifold-with-corners structure on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$.

Let $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}$ be the singular (constructible) sub-orbifold-with-corners of the Banach orbifold-with-corners $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ whose system of local singular orbifold-with-corners charts consists of

$$\left\{ (V',\Gamma_{V'}) \;\middle|\; \begin{array}{l} \text{There exists a Banach orbifold-with-corners local chart } (V'',\Gamma_{V''}) \text{ of} \\ \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu) \text{ such that} \\ \quad \cdot\ V' \text{ is the subset of } V'' \text{ parameterizing all those } \check{W}^{1,p}\text{-maps to} \\ \quad\ (W[k],L[k]) \text{ parameterized by } V'' \text{ whose image lies completely} \\ \quad\ \text{in a fiber of } (W[k],L[k])/B[k] \text{ and which are stable in the sense} \\ \quad\ \text{of Definition 4.2.1;} \\ \quad \cdot\ \Gamma_{V'}=\Gamma_{V''}\,. \end{array} \right\}.$$

Note that V' is a locally closed subset of the corresponding V''. The gluing of the system of local charts $\{(V',\Gamma_{V'})\}_\bullet$ for $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}$ follows from the restriction of the gluing of the subsystem $\{(V'',\Gamma_{V''})\}_\bullet$ of charts for $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$. The natural map from $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}$ to $B[k]$ defines the notation $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}/B[k]$. A singular orbifold-with-corners structure on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ can be obtained by gluing a system of singular charts from further orbifolding appropriate subsets of the singular local charts of $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}$, $k\in\mathbb{Z}_{\geq 0}$, as follows.

Let $\rho\in\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ be represented by $h:(\Sigma,\partial\Sigma)/pt\to(Y_{[k]},L_{[k]})/\{0\}\subset(W[k],L[k])/B[k]$. (The case the target is a smooth W_λ, $\lambda\neq 0$, is immediate and will be omitted.) For our final purpose of studying $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$, we will assume that the image of h has non-empty intersection with each irreducible component of $Y_{[k]}$. The following discussion can be adapted to the situation when this is not the case as well. As an element in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}$, let $(V',\Gamma_{V'})$ be in the form of a singular local chart-with-corners $(V'_h,Aut(h)^{\,\mathrm{rigid}})$ centered at h. The equivariant pseudo-$\mathbb{G}_m[k]$-action on $W[k]/B[k]$ induces an equivariant pseudo-$\mathbb{G}_m[k]$-action on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}/B[k]$ via post-composition with maps. Locally this is a (pseudo) $\mathbb{G}_m[k]$-action on the singular local chart-with-corners V' that commutes with the $\Gamma_{V'}$-action on V'. A $\Gamma_{V'}$-invariant slice V_h through h in V' to rigidify this $\mathbb{G}_m[k]$-action can be constructed as follows.

Let $\mathcal{C}'/V'$ be the universal bordered Riemann surface with marked points over V' and $F':\mathcal{C}'/V'\to W[k]/B[k]$ be the universal map. Both are built-in from the construction of Siebert. Let $U_\Sigma\subset\Sigma$ be an $Aut(h)^{\,\mathrm{rigid}}$-invariant sub-surface of Σ by removing an appropriate small neighborhood of all the nodes of Σ. Then, for V' small enough, the $Aut(h)^{\,\mathrm{rigid}}$-equivariant embedding $U_\Sigma\hookrightarrow\Sigma$ extends to an $Aut(h)^{\,\mathrm{rigid}}$-equivariant embedding $V'\times U_\Sigma\hookrightarrow\mathcal{C}'$ over V'. This implies that there exist global sections s_i, $i=1,\ldots,k$, of $\mathcal{C}'\to V'$ such that

- α^*s_i, $\alpha\in Aut(h)^{\,\mathrm{rigid}}$, takes values in the image of $V'\times U_\Sigma$;
- the image of $F'\circ\alpha^*s_i$, $\alpha\in Aut(h)^{\,\mathrm{rigid}}$, lies in a neighborhood $U_i[k]$ of $h\circ s_i(0)$ in $Trunk\,[k]_i\simeq B[k]\times(\Delta_i-N_\varepsilon(D_{i-1}\cup D_i))$ of $W[k]$, cf. Remark 1.1.1.6;

· the finite set $\left\{ \pi_{2,i} \circ F \circ \alpha^* s_i(0) : i = 1, \ldots, k;\, \alpha \in Aut(h)^{\,\mathrm{rigid}} \right\}$ lies in $\mathbb{C} - \mathbb{R}_{\leq 0}$, where $\pi_{2,i} : U_i[k] \to \mathbb{C} - \{0\}$ is the projection map to the fiber of $\mathbb{L}$ from a local trivialization of $\mathbb{L}$, as an embedded submanifold in Δ_i, $i = 1, \ldots, k$, cf. Sec. 1.1.1.

Define the average function *Average* for a finite subset S in $\mathbb{C} - \mathbb{R}_{\leq 0}$ by

$$Average\,(S) \;=\; e^{\frac{1}{|S|}\sum_{w\in S}\left(\log(|w|)+\sqrt{-1}\arg(w)\right)}\,,$$

where $\arg(w) \in (-\pi, \pi)$. Let

$$\bar{s}_i \;:=\; Average\left(\pi_{2,i} \circ F \circ \alpha^* s_i \,:\, \alpha \in Aut(h)^{\,\mathrm{rigid}}\right).$$

For V' small enough, this is a well-defined $Aut(h)^{\,\mathrm{rigid}}$-invariant function on V' for $i = 1, \ldots, k$, with values in $\mathbb{C} - \mathbb{R}_{\leq 0}$. The k-tuple

$$R \;:=\; (\bar{s}_1, \ldots, \bar{s}_k) \;:\; V' \;\longrightarrow\; (\mathbb{C} - \{0\})^k$$

defines thus an $Aut(h)^{\,\mathrm{rigid}}$-invariant $\mathbb{G}_m[k]$-equivariant map, where $\mathbb{G}_m[k]$ acts on $(\mathbb{C} - \{0\})^k$ by $(w_1, , \ldots, w_k) \mapsto (\sigma_1 w_1, \ldots, \sigma_k w_k)$, $(\sigma_1, \ldots, \sigma_k) \in \mathbb{G}_m[k] = (\mathbb{C}^\times)^k$. Let $V_h = R^{-1}(R(0))$. Then $V_h \subset V'$ is a rigidifying slice through h to the $\mathbb{G}_m[k]$-action on V' and is invariant under the $\Gamma_{V'}$-action.

By construction, the residual discrete subgroup Γ_{V_h} of $Aut(\Sigma) \times \mathbb{G}_m[k]$ that pseudo-acts on V_h is an extension of $Aut(h)^{\,\mathrm{rigid}}$ by a discrete subgroup of $\mathbb{G}_m[k]$ whose elements fix $[h]$ when they descend to pseudo-act on the quotient space $V_h/Aut(h)^{\,\mathrm{rigid}}$. In other words, $(\alpha, \beta) \in \Gamma_{V_h}$ if and only if $\beta \circ h = h \circ \alpha$. By shrinking V_h if necessary, one can render the pseudo Γ_{V_h}-action to an honest group action. This shows that indeed $\Gamma_{V_h} = Aut(h)$. Stability of h says that Γ_{V_h} is finite. Thus, (V_h, Γ_{V_h}) defines a singular orbifold local chart-with-corners at $\rho = [h] \in \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$. Re-write h above as h_ρ to manifest its representing ρ and denote the map $V_{h_\rho} \to \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$ that identifies $V_{h_\rho}/Aut(h_\rho)$ with a neighborhood of ρ in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$ by ψ_ρ, then a system of singular local charts-with-corners on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$ is given by $\{(V_{h_\rho}, Aut(h_\rho), \psi_\rho)\}_\rho$. We will identify each $V_{h_\rho}/Aut(h_\rho)$ directly as a subset in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$.

We next construct the transition data for the local charts. Given a pair (p, q) with $p \in \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$ and $q \in V_{h_p}/\Gamma_{V_{h_p}} \subset \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$, there is a $\Gamma_{V_{h_q}}$-invariant neighborhood V_{qp} of h_q in V_{h_q} such that $\psi_q(V_{qp}) \subset V_{h_p}/\Gamma_{h_p}$. The set of embeddings $\{h_q\} \hookrightarrow V_p$ is parameterized by a Γ_{h_p}-orbit in V_{h_p}. Fixing a such embedding determines an embedding $h_{qp} : V_{qp} \to V_{h_p}$ up to a pre-composition with the Γ_{h_q}-action on V_{qp}. The map h_{qp} determines then an embedding $\phi_{qp} : \Gamma_{V_{h_q}} \to \Gamma_{V_{h_p}}$. The orbifold cocycle condition (cf. Definition 5.1.2 (2)) for a triple (p, q, r) with (p, q) as above and $r \in V_q/\Gamma_{h_q}$ follows immediately. Thus, the system $\{(V_{qp}, h_{qp}, \phi_{qp})\}_{(p,q)}$ gives a required orbifold transition data.

The two systems $\{(V_{h_\rho}, Aut(h_\rho), \psi_\rho)\}_\rho$ and $\{(V_{qp}, h_{qp}, \phi_{qp})\}_{(p,q)}$ together give a singular orbifold-with-corners structure on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$. The induces topology on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$ from these charts will be called the $\check{W}^{1,p}$-*topology* on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}($ $(\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$. Theorem 4.1.2 together with the detail above implies:

Proposition 4.2.2 [Hausdorffness]. $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W}, \widehat{L})/\widehat{B}\,|\,[\beta], \vec{\gamma}, \mu)$ *with the* $\check{W}^{1,p}$*-topology is Hausdorff.*

Note that there is a natural morphism (as topological spaces with a system of local charts and gluing data) from $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ to $\tilde{\mathcal{M}}_{(g,h),(n,\vec{m})}\times\widehat{B}$ that forgets the map, keeping only the domain and the target in a stable-map data. It is with respect to this morphism that we denote $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)/(\tilde{\mathcal{M}}_{(g,h),(n,\vec{m})}\times\widehat{B})$.

The relative $\check{W}^{1,p}$-tangent-obstruction fibration complex on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$.

The total space of the Banach orbi-bundle $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ is itself a Banach orbifold-with-corners. The system of local trivializations of $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ over the system[17] $\{(V'',\Gamma'')\}_\bullet$ of local charts on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ provides the Banach orbifold-with-corners charts for $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ with the system of gluing data. After a refinement if necessary, we may assume that all V'' are small enough, so that the collection

$$\left\{\left(\,T^1_{V''}:=T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,\big|_{V''}\,,\,\Gamma_{V''}\right)\right\}_\bullet$$

gives the Banach-orbifold-with-corners local charts for $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$.

Let $\{(V,\Gamma_V)\}_\bullet$ be a (fine enough) system of local charts on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ as constructed in the previous theme, with each V admitting

$$V\;\subset\;V'\;\subset\;V''\,,$$

where, recall that, $(V',\Gamma_{V'})$ is a local chart on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)^{W[k]/B[k]}$, and $(V'',\Gamma_{V''})$ is a local chart on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$, for some $k\in\mathbb{Z}_{\geq 0}$ depending on V. Consider the fiberwise-closed singular (constructible) subset T^1_V of $T^1_{V''}$ defined by

$$T^1_V\;:=\;\left\{\begin{array}{l}([h:(\Sigma,\partial\Sigma)\to W[k]/B[k]],\xi)\;\in\;T^1_{V''}|_V\\ \qquad\qquad :\;\xi\in\check{W}^{1,p}(\Sigma,\partial\Sigma;h^*T_{W[k]/B[k]},(h|_{\partial\Sigma})^*T_*L)\end{array}\right\},$$

where $\check{W}^{1,p}(\Sigma,\partial\Sigma;h^*T_{W[k]/B[k]},(h|_{\partial\Sigma})^*T_*L)$ is the closed Banach subspace of $\check{W}^{1,p}(\Sigma,\partial\Sigma;h^*T_*W[k],(h|_{\partial\Sigma})^*T_*L[k])$ that consists of $\check{W}^{1,p}$-sections of $(h^*T_*W[k],(h|_{\partial\Sigma})^*T_*L[k])$ that are projected to 0 under $\pi[k]_*:T_*W[k]\to T_*B[k]$. Then the Γ_V-action on V canonically lifts to an action on T^1_V. The gluing data of the system $\{T^1_{V''}\}_\bullet$ extends to the lifting the gluing data on the system $\{(V,\Gamma_V)\}_\bullet$ to on $\{(T^1_V,\Gamma_V)\}_\bullet$. This gives rise to a *singular orbifold-with-corners* $T^1_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}|\bullet)/\widetilde{M}_\bullet}$. The system of maps $\{(T^1_V\;\to\;V)\}_\bullet$ descends to a morphism of orbifolds

$$T^1_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\;\longrightarrow\;\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)\,,$$

whose fiber at ρ, represented by $h:\Sigma\to Y_{[k]}$, is given by the Banach $Aut(\rho)$-space $\check{W}^{1,p}(\Sigma,\partial\Sigma;h^*T_*Y_{[k]},(h|_{\partial\Sigma})^*T_*L)$ (:= is the closed Banach subspace of $\check{W}^{1,p}(\Sigma,\partial\Sigma;h^*T_*W[k],(h|_{\partial\Sigma})^*T_*L[k])$ that consists of $\check{W}^{1,p}$-sections of $(h^*T_*W[k],(h|_{\partial\Sigma})^*T_*L[k])$ that are projected to 0 under $\pi[k]_*:T_*W[k]\to T_*B[k]$).

The same restrict-and-descend construction applied to the collection of orbi-bundles:

$$\left\{\;T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\;\text{over}\;\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)\;\right\}_{k\in\mathbb{Z}_{\geq 0}}$$

[17] The notation $\{\cdots\}_\bullet$ to indicate a system of objects of the form $\cdots$ will be used in many places of the work.

gives rise to the singular orbifold-with-corners $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)\widetilde{\mathcal{M}}_\bullet}$ with a built-in orbifold morphism

$$T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet} \longrightarrow \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)\,,$$

whose fiber at ρ, represented by $h:(\Sigma,\partial\Sigma)\to(Y_{[k]},L)$, is given by the Banach $Aut(\rho)$-space $\check{L}^p(\Sigma,\partial\Sigma;\Lambda^{0,1}\Sigma\otimes_J h^*T_*Y_{[k]})$.

Let $\widehat{B}=(B-\{0\})\amalg\mathbb{Z}_{\geq 0}$ be the stratification of $\widehat{B}$ by the homeomorphism type of the fibers of $\widehat{W}/\widehat{B}$ (with the stratum $B-\{0\}$ labelled by -1). It induces a stratification $\{\mathcal{S}_k\}_{k\in\mathbb{Z}_{\geq -1}}$ on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu)$ by taking the preimage under the forgetful morphism

$$\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu) \longrightarrow \widehat{B}$$

of each stratum in $\widehat{B}$. The restriction of $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ and $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ to over $\mathcal{S}_i$ are orbi-bundles on $\mathcal{S}_i$. We say that $\{\mathcal{S}_i\}_{i\in\mathbb{Z}_{\geq -1}}$ gives a common *flattening stratification* for both $T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ and $T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$, as for a coherent sheaf in algebraic geometry[18].

The system of sections

$$\Big\{s_{\bar\partial_J}\,:\,\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}(W[k],L[k]\,|\,[\beta],\vec{\gamma},\mu) \longrightarrow T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\Big\}_{k\in\mathbb{Z}_{\geq 0}}$$

restricts and descends to a section (as a morphism of orbifolds)

$$s_{\bar\partial_J}:\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu) \longrightarrow T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$$

of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$. In contrast, as the connection ∇ on $W[k]$ that defines the orbi-bundle homomorphism

$$D\bar\partial_J\,:\,T^1_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet} \longrightarrow T^2_{\check{\mathcal{W}}^{1,p}_\bullet(W[k],L[k]\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,.$$

is not $\mathbb{G}_m[k]$-invariant, the system of these linearizations does not restrict and descend to a linearization of $\bar\partial_J$ from $T^1_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ to $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$. However, as the restriction of the linearization $D\bar\partial_J$ over the J-holomorphy locus

$$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu) \subset \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$$

is independent of ∇, one does have a morphism as an orbifold map between fibered orbifolds:

$$T^1_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,\big|\,\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet) \xrightarrow{D\bar\partial_J} T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,\big|\,\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet)\,.$$

We will call this the *relative $\check{W}^{1,p}$-tangent-obstruction fibration complex* on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$.

This concludes our discussion for these auxiliary ∞-dimensional Banach-type orbifolds. To give an orientation for next, we remark that to go from these spaces to a finite-dimensional object that serves as local charts for $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$ in a generalized sense and is flat over B, there are three transversality issues one has to deal with:

[18] In the algebro-geometric setting, the parallel to the various $T^1_\bullet$ and $T^2_\bullet$ here will be constructed as a coherent sheaf on a Deligne-Mumford moduli stack from the deformation-obstruction theory of the moduli problem in question. See, e.g. [L-T1: Sec. 1] and [Li2: Sec. 1.2, Sec. 1.3]. In the analytic category that involves Banach orbifolds, it is simpler to construct directly the associated total space, which are themselves (singular) orbifolds, of the would-be sheaves rather than to construct these sheaves.

· transversality of the operator $\bar{\partial}_J$, or equivalently the section $s_{\bar{\partial}_J}$;

· transversality of matching conditions at distinguished nodes;

· transversality/S of the pre-deformability condition at distinguished nodes.

The construction of a Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ is guided by the attempt to achieve all three transversality conditions simultaneously and in a way that is flat over B. This can be realized by a modification of $\bar{\partial}_J$ and the take of a system of finite-dimensional orbifold-structure-group-invariant subsets in the local charts of these auxiliary orbifolds.

5 Construction of a family Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ over B.

In this rather long section, we construct a family Kuranishi structure over B for the moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ of stable maps to fibers of $(\widehat{W},\widehat{L})/\widehat{B}$. This answers incidentally the simplest case, namely the degeneration from a symplectic cut, of the question posed in [F-O: p. 962] on the family version of Kuranishi structure for a degeneration. The detail merges [F-O], [Liu(C)] with [I-P2], [L-R], [Li1]; and the result gives an almost-complex/analytic/symplectic parallel to the algebraic [Li1] and [Li2] when curves are closed.

5.1 Family Kuranishi structure modelled in the category $\mathcal{C}_{\text{spsccw}}/\mathbb{C}$.

We extend the notion of Kuranishi structure in [F-O: Sec. 5] (and also [F-O-O-O: Appendix 2] and [Liu(C): Sec. 6.1]) and define a Kuranishi structure modelled in a specific category of topology/geometry that appears in our problem; see also [Sat: Sec. 1] and [Th: Chapter 13] for related discussions on orbifolds.

Kuranishi structure modelled in a category of topology/geometry.

Let $\mathcal{C}$ be a category of topology/geometry – e.g. smooth manifolds with corners, complex spaces of specified type of singularities, or fibrations over a fixed topological space – in which the notion of morphisms, embeddings, isomorphisms, bundles, and groups actions make sense. Then, the notion of *orbifolds*, *orbi-bundles* (see also [Th]), *Kuranishi neighborhoods*, *equivalence of Kuranishi neighborhoods*, and *Kuranishi structures* in [F-O] can be generalized by replacing the model topology/geometry in a local chart from domains in $\mathbb{R}^n$ to objects in $\mathcal{C}$, with diffeomorphisms (resp. embeddings; bundle isomorphisms, bundle embeddings) that appear in the data of gluing replaced by isomorphisms between (resp. embeddings of, isomorphisms of bundles over, embeddings of bundles over) objects in $\mathcal{C}$.

Definition 5.1.1 [Kuranishi neighborhood-in-$\mathcal{C}$]. Let M be a Hausdorff topological space and $\mathcal{C}$ be a category of topology/geometry. A *Kuranishi neighborhood-in-$\mathcal{C}$* of $p \in M$ is a 5-tuple $(V_p, \Gamma_{V_p}, E_{V_p}\,; s_p, \psi_p)$ (collectively denoted also by V_p for simplicity of notation) such that

(1) [*neighborhood model*] V_p is an object in $\mathcal{C}$, Γ_{V_p} is a finite group that acts on V_p (as isomorphisms in $\mathcal{C}$) effectively; Γ_{V_p} is called the *structure group* of the Kuranishi neighborhood;

(2) [*obstruction bundle*] E_{V_p} is a Γ_{V_p}-equivariant vector bundle over V_p;

(3) [*Kuranishi map*] $s_p : V_p \to E_{V_p}$ is a Γ_{V_p}-equivariant continuous section of E_{V_p};

(4) [*local coordinate map*] $\psi_p : s_p^{-1}(0) \to M$ is a continuous map which induces a homeomorphism from $s_p^{-1}(0)/\Gamma_{V_p}$ to a neighborhood of p in M.

Two Kuranishi neighborhoods-in-$\mathcal{C}$ $(V_{1,p}, \Gamma_{V_{1,p}}, E_{V_{1,p}}; s_{1,p}, \psi_{1,p})$, $(V_{2,p}, \Gamma_{V_{2,p}}, E_{V_{2,p}}; s_{2,p}, \psi_{2,p})$ of $p \in M$ are said to be *equivalent*, in notation $(V_{1,p}, \Gamma_{V_{1,p}}, E_{V_{1,p}}; s_{1,p}, \psi_{1,p}) \sim (V_{2,p}, \Gamma_{V_{2,p}}, E_{V_{2,p}}; s_{2,p}, \psi_{2,p})$, if

(1) $dim\, V_{1,p} - rank\, E_{V_{1,p}} = dim\, V_{2,p} - rank\, E_{V_{2,p}} =: d$;

(2) their exists another Kuranishi neighborhood-in-$\mathcal{C}$ $(V_p, \Gamma_{V_p}, E_{V_p}; s_p, \psi_p)$ of p such that

- $dim\, V_p - rank\, E_{V_p} = d$,
- there exists a group homomorphism $h_i : \Gamma_{V_{i,p}} \to \Gamma_{V_p}$ and an h_i-equivariant vector-bundle embedding $\hat{\phi}_i/\phi_i : (E_{V_{i,p}}|_{V_{i,p}^\flat})/V_{i.p}^\flat \to E_{V_p}/V_p$ of the restriction of $E_{V_{i,p}}$ to a neighborhood $V_{i,p}^\flat$ of $\psi_{i,p}^{-1}(p)$ in $V_{i,p}$ so that $\hat{\phi}_i \circ s_{i,p} = s_p \circ \phi_i$ on $V_{i,p}^\flat$ and $\psi_{i,p} = \psi_p \circ \phi_i$ on $s_{i,p}^{-1}(0) \cap V_{i,p}^\flat$; $i = 1, 2$.

(When this happens, we say that $(V_p, \Gamma_{V_p}, E_{V_p}; s_p, \psi_p)$ *dominates* $(V_{i,p}, \Gamma_{V_{i,p}}, E_{V_{i,p}}; s_{i,p}, \psi_{i,p})$ or $(V_{i,p}, \Gamma_{V_{i,p}}, E_{V_{i,p}}; s_{i,p}, \psi_{i,p})$ is *subordinate to* $(V_p, \Gamma_{V_p}, E_{V_p}; s_p, \psi_p)$ via $(h_i, \phi_i, \hat{\phi}_i)$, $i = 1, 2$.)

The following definition of Kuranishi structure is modified from [F-O-O-O: A.2.1.5 - A.2.1.11], [Liu(C): Definition 6.3], and [Th: Sec. 13.2]. It is based on the original definition of orbifolds ([Sat] and [Th]) and the notion of a "good coordinate system" ([F-O: Definition 6.1]) extracted from a Kuranishi structure that is originally defined in a functorially-more-natural and closer-to-stack way in [F-O: Definition 5.3] (if [F-O: Definition 5.2] is replaced by the equivalence relation $\sim$ in Definition 5.1.1 above).

Definition 5.1.2 [Kuranishi structure-in-$\mathcal{C}$]. Let M be a Hausdorff topological space and $\mathcal{C}$ be a category of topology/geometry. A *Kuranishi structure-in-$\mathcal{C}$* $\mathcal{K}$ on M consists of the following data/assignment:

(1) a system
$$\mathfrak{N}^{(0)} := \{(V_p, \Gamma_{V_p}, E_{V_p}; s_p, \psi_p)\}_{p \in M}$$
of Kuranishi neighborhoods-in-$\mathcal{C}$, one for each $p \in M$;

(2) a system
$$\mathfrak{N}^{(1)} := \{(V_{qp}; h_{qp}, \phi_{qp}, \hat{\phi}_{qp})\}_{p,q}$$
of 4-tuple *transition data* $(V_{qp}, h_{qp}, \phi_{qp}, \hat{\phi}_{qp})$, one each pair (p, q) with $p \in M$ and $q \in \psi_p(s_p^{-1}(0))$, such that

- (*transition function*): V_{qp} is an open neighborhood of $\psi_q^{-1}(q)$ in V_q, $h_{qp} : \Gamma_{V_q} \to \Gamma_{V_p}$ is an injective group homomorphism, $\hat{\phi}_{qp}/\phi_{qp} : (E_{V_q}|_{V_{qp}})/V_{qp} \to E_{V_p}/V_p$ is an h_{qp}-equivariant vector-bundle embedding such that $(V_p, \Gamma_{V_p}, E_{V_p}; s_p, \psi_p)$ dominates the restriction of $(V_q, \Gamma_{V_q}, E_{V_q}; s_q, \psi_q)$ to V_{pq} via $(h_{qp}, \phi_{qp}, \hat{\phi}_{pq})$;
- (*orbifold cocycle condition*): if $r \in \psi_q(s_q^{-1}(0) \cap V_{qp})$, then there exists a $\gamma \in \Gamma_{V_p}$ such that $\phi_{qp} \circ \phi_{rq} = \gamma\, \phi_{rp}$ on a neighborhood V_{rqp} of $\psi_r^{-1}(r)$ in V_r, $\hat{\phi}_{qp} \circ \hat{\phi}_{rq} = \gamma\, \hat{\phi}_{rp}$ over V_{rqp}, and $h_{qp} \circ h_{rq}(g) = \gamma \cdot h_{rp}(g) \cdot \gamma^{-1}$ for each $g \in \Gamma_{V_r}$.

If furthermore $\mathcal{C}$ allows a well-defined notion of dimensions to its objects and we require that $dim\, V_p - rank\, E_{V_p}$ be a constant d independent of p in the above data, then we say that the Kuranishi structure-in-$\mathcal{C}$ $\mathcal{K}$ on M has *virtual dimension* d.

Two Kuranishi structures-in-$\mathcal{C}$

$$\mathcal{K}_1 \;=\; \Big(\mathfrak{N}_1^{(0)} = \{(V_{1,p}, \Gamma_{V_{1,p}}, E_{V_{1,p}}\,; s_{1,p}, \psi_{1,p})\}_{p\in M}\;,\; \mathfrak{N}_1^{(1)} = \{(V_{1,qp}; h_{1,qp}, \phi_{1,qp}, \hat{\phi}_{1,qp})\}_{p,q}\Big)\;,$$

$$\mathcal{K}_2 \;=\; \Big(\mathfrak{N}_2^{(0)} = \{(V_{2,p}, \Gamma_{V_{2,p}}, E_{V_{2,p}}\,; s_{2,p}, \psi_{2,p})\}_{p\in M}\;,\; \mathfrak{N}_2^{(1)} = \{(V_{2,qp}; h_{2,qp}, \phi_{2,qp}, \hat{\phi}_{2,qp})\}_{p,q}\Big)$$

on M are said to be *equivalent*, in notation $\mathcal{K}_1 \sim \mathcal{K}_2$, if there exist another Kuranishi structure-in-$\mathcal{C}$ on M

$$\mathcal{K} \;=\; \Big(\mathfrak{N}^{(0)} = \{(V_p, \Gamma_{V_p}, E_{V_p}\,; s_p, \psi_p)\}_{p\in M}\;,\; \mathfrak{N}^{(1)} = \{(V_{qp}; h_{qp}, \phi_{qp}, \hat{\phi}_{qp})\}_{p,q}\Big)$$

and a system of triples of (group , space , bundle)-embedding

$$\Big\{\Big(h_{i,p} : \Gamma_{V_{i,p}} \to \Gamma_{V_p}\;,\; \phi_{i,p} : V_{i,p}^{\flat} \to V_p\;,\; \hat{\phi}_{i,p} : E_{V_{i,p}}|_{V_{i,p}^{\flat}} \to E_{V_p}\Big)\Big\}_{p\in M}\;,$$

where $V_{i,p}^{\flat}$ is a neighborhood of $\psi_{i,p}^{-1}(p)$ in $V_{i,p}$ and $\hat{\phi}_{i,p}$ covers $\phi_{i,p}$, such that

- (*morphism between Kuranishi neighborhoods*)
  $(V_p, \Gamma_{V_p}, E_{V_p}\,; s_p, \psi_p)$ dominates $(V_{i,p}, \Gamma_{V_{i,p}}, E_{V_{i,p}}\,; s_{i,p}, \psi_{i,p})$ via $(h_{i,p}, \phi_{i,p}, \hat{\phi}_{i,p})$;
- (*compatibility with gluing*) $h_{i,p} \circ h_{i,qp} = h_{qp} \circ h_{i,q}\,,\;\; \phi_{i,p} \circ \phi_{i,qp} = \phi_{qp} \circ \phi_{i,q}$ on $V_{i,qp}$,
  $\hat{\phi}_{i,p} \circ \hat{\phi}_{i,qp} = \hat{\phi}_{qp} \circ \hat{\phi}_{i,q}$ over $V_{i,qp}$;

$i = 1,\, 2$.

Remark 5.1.3 [*orbifold cocycle condition*]. Though we are not generally looking at a space locally modelled on some $\mathbb{R}^n$ modulo faithful finite group actions as in the definition of an orbifold, the fact that all the maps h_{qp} on Kuranishi neighborhoods are regarded as being defined up to composition with elements in the structure finite group Γ_{V_p} and the morphism h_{qp} of the structure groups are defined up to a conjugation in Γ_{V_p} remain to hold in the definition of Kuranishi structure-in-$\mathcal{C}$. The expression of the compatibility of gluings via the transitions functions $\{(\phi_{qp}, \hat{\phi}_{qp})\}_{p,q}$ in terms of the orbifold cocycle condition, rather than the ordinary cocycle condition, reflects particularly this fact. It is in such form that the setting re-phrases the gluing in a Deligne-Mumford stack.

We should remark that a Hausdorff topological space with a Kuranishi structure is a topological analogue to a Deligne-Mumford moduli stack with a perfect tangent-obstruction complex ([B-F] and [L-T1]) and a coarse moduli space.

Example/Definition 5.1.4 [Kuranishi structure with corners]. Let $\mathcal{C}$ be the category of smooth manifolds with corners, locally modelled on open sets in some $\mathbb{R}^{n_1} \times (\mathbb{R}_{\geq 0})^{n_2}$, or more generally, $\mathbb{R}^{n_1} \times (\text{ cone in } \mathbb{R}^{n_2})$, ($n_1$ and n_2 are allowed to vary). This gives the notion of *Kuranishi structures with corners* in [F-O-O-O: Sec. A.2] and [Liu(C): Sec. 6.1].

Example/Definition 5.1.5 [family Kuranishi structure]. Let M be a Hausdorff topological space fibered over a base Hausdorff topological space B, in notation $\pi : M \to B$ or M/B, and $\mathcal{C}$ be a category of topological spaces all of whose objects and morphisms are over B (as in the category of schemes over a base scheme in algebraic geometry). A (*family*) *Kuranishi structure-in-*$\mathcal{C}$ *on* M/B is a Kuranishi structure-in-$\mathcal{C}$ $\mathcal{K}$ on M, for which all the data in Definition 5.1.1

and Definition 5.1.2 are over B. By construction, there is a natural morphism $\tilde{\pi} : \mathcal{K} \to B$, which restricts to the defining map $V_p \to B$ on each Kuranishi neighborhood V_p. The *fiber* $\mathcal{K}_b := \tilde{\pi}^{-1}(b)$ of $\mathcal{K}$ over $b \in B$ gives a Kuranishi structure-in-$\mathcal{C}_b$ on the fiber $M_b := \pi^{-1}(b)$, where $\mathcal{C}_b$ is the category whose objects and morphisms are from taking the restriction of objects and morphisms in $\mathcal{C}$ to over b. We will denote such $\mathcal{K}$ on M by $\mathcal{K}/B$ on M/B when the family notion is emphasized. If, furthermore, there exists an open dense subset B_0 of B and a category C' of topology/geometry such that each $\mathcal{K}_b$ is a Kuranishi structure-in-$\mathcal{C}'$ on M_b for $b \in B_0$, then we say that $\mathcal{K}/B$ has *general fibers in* $\mathcal{C}'$ and M/B has general fibers with a Kuranishi structure-in-$\mathcal{C}'$.

Morphisms and fibered product.

In any category of geometry, once the geometric objects are defined, the notion of *morphisms* and *fibered products* between them have to be defined accordingly/compatibly as well since these two are the foundation of many other notions and constructions. We will postpone them until Sec. 7.1, where we will define these two notions in a way that works for the specific type of topological spaces-with-a-Kuranishi-structure from the current moduli problem. We won't need them until then.

The category $\mathcal{C}_{\text{spsccw}}$.

We now describe the category $\mathcal{C}_{\text{spsccw}}$ over the complex line $\mathbb{C}$ in which our Kuranishi structure will model. An object in $\mathcal{C}_{\text{spsccw}}$ is a specific kind of *s*tratified *p*iecewise-*s*mooth-with-*c*orners topological space with complex *CW*-complex singularities and is fibered over $\mathbb{C}$ with smooth-with-corner fibers except at 0, constructed as follows.

First, we introduce a complex stratified space $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ over the complex line $\mathbb{C}$. Let $\vec{s}_i = (s_{i1}, \ldots, s_{i,I_i}) \in \mathbb{N}^{I_i}$, $\vec{\mu}_i = (\mu_{i1}, \ldots, \mu_{i,I_i}) \in \mathbb{C}^{I_i}$, for $i = 0, \ldots, k$, and $\vec{\lambda} = (\lambda_0, \ldots, \lambda_k) \in B[k] := \mathbb{C}^{k+1}$. As an affine variety, $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ is defined as the subvariety in $\mathbb{C}^{(I_0+\cdots+I_k)+(k+1)}$:

$$\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \;=\; \{(\vec{\mu}_0, \ldots, \vec{\mu}_k; \vec{\lambda}) \,:\, {\mu_{ij}}^{s_{ij}} \,=\, \lambda_i,\; i = 0, \ldots, k,\; j = 1, \ldots, I_i\}\,.$$

It has complex dimension $dim_{\mathbb{C}}(\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}) = k+1$. The projection

$$\mathbb{C}^{(I_0+\cdots+I_k)+(k+1)} \;\longrightarrow\; \mathbb{C}^{k+1}\,, \quad (\vec{\mu}_0, \ldots, \vec{\mu}_k; \vec{\lambda}) \;\longmapsto\; \vec{\lambda}\,,$$

induces a finite flat morphism from $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ onto $\mathbb{C}^{k+1}$ of degree $\prod_{i=0}^{k}\prod_{j=1}^{I_i} s_{ij}$, and is étale over the complement $\{\vec{\lambda} \,:\, \lambda_i \neq 0,\; i = 0, \ldots, k\}$ of coordinate subspaces in $B[k]$. After the post-composition with the flat morphism $\mathbf{p}[k] : B[k] \to \mathbb{C}$, $\vec{\lambda} \mapsto \lambda_0 \cdots \lambda_k$ (cf. Sec. 1.1.1), one has a flat morphism $p : \Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \to \mathbb{C}$ that is smooth over $\mathbb{C} - \{0\}$.

From the system of defining equations, $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ is the product $\prod_{i=0}^{k} \Xi_{\vec{s}_i}$, where

$$\Xi_{\vec{s}_i} \;=\; \{(\vec{\mu}_i; \lambda_i) \,:\, {\mu_{ij}}^{s_{ij}} \,=\, \lambda_i,\; j = 1, \ldots, I_i\}\,.$$

$\Xi_{\vec{s}_i}$ is the fibered product of the morphisms $f_j : \mathbb{C} \to \mathbb{C}$, $z \to z^{s_{ij}}$, $j = 1, \ldots, I_i$. Its (C^0-)topology is thus the gluing $\vee_{n_i}\mathbb{C}$ of n_i copies of $\mathbb{C}$'s at the origin, where n_i is the number of orbits in the group $(\mathbb{Z}/s_{i1}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/s_{i,I_i}\mathbb{Z})$ under the action generated by the translation $(e_1, \ldots, e_{I_i}) \mapsto (e_1 + 1, \ldots, e_{I_i} + 1)$. It follows that

- the (C^0-)topology of $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ is the product $\prod_{i=0}^{k}(\vee_{n_i}\mathbb{C})$, which is a gluing of $n_0 \cdots n_k$ copies of $\mathbb{C}^{k+1}$ along proper coordinate subspaces, and

· $p^{-1}(t)$, $t \neq 0$, is a disjoint union of $n_0 \cdots n_k$ copies of $(\mathbb{C}^\times)^k$ with each $(\mathbb{C}^\times)^k$ étale over $\mathbf{p}[k]^{-1}(t) \subset B[k]$.

Denote by H_I the coordinate subspace of $B[k]$ whose points have coordinates $\lambda_i = 0$ for $i \in I$. It follows from the topology of $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ that $p^{-1}(0) = \mathbf{p}[k]^{-1}(H_{\{0\}}) \cup \cdots \cup \mathbf{p}[k]^{-1}(H_{\{k\}})$ has $n_0 \cdots n_k\,(\frac{1}{n_0} + \cdots + \frac{1}{n_k})$ irreducible components, with $n_0 \cdots n_{i-1}\, n_{i+1} \cdots n_k$ of them contained in $\mathbf{p}[k]^{-1}(H_{\{i\}})$. Each of these irreducible components has (C^0-)topology isomorphic to $\mathbb{C}^k$. Let $[\mathbf{p}[k]^{-1}(H_{\{i\}})]_0$ be the formal sum of the subvarieties of $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ that appear as irreducible components of $p^{-1}(0)$. It follows from the defining equation, $\lambda_i = 0$, of $\mathbf{p}[k]^{-1}(H_{\{i\}})$ in $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$ that

$$[p^{-1}(t)]\,,\ t \neq 0\,,\ =\ [p^{-1}(0)]\ =\ \sum_{i=0}^{k} (s_{i1} \cdots s_{i,I_i})\,[\mathbf{p}[k]^{-1}(H_{\{i\}})]_0$$

in the Chow group $A_k(\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)})$.

The composition of the projection map with p

$$\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \times \mathbb{R}^{n_1} \times (\mathbb{R}_{\geq 0})^{n_2} \ \longrightarrow\ \Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \ \xrightarrow{p}\ \mathbb{C}$$

gives a flat[19] fibration of $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \times \mathbb{R}^{n_1} \times (\mathbb{R}_{\geq 0})^{n_2}$ over $\mathbb{C}$. Let $\mathcal{C}_{\text{spsccw}}$ be the category of Hausdorff topological spaces fibered over $\mathbb{C}$ that are locally modelled on an open set in $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \times \mathbb{R}^{n_1} \times (\mathbb{R}_{\geq 0})^{n_2}$ as stratified piecewise-smooth-with-corner spaces, with the gluing maps isomorphisms over $\mathbb{C}$. Here, k, $(\vec{s}_0, \ldots, \vec{s}_k)$, n_1, n_2 are all allowed to vary.

We can now state the main theorem of the current work, which gives the foundation of the degeneration axiom and the gluing axiom of open Gromov-Witten invariants. Its proof takes Sec. 5.3 - Sec. 5.4.

Theorem 5.1.6 [family Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$]. *There is a family Kuranishi structure $\mathcal{K}$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ over B that is modelled in $\mathcal{C}_{\text{spsccw}}/\mathbb{C}$, (recall that $B \subset \mathbb{C}$). $\mathcal{K}/B$ is fiberwise of the same virtual dimension*

$$\textit{vdim}^{\,\text{fiber}}\,\overline{\mathcal{M}}_\bullet(W/B, L\,|\,\bullet\,)/B\ :=\ \mu + (N-3)(2-2g-h) + 2n + (m_1 + \cdots + m_h)\,,$$

where $2N$ is the dimension of X (as a fiber of W/B). The family Kuranishi neighborhood-in-$\mathcal{C}_{\text{spsccw}}$ $(V_\rho, \Gamma_{V_\rho}, E_{V_\rho}; s_\rho, \psi_\rho)$ at $\rho = [f : (\Sigma, \partial\Sigma) \to (Y_{[k]}, L_{[k]})]$ has V_ρ/B [20] isomorphic to a neighborhood of the origin in the total space of the flat fibration

$$\big(\,\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \times \mathbb{R}^{n_1} \times (\mathbb{R}_{\geq 0})^{n_2}\,\big)\big/\,\mathbb{C}\,,$$

where

· *$\vec{s}_i$ is the contact order of f along D_i at the ordered set of distinguished nodes in $f^{-1}(D_i)$, $i = 0, \ldots, k$, (and recall that $\dim \Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} = 2k+2$);*

· *$n_1 \ =\ \textit{vdim}^{\,\text{fiber}}\,\overline{\mathcal{M}}_\bullet(W/B, L\,|\,\bullet\,)/B\ + \dim E_\rho - (2k + n_2)$; and*

· *$n_2 \ =$ the total number of boundary nodes and free marked points that land on $\partial\Sigma$.*

[19] For non-algebraic-geometers: the flatness of the fibration $p : \Xi_{(\vec{s}_0,\ldots,\vec{s}_k)} \to \mathbb{C}$ is in the sense of morphisms of schemes over the ground field $\mathbb{C}$. The fiber subscheme $p^{-1}(0)$ over $0 \in \mathbb{C}$ is non-reduced; each of the irreducible components of $p^{-1}(0)$ carries a multiplicity in the sense of [Fu].

[20] Here we use V_ρ/B to indicate that there is built-in map $V_\rho \to B$. The map is not necessarily surjective.

The homeomorphism-type $\{Y_{[k']}\}_{0\leq k'\leq k}$ of the targets of maps gives a Γ_{V_ρ}-invariant stratification $\{S_{k'}\}_{0\leq k'\leq k}$ on the fiber $V_{\rho;0}$ of V_ρ/B over $0\in B$; each connected component of $S_{k'}$ is a manifold of codimension $2k'$ in $V_{\rho;0}$. This stratification coincides with the induced stratification on $V_{\rho;0}$ from the stratification[21] *of $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$.*

5.2 Local transversality and locally regular almost-complex structures.

There are three types of local transversality issues in our moduli problem that have to be understood before one can choose a good obstruction space to work on: (for a *fixed* J)

(T1) local surjectivity of $D_f\bar{\partial}_J$,

(T2) local transversality of evaluation maps, and

(T3) local transversality of the contact order condition along D and local transversality of the pre-deformability conditions at a distinguished node with a specified contact order.

Global such issues have been discussed in related symplectic Gromov-Witten theories, e.g. [MD-S1; MD-S3], [R-T1; R-T2], and [I-P1; I-P2] (particularly for Item (3)). In dealing with transversality issues, it is a standard procedure by now that one first show the sought-for transversality properties on the related universality moduli space $\mathcal{U}\overline{\mathcal{M}}$ of extended tuples $(J, f:\Sigma\to X)$ (or $(J,\nu,f:\Sigma\to X)$ where ν is an additional perturbation in [I-P1], [R-T1], [R-T2]) that contains a choice of an almost-complex structure J and a J-holomorphic map f of a fixed class. One shows that $\mathcal{U}\overline{\mathcal{M}}$ is a smooth Banach manifold and then apply the Sard-Smale Theorem to the fibration of $\mathcal{U}\overline{\mathcal{M}}$ over the Banach manifold $\mathcal{J}$ of allowed almost-complex structures to obtain the sought-for transversality property for the fiber moduli space $\overline{\mathcal{M}}^J$ over a regular value $J\in\mathcal{J}$ of the fibration $\mathcal{U}\overline{\mathcal{M}}\to\mathcal{J}$. A good feature for such a setting is that the moduli space $\overline{\mathcal{M}}^J$ for J regular is a smooth orbifold of the correct dimension as expected from deformation theory. However, it can happen that no regular J's are integrable.

In our current moduli problem, approached along [F-O], the effect of allowing J to vary to obtain a sought-for transversality property is absorbed into a choice of a large enough subspace E in $L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*X)$ so that its preimage $(D_f\bar{\partial}_J)^{-1}(E)$ in $W^{1,p}(\Sigma,\partial\Sigma;f^*T_*X,(f|_{\partial\Sigma})^*T_*L)$ can fit into the related transversality statement. This is because the infinitesimal deformations of J give rise to elements in $L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*X)$ as well, after the pre-composition with $df\circ j$; and, similarly, for the additional ν in [I-P1], [R-T1], [R-T2]. The larger-than-expected dimension and the possibly-worse singularities of $\overline{\mathcal{M}}^J$ for a fixed J that is not regular now have to be compensated in the construction of Kuranishi structure. However, in doing so, we may retain a good J to work on, The latter can be important for other parts in the theory, cf. Example 5.2.3.

With these highlights in mind, we now give the precise respective statement of Conditions (T1), (T2), and (T3) in the setting of Kuranishi structures. The domain unit disc or half unit-disc in the following discussion is considered *fixed*.

(1) *Local surjectivity of $D_f\bar{\partial}_J$.* This condition says that:

(T1) The map

$$D_f\bar{\partial}_J\ :\ W^{1,p}(\Sigma,\partial\Sigma;f^*T_*X,(f|_{\partial\Sigma})^*T_*L)\ \longrightarrow\ L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*X)$$

[21] Recall the map $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}\to\mathbb{C}^{k+1}$. The coordinate-subspace stratification of $\mathbb{C}^{k+1}$ induces a stratification on $\Xi_{(\vec{s}_0,\ldots,\vec{s}_k)}$.

is *surjective* for any non-constant J-holomorphic maps on the unit disc $f : D^2 := \{z \in \mathbb{C} : |z| \leq 1\} \to X$ or on a half unit disc $f : (D^2_+ , \partial_0 D^2_+) := (\{z \in \mathbb{C} : |z| \leq 1, \mathrm{Im}\,(z) \geq 0\}, [-1, 1]) \to (X, L)$.

Any almost-complex structure J that is C^1 close to a complex structure has this property, cf. Example 5.2.3.

(2) *Local transversality of evaluation maps.* This condition says that:

(T2) Given a J-holomorphic map on the marked disc $f : (D^2; 0) \to X$ (resp. on the marked half unit disc $f : (D^2_+ , \partial_0 D^2_+; 0) \to (X, L)$), there exists a (finite dimensional) subspace $E \subset L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*X)$ such that the differential of the evaluation map ev associated to the marked point

$$D_f ev \ : \ (D_f \bar\partial_J)^{-1}(E) \ \longrightarrow \ T_{f(0)}X$$

(resp. $D_f ev : (D_f \bar\partial_J)^{-1}(E) \to T_{f(0)}L$) is surjective.

This is the local Kuranishi statement for [MD-S1: Lemma 6.1.2]. Note that, in the above expression, $D_f ev$ is defined on the whole $W^{1,p}(\Sigma, \partial\Sigma; f^*T_*X, (f|_{\partial\Sigma})^*T_*L)$ by $(D_f ev)(\xi) = \xi(0)$, where $\Sigma = (D^2, 0)$ or $(D^2_+, 0)$.

(3) *Local transversality of the contact order and the pre-deformability condition.* To describe these conditions in the Kuranishi setting, we have to introduce the objects from [I-P1] (with a notation change: V there $= D$ here):

- $\mathcal{J}^D$: the space of pairs (J', ν') where J' is an admissible almost complex structure on the relative pair $(X; D)$ and ν' is an element in $Hom\,(\pi_2^*T_*D^2, \pi_1^*X)$ (of the lifted bundles on $X \times D^2$) that is anti-J'-linear: $\nu \circ j = -J \circ \nu$, (the set of all such ν will be denoted by $Hom^{\,J}(\pi_2^*T_*D^2, \pi_1^*T_*X)$);
- $\mathcal{UM}$: the universal moduli space of (J', ν')-holomorphic maps (i.e. $(f', \phi') : D^2 \to X \times D^2$ such that $\bar\partial_{J'} f' = \nu'$) for some (J', ν').

Let $(J, 0) \in \mathcal{J}^D$ and $f : D^2 \to (X, D)$ be a J-holomorphic disc in X with $f^{-1}(D) = s \cdot (0)$. (we set $\phi = Id_{\,D^2}$ for such f by convention.) Then, [I-P1: Lemma 3.4] implies that there is a *divisor map div* from a neighborhood of $[f] \in \mathcal{UM}$ to the space $\mathrm{Div}^{\,s}(D^2) \subset \mathbb{C}^s$ of degree s divisors on the unit disc D^2, defined by $f' \mapsto f'^{-1}(D)$. Let $End^{\,J}(T_*X)$ be the space of anti-J-linear endomorphisms of T_*X. Then, there is a map

$$\begin{array}{rcccl} T_{(J,0)}\mathcal{J}^D & = & End^{\,J}(T_*X) \oplus Hom^{\,J}(\pi_2^*T_*D^2, \pi_1^*T_*X) & \longrightarrow & L^p(D^2; \Lambda^{0,1}D^2 \otimes_J f^*T_*X) \\ & & (\delta J \,,\, \delta\nu) & \longmapsto & \frac{1}{2}(\delta J) \circ df \circ j \ - \ \delta\nu \end{array} \quad ,$$

where $(\delta J, \delta\nu)$ denotes an infinitesimal deformation of $(J, 0)$. Denote the image of the above map by $\mathcal{H}$. Then $D_f div$ is defined on the subspace $(D_f \bar\partial_J)^{-1}(\mathcal{H})$ of $W^{1,p}(D^2, f^*T_*X)$. Recall the holomorphic coordinate z on D^2 and fix a complex normal coordinate to D around $f(0)$ in $D \subset X$ that is compatible with $J|_{f(0)}$. For ξ in the subspace $V_0 := (D_f div)^{-1}(0)$ of $(D_f \bar\partial_J)^{-1}(\mathcal{H})$, let ξ^n be its normal component with respect to the normal coordinate to D. Then there is a linear $s(0)$-jet-at-0 map

$$\begin{array}{rcccl} jet_0^{s(0)} & : & V_0 & \longrightarrow & \mathbb{C} \\ & & \xi & \longmapsto & d^{\,s(0)}\xi^n(0)/dz^{s(0)} \,. \end{array}$$

With these preparations, the local transversality of contact order condition says that:

(T3.1) Given a J-holomorphic map $f : D^2 \to (X; D)$ such that $f^{-1}(D)$ is a divisor $s\cdot(0)$ on D^2, there exists a (finite dimensional) subspace $E \subset \mathcal{H} \subset L^p(D^2; \Lambda^{0,1}D^2 \otimes_J f^*T_*X)$ such that

- $D_f div : (D_f\bar\partial_J)^{-1}(E) \to T_{s\cdot(0)}\mathrm{Div}^{\,s}(D^2) \simeq T_{\mathbf{0}}\mathbb{C}^s$ is surjective;
- $jet_0^{s(0)}$ on the subspace $(D_f div)^{-1}(0)$ of $(D_f\bar\partial_J)^{-1}(E)$ is also surjective.

This is the local Kuranishi statement for the combination of the related part of [I-P1: proof of Lemma 4.2] and [I-P1: Lemma 3.4].

For the local transversality of the pre-deformability condition, consider first the *fixed* unit disc D^2 with the marked point 0 and restrict the above discussion to maps with 0 sent to D in X. Denote the related universal moduli space by $\mathcal{UM}^0(X; D)$ and let $f : (D^2, 0) \to (X, D)$ be a J-holomorphic map with $f^{-1}(D) = s\cdot(0)$. Then there are the evaluation map $ev_0 : \mathcal{UM}^0(X; D) \to D$ associated to the marked point 0 and the divisor map div_0 from a neighborhood of $[f] \in \mathcal{UM}^0(X; D)$ to the space $\mathrm{Div}^{\,s-1}(D^2) \subset \mathbb{C}^{s-1}$, defined by $f' \mapsto f'^{-1}(D) - (0)$. Their differential, $D_f ev_0$ and $D_f div_0$, are both defined on the subspace $(D_f\bar\partial_J)^{-1}(\mathcal{H})$ of $W^{1,p}(D^2, f^*T_*X)$, where $\mathcal{H}$ is from the previous discussion. Again, for the complex coordinate z on D^2 and a fixed normal coordinate to D in X, has the $s(0)$-jet-at-0 map $jet_0^{s(0)}$ from the subspace $(D_f div_0)^{-1}(0)$ of $(D_f\bar\partial_J)^{-1}(\mathcal{H})$ to $\mathbb{C}$.

Next consider a pre-deformable J-holomorphic map

$$f \;=\; f_1 \cup f_2 \;:\; \Sigma \;:=\; D_1^2 \cup_0 D_2^2 \;\longrightarrow\; Y \;=\; Y_1 \cup_D Y_2$$

of contact order s along D at the distinguished node 0. Define

$$\begin{aligned} &W^{1,p}(\Sigma; f^*T_*Y) \\ &\quad:=\; \{\, (\xi_1, \xi_2) \in W^{1,p}(D_1^2; f_1^*T_*Y_1) \oplus W^{1,p}(D_2^2; f_2^*T_*Y_2) \;:\; \xi_1(0) = \xi_2(0) \,\} \end{aligned}$$

and

$$L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y) \;:=\; L^p(D_1^2; \Lambda^{0,1}D_1^2 \otimes_J f_1^*T_*Y_1) \oplus L^p(D_2^2; \Lambda^{0,1}D_2^2 \otimes_J f_2^*T_*Y_2)\,.$$

Gluing of evaluation maps and their differential defines

$$D_f ev_0 \;:\; W^{1,p}(\Sigma; f^*T_*Y) \;\longrightarrow\; T_{f(0)}D\,.$$

Let $\mathcal{H}_i \subset L^p(D_i^2; \Lambda^{0,1}D_i^2 \otimes_J f_i^*T_*Y_i)$ be the subspace that encodes the infinitesimal deformation of (J, ν) as in the previous discussion and set $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \subset L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y)$. Then, gluing of the divisor map $div_{0,i}$, $i = 1, 2$, and their differential gives $D_f div_0 : (D_f\bar\partial_J)^{-1}(\mathcal{H}) \to T_{(s-1)\cdot(0)}\mathrm{Div}^{\,s-1}(D_1^2) \oplus T_{(s-1)\cdot(0)}\mathrm{Div}^{\,s-1}(D_2^2)$. Again, recall the complex coordinates z_1 and z_2 on D_1^2 and D_2^2 and fixed normal coordinates on (Y_1, D) and (Y_2, D). that is compatible with $J|_{f(0)}$. Then, for $\xi = (\xi_1, \xi_2)$ in the subspace $V^{\mathrm{pd}} := (D_f div)^{-1}(0)$ of $(D_f\bar\partial_J)^{-1}(\mathcal{H})$, let $\xi^n = (\xi_1^n, \xi_2^n)$ be its normal component with respect to the normal coordinate to D. Then there is a linear $s(0)$-jet-at-0 map

$$\begin{aligned} jet_0^{s(0)} \;:\; V^{\mathrm{pd}} &\longrightarrow \mathbb{C}^2 \\ \xi &\longmapsto \left(d^{\,s(0)}\xi_1^n(0)/dz_1^{s(0)} \,,\, d^{\,s(0)}\xi_2^n(0)/dz_2^{s(q)} \right). \end{aligned}$$

In terms of these, the local transversality of the pre-deformability condition says that:

(T3.2) Given a pre-deformable J-holomorphic map $f = f_1 \cup f_2 : \Sigma := D_1^2 \cup_0 D_2^2 \to Y = Y_1 \cup_D Y_2$ of contact order s along D at the distinguished node 0, let $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$ be the subspace of $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y) := L^p(D_1^2; \Lambda^{0,1}D_1^2 \otimes_J f_1^*T_*Y_1) \oplus L^p(D_2^2; \Lambda^{0,1}D_2^2 \otimes_J f_2^*T_*Y_2)$ from the previous discussion on which $D\,div_0 := D\,div_{0,1} \oplus D\,div_{0,2}$ is defined. Then, there exists a (finite dimensional) subspace $E \subset \mathcal{H}$ such that

- $D_f\,div_0 : (D_f\bar{\partial}_J)^{-1}(E) \to T_{(s-1)\cdot(0)}\mathrm{Div}^{\,s-1}(D^2) \oplus T_{(s-1)\cdot(0)}\mathrm{Div}^{\,s-1}(D^2)$ is surjective;
- denote the subspace $(D_f\,div_0)^{-1}(0)$ of $(D_f\bar{\partial}_J)^{-1}(E)$ by $(D_f\bar{\partial}_J)^{-1}(E)^{\mathrm{pd}}$, then $D_f\,ev_0 \oplus jet_0^{s(0)} : (D_f\bar{\partial}_J)^{-1}(E)^{\mathrm{pd}} \to T_{f(0)}D \oplus \mathbb{C}^2$ is surjective.

This is the local Kuranishi statement for the combination of [I-P2: Lemma 3.5] and [I-P1: Lemma 4.2].

Note that, in both (T3.1) and (T3.2), though the map $jet_0^{s(0)}$ depends on the choice of a local coordinate around 0 in the domain and a local normal coordinate to D around $f(0)$ in the target, the surjectivity condition stated is independent of the choices. of such coordinates.

Definition 5.2.1 [(strongly) locally regular almost-complex structure]. An almost-complex structure J on (X, L) with L a maximal totally real submanifold (resp. on $(X, L; D)$ with D a *codim* $_{\mathbb{R}}$-2 almost-complex submanifold) is called *locally regular* if the transversality conditions $(T1)$, $(T2)$ (resp. in addition, $(T3)$) hold for sufficiently small holomorphic discs and half-discs in X. Such J is called *strongly locally regular* if, in addition, E in Condition (T2) and Condition $(T3)$, can be chosen to be supported in a compact set away from the marked point and the distinguished node respectively.

Remark 5.2.2. Condition (T1) is said to be true for all smooth J in [F-O: (12.7.3)] and [Liu(C): proof of Lemma 6.18]. The proof of [MD-S1: Lemma 6.1.2] can be adapted to show that Condition (T2) always holds and E can be chosen to be supported in a compact set away from the marked point. The proof of [I-P1: Lemma 4.2] can be adapted to show that Condition (T3.1) also always holds. Since the domain D^2 is unstable, the perturbation ν in [I-P1: proof of Lemma 4.2] can be set to be 0. The argument in the proof of [I-P1: Lemma 4.2] implies then that the E in Condition (T3.1) can be chosen to be 0. Similarly for the case of $Y = Y_1 \cup_D Y_2$ and Condition (T3.2).

Example 5.2.3 [complex structure]. Let $(X, L; D)$ be a complex manifold (X, J) with a maximal totally real submanifold L and a smooth divisor D. Then the local study of [Sie1], [Sik], [Ve] implies that Condition (T1) is satisfied and a right inverse Q of $D_f\bar{\partial}_J$ is given as an singular integral operator. Conditions (T2) and (T3) can be directly checked by constructing a family of local holomorphic discs or half-discs whose associated deformation vectors map surjectively to $T_{f(0)}X$, $T_{f(0)}L$, and $T_{s(0)}\mathrm{Div}^{\,s}(D^2)$ respectively, e.g. using the local pseudo-automorphism group action on X around $f(0)$. One can also choose E in Conditions (T2) and (T3) to be 0 as long as the holomorphic disc or half-disc is small enough. This shows directly that the complex X is strongly locally regular. Similarly for a complex manifold-divisor relative pair $(Y; D)$ and the singular complex space $Y = Y_1 \cup_D Y_2$.

Assumption. *From now on, we assume that the fixed smooth (C^∞) almost-complex structure on targets of types X, $W[k]/B[k]$, $(Y[k]; D[k])$, $k \in \mathbb{Z}_{\geq 0}$, are all strongly locally regular.*

5.3 Construction of family Kuranishi neighborhoods.

The foundation of the construction is the following two facts, applied in a continuous way to Banach-space fibers of a family over a finite dimensional base.

Proposition 5.3.0.1 [Newton-Picard iteration]. ([MD-S3: Proposition A.3.4].) *Let X and Y be Banach spaces, $U \subset X$ be an open set, and $f : U \to Y$ be a continuous differentiable map. Let $x_0 \in U$ be such that $D := df(x_0) : X \to Y$ is surjective and has a bounded linear right inverse $Q : Y \to X$. Choose positive constants δ and c such that $\|Q\| \leq c$, $B_\delta(x_0; X) \subset U$, and*

$$\|x - x_0\| \ < \ \delta \quad \Longrightarrow \quad \|df(x) - D\| \leq \frac{1}{2c}\,.$$

Suppose that $x_1 \in X$ satisfies

$$\|f(x_1)\| \ < \ \frac{\delta}{4c}\,, \quad \|x_1 - x_0\| \ \leq \ \frac{\delta}{8}\,.$$

Then there exists a unique $x \in X$ such that

$$f(x) \ = \ 0\,, \quad x - x_1 \ \in \ Im\,Q\,, \quad \|x - x_0\| \ \leq \ \delta\,.$$

Moreover, $\|x - x_1\| \ \leq \ 2c\,\|f(x_1)\|$.

Theorem 5.3.0.2 [implicit function theorem]. ([MD-S3: Theorem A.3.3].) *Let X and Y be Banach spaces, $U \subset X$ be an open set, and l be a positive integer. If $f : U \to Y$ is of class C^l and y is a regular value of f (i.e. $df(x)$ surjective with a right inverse for every $x \in f^{-1}(y)$). then $\mathcal{M} := f^{-1}(y) \subset X$ is a C^l Banach manifold and $T_x\mathcal{M} = Ker\,df(x)$ for every $x \in \mathcal{M}$.*

With notations therein, Proposition 5.3.0.1 and Theorem 5.3.0.2 together imply that, for $x_0 \in \mathcal{M}$, there is a homeomorphism from a neighborhood of $0 \in T_{x_0}\mathcal{M}$ to a neighborhood of $x_0 \in \mathcal{M}$. We now resume our study and notations.

The construction of a Kuranishi neighborhood involves the construction of a (continuous) family of objects and maps that fit into Proposition 5.3.0.1 and Theorem 5.3.0.2. Relevant techniques and results in (closed) [MD-S1: Sec. 3.3, Appendix A], [MD-S3: Sec. 3.5, Chapter 10], [F-O: Sec. 12 - Sec. 14]; (closed relative and closed degeneration) [I-P1: Sec.3, Sec. 4, Sec. 6, Sec. 7], [I-P2: Sec. 5 - Sec. 9], [L-R: Sec. 4]; and (open) [Liu(C): Sec. 6.4] for various related symplectic Gromov-Witten theories will be adapted and used to construct a family Kuranishi neighborhood-in-$\mathcal{C}_{\rm spsccw}$ V_ρ at each $\rho = [f : (\Sigma, \partial\Sigma) \to (Y[k], L)] \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L \,|\, [\beta], \vec{\gamma}, \mu)$ for an open Gromov-Witten theory of the degeneration family W/B. The following diagram/flow-chart outlines the construction:

∘ *step* (1)

choice of a *saturated obstruction space* E_ρ at ρ

∘ *step* (2)

$\Downarrow$ • linearized (J, E_ρ)-stability condition

$Ker(\pi_{E_\rho} \circ D_f\bar{\partial}_J)^{\rm pd}$ in $W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$

$\Downarrow$ • upper semi-continuity of $index(D\bar{\partial}_J)$ w.r.t. $B[k]$

the product space $Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar{\partial}_J)^{\rm pd}$ is large enough

$\Downarrow$ • system of algebraic equations for *target-deformation-driven deformations* of Σ

algebraic subset $\widetilde{V}_\rho$ of $Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$, which projects to a constructible subset $\pi_\bullet(\widetilde{V}_\rho)$ in $Def(\Sigma)\times B[k]$

∘ *step* (3)

• piecewise-continuous section $\pi_\bullet(\widetilde{V}_\rho)\to\widetilde{V}_\rho$ with image closure Θ_ρ

$\Downarrow$ • gluing construction around three types - ordinary interior, boundary, and distinguished interior - of nodes on Σ

• exponential-map construction

piecewise-continuous-$\pi_\bullet(\widetilde{V}_\rho)$-family, which extends to a *continuous-$\widetilde{V}_\rho$-family*, of pre-deformable approximate-J-stable C^∞ maps $h_{\,\rm approx\,,\,\bullet}$ from $\Sigma_\bullet$ to fibers of $W[k]/B[k]$

∘ *step* (4)

• E_ρ induces a trivialized obstruction bundle $E_{\widetilde{V}_\rho}$ over $\widetilde{V}_\rho$ with fiber $E_\bullet\subset L^p(\Sigma_\bullet;\Lambda^{0,1}\Sigma_\bullet\otimes_J h_\bullet^*T_*(W[k]_\bullet))$

$\Downarrow$ • construction of a $\pi_\bullet(\widetilde{V}_\rho)$-family of uniformly bounded right inverse $Q_\bullet$ to $\pi_{E,\bullet}\circ D_{h_\bullet}\bar\partial_J$

• Proposition 5.3.0.1 + Theorem 5.3.0.2 : Newton's iteration method to deform approximate solutions to exact solutions to the (J,E)-holomorphy equation

$\widetilde{V}_\rho$-family of (exact) (J,E)-*stable maps* $f_\bullet$ from $\Sigma_\bullet$ to fibers of $W[k]/B[k]$

∘ *step* (5) [*rigidification*]

$\Downarrow$ • the J-holomorphy of the $Aut(\Sigma)\times\mathbb{G}_m[k]$-action

a maximal subset V_ρ in $\widetilde{V}_\rho$ through ρ, transverse to the $Aut(\Sigma)\times\mathbb{G}_m[k]$-orbit of ρ

(This converts ' maps to fibers of $W[k]/B[k]$ ' to ' maps to fibers of $\widehat{W}/\widehat{B}$ '.)

• Kursnishi map $s_\rho: V_\rho\to E_{V_\rho}$ from the $\bar\partial_J$-operator

$\Downarrow$ • stability of ρ, $\Gamma_{V_\rho}=Aut(\rho)$

• $\psi: s_\rho^{-1}(0)\to U_\rho$: orbifold quotient map to a neighborhood of ρ

V_ρ/B : a Kuranishi neighborhood-in-$\mathcal{C}_{\rm spsccw}$ of ρ on $\overline{\mathcal{M}}_{(g,h),(n,\vec m)}(W/B, L\,|\,[\beta],\vec\gamma,\mu)\,/B$

Step (4) is the analytical core in the construction. The algebraic system in Step (2), the distinguished nodes in Step (3), and the conversion in Step (5) from 'maps to fibers of $W[k]/B[k]$' back to 'maps to fibers of W^+/B^+' are the main substeps for which the singularity of the degenerate fiber $W_0=Y_1\cup_D Y_2$ plays a role.

Throughout this subsection, we let $\rho = (\Sigma, \dot\partial\Sigma; \vec p, \vec p_1, \ldots, \vec p_h; f)$ be a stable map to the central fiber $(Y_{[k]}, L_{[k]})$ of $W[k]/B[k]$, and $\rho_{(i)} := (\Sigma_{(i)}, (\dot\partial\Sigma)_{(i)}; \vec p_{(i)}, \vec p_{1,(i)}, \ldots, \vec p_{h,(i)}; f_{(i)})$ be the associated submap to the irreducible component Δ_i of $Y_{[k]}$, for $i=0,\ldots,k{+}1$. (By construction,

$(\dot\partial\Sigma)_{(i)}$, $\vec p_{j,(i)}$ can be non-empty only for $i = 0$ and $k+1$.) We denote the labelled-bordered Riemann surface with marked points $(\Sigma, \dot\partial\Sigma; \vec p, \vec p_1, \ldots, \vec p_h)$ also simply by Σ. The corresponding point of ρ in $\overline{\mathcal{M}}_{(g,h),(n,\vec m)}(W/B, L\,|\,[\beta], \vec\gamma, \mu)$ will also be denoted by ρ. Let $\Lambda_i = f^{-1}(D_i)$ and $\Lambda = \amalg_{i=0}^k \Lambda_i$ be the set of distinguished nodes on Σ under f. Let $\mathbf{s} = (\vec s_0, \cdots, \vec s_k)$ be the tuple of contact orders of f at Λ. Both $Aut(\rho)$ and $Aut(f)$ mean the same. Denote by $Aut(\rho)^{\rm domain}$ (resp. $Aut(\rho)^{\rm target}$) the subgroup of $Aut(\Sigma)$ (resp. $\mathbb{G}_m[k]$) that consists of α (resp. β) such that there is an (α,β) in $Aut(f)$. These groups are all finite. With a re-adjustment, we assume that the auxiliary Kähler metric on $\mathcal{C}/Def(\Sigma)$ is $Aut(\rho)^{\rm domain}$-invariant and the symplectic and, hence, the metric structure on $W[k]$ are $Aut(\rho)^{\rm target}$-invariant.

5.3.1 Choice of obstruction space E_ρ of $\overline{\mathcal{M}}_{(g,h),(n,\vec m)}(W/B, L\,|\,[\beta], \vec\gamma, \mu)$ at ρ.

The index of the linearized operator $D_f\bar\partial_J$ of $\bar\partial_J$ at f.

The fiber of the $\check W^{1,p}$-tangent-obstruction fibration complex

$$T^1_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat W,\widehat L)/\widehat B\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,\big|\,\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet) \stackrel{D\bar\partial_J}{\longrightarrow} T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat W,\widehat L)/\widehat B\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}\,\big|\,\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet)$$

at ρ has a C^l-, C^∞-, and $W^{1,p}$-parallel as follows: (by convention, $\partial\Sigma_{(i)} = \emptyset = L_{[k]_{(i)}}$ for $i = 1, \ldots, k$)

$$\begin{aligned} &C^l(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \\ &:= \Big\{ (\xi_{(i)})_{i=0}^{k+1} \in \oplus_{i=0}^{k+1} C^l(\,\Sigma_{(i)}, \partial\Sigma_{(i)}; f_{(i)}^*T_*\Delta_{(i)}, (f_{(i)}|_{\partial\Sigma_{(i)}})^*T_*L_{[k]_{(i)}}\,) \\ &\qquad\qquad : \xi_{(j)}|_{\Lambda_j} = \xi_{(j+1)}|_{\Lambda_j} \in (f|_{\Lambda_j})^*T_*D_j\,, j = 0, \ldots, k \Big\}\,, \end{aligned}$$

$$C^l(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]}) := \oplus_{i=0}^{k+1} C^l(\Sigma_{(i)}\,, \Lambda^{0,1}\Sigma_{(i)} \otimes_J f_{(i)}^*T_*\Delta_{(i)})\,,$$

and

$$\begin{aligned} D_f\bar\partial_J &: C^\infty(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \longrightarrow C^\infty(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\,, \\ D_f\bar\partial_J &: W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \longrightarrow L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\,. \end{aligned}$$

For ∇ the Levi-Civita connection of the metric on $Y_{[k]}$ induced by (ω, J), the linearization $D\bar\partial_J$ of $\bar\partial_J$ is given by

$$(D_f\bar\partial_J)(\xi) = \frac{1}{2}\Big(\nabla\xi \circ df + J\circ\nabla\xi\circ df\circ j + \nabla_\xi J\circ df\circ j\Big)\,,$$

on the irreducible components of Σ for which f is not constant, cf. [Liu(C): Proposition 6.12]; see also [MD-S1: Eq. (3.2) and Remark 3.3.1]. For an irreducible component of Σ on which f is a constant map. the related bundles, $(f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ and $\Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]}$, on that component are of the respective forms, $\mathcal{O}_\Sigma \otimes_{\mathbb{C}} \mathbb{C}^m$ and $\Lambda^{0,1}\Sigma \otimes_{\mathbb{C}} \mathbb{C}^m$, and have the canonical holomorphic structure from the complex structure on Σ. $D_f\bar\partial_J$ for such component is the restriction to that component of the operator $\bar\partial : C^\infty(\Sigma, \mathcal{O}_\Sigma \otimes_{\mathbb{C}} \mathbb{C}^m) \to C^\infty(\Sigma, \Lambda^{0,1}\Sigma \otimes_{\mathbb{C}} \mathbb{C}^m)$ associated to the canonical holomorphic structure. The following lemma should be compared to [I-P2: Lemma 7.2] and [L-R: Theorem 5.1].

Lemma 5.3.1.1 [index of $D_f\bar\partial_J$ for rigid target]. *Let $f : (\Sigma, \partial\Sigma) \to (Y_{[k]}, L_{[k]})$ be a stable map to the specified expanded target space as above. Then the restriction*

$$D_f\bar\partial_J : C^\infty(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \longrightarrow C^\infty(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$$

is a Fredholm operator of index

$$ind(D_f\bar\partial_J) \;=\; \mu(f) \;+\; dim Y\cdot(1-\tilde g) \;-\; 2\sum_{i=0}^{k} l(\vec s_i) \;+\; 4\sum_{i=0}^{k} deg\,\vec s_i\,,$$

where $\tilde g$ is the arithmetic genus of $\Sigma_{\mathbb{C}}$.

Proof. Let $f = \cup_{i=0}^{k+1} f_{(i)} : \Sigma = \cup_{i=0}^{k+1}\Sigma_{(i)} \to Y_{[k]} = \cup_{i=0}^{k+1}\Delta_i$ be the decomposition of f into submaps. Then, it follows from the Riemann-Roch Theorem (e.g. [F-O: Lemma 12.2], [Liu(C): Lemma 6.13], and [MD-S3: Appendix C]) that each of

$$D_{f_{(i)}}\bar\partial_J : C^\infty(\Sigma_{(i)},\partial\Sigma_{(i)}; f_{(i)}^*T\Delta_i, (f_{(i)}|_{\partial\Sigma})^*TL) \longrightarrow C^\infty(\Sigma_{(i)};\Lambda^{0,1}\Sigma_{(i)}\otimes_J f_{(i)}^*T\Delta_i)\,,$$

for $i = 0,\, k+1$, and

$$D_{f_{(i)}}\bar\partial_J : C^\infty(\Sigma_{(i)}; f_{(i)}^*T\Delta_i) \longrightarrow C^\infty(\Sigma_{(i)};\Lambda^{0,1}\Sigma_{(i)}\otimes_J f_{(i)}^*T\Delta_i)\,,$$

for $i = 1, \ldots, k$, is a Fredholm operator of index

$$ind(D_{f_{(i)}}\bar\partial_J) \;=\; \mu(f_{(i)}) \;+\; dim Y\,\tilde\chi_i/2\,, \quad \text{for } i = 0,\, k+1\,,$$

and

$$ind(D_{f_{(i)}}\bar\partial_J) \;=\; -2K_{\Delta_i}\cdot\beta_i + dim Y\chi_i/2\,, \quad \text{for } i = 1, \ldots, k\,.$$

This implies, in particular, that $D_f\bar\partial_J$ is Fredholm.

The matching condition along TD_i at each distinguished node of Σ imply that

$$\begin{aligned} &C^\infty(\Sigma,\partial\Sigma; f^*TY_{[k]}, (f|_{\partial\Sigma})^*TL_{[k]}) \\ &\quad\hookrightarrow \oplus_{i=0,k+1} C^\infty(\Sigma_{(i)},\partial\Sigma_{(i)}; f_{(i)}^*T\Delta_i, (f_{(i)}|_{\partial\Sigma})^*TL) \bigoplus \oplus_{i=1}^{k} C^\infty(\Sigma_{(i)}; f_{(i)}^*T\Delta_i)\end{aligned}$$

has codimension

$$\sum_{i=0}^{k} l(\vec s_i)\,(dim Y + 2)\,.$$

Denote the quotient vector space of this inclusion by V, then one has the following short exact sequence of 2-term complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0 & \longrightarrow & \oplus_{i=0}^{k+1} C_{(i)}^0 & \longrightarrow & V & \longrightarrow & 0 \\ & & {\scriptstyle D_f\bar\partial_J}\downarrow & & {\scriptstyle \oplus_i D_{f_{(i)}}\bar\partial_J}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1 & \longrightarrow & \oplus_{i=0}^{k+1} C_{(i)}^1 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}\,,$$

where

$$\begin{aligned} C^0 &= C^\infty(\Sigma,\partial\Sigma; f^*TY_{[k]}, (f|_{\partial\Sigma})^*TL_{[k]})\,, \\ C^1 &= C^\infty(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*TY_{[k]})\,, \\ C_{(i)}^0 &= \begin{cases} C^\infty(\Sigma_{(i)},\partial\Sigma_{(i)}; f_{(i)}^*T\Delta_i, (f_{(i)}|_{\partial\Sigma})^*TL)\,, & \text{for } i = 0, k+1\,, \\ C^\infty(\Sigma_{(i)}; f_{(i)}^*T\Delta_i)\,, & \text{for } i = 1, \ldots, k\,, \end{cases} \\ C_{(i)}^1 &= C^\infty(\Sigma_{(i)};\Lambda^{0,1}\Sigma_{(i)}\otimes_J f_{(i)}^*T\Delta_i)\,. \end{aligned}$$

The Snake Lemma, together with the additivity property of (relative) Maslov index under joining of submaps (Definition 3.1.2) and of the Euler characteristic of Riemann surfaces under gluing along boundaries from removing small discs around distinguished nodes, implies then

$$ind(D_f\bar\partial_J) \;=\; \mu(f) \;+\; dim\, Y\cdot(1-\tilde g) \;-\; 2\sum_{i=0}^{k} l(\vec{s_i}) \;+\; 4\sum_{i=0}^{k} deg\,\vec{s}_i\,.$$

□

Remark 5.3.1.2 [*class independence*]. ([MD-S1: Remark. 3.2.3].) Lemma 5.3.1.1 holds also for

$$D_f\bar\partial_J \,:\, W^{l,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \;\longrightarrow\; W^{l-1,p}(\Sigma,\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})\,,$$

$$D_f\bar\partial_J \,:\, \check W^{l,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \;\longrightarrow\; \check W^{l-1,p}(\Sigma,\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})\,,$$

and

$$D_f\bar\partial_J \,:\, C^{l}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \;\longrightarrow\; C^{l-1}(\Sigma,\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})\,.$$

We have taken J to be of class C^∞ on each irreducible component of $Y_{[k]}$. Thus, elliptic regularity implies that $Ker(D_f\bar\partial_J)$ always lies in $C^\infty(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$, independent of the choice of the space on which $D_f\bar\partial_J$ is defined.

Existence of a saturated obstruction space E_ρ at ρ.

For a small enough neighborhood $U_{\Lambda^+} = (\amalg_{q\in\Lambda}U_q)\amalg(\amalg_{p_i}U_{p_i})\amalg\left(\amalg_{q_{ij}}U_{q_{ij}}\right)$ of the set $\Lambda^+ := \Lambda\cup\mathbf{p}\cup\cup_{j=1}^h\mathbf{q}_j$ of the distinguished nodes and the marked points on Σ, recall from Sec. 5.2 (with 0 there replaced by q here) the associated subspace $\mathcal{H}_q$ in $L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})|_{U_q}$, $q\in\Lambda$, such that $D_f div_q$ is defined on $(D_f\bar\partial_J)^{-1}(\mathcal{H}_q)$ with values in $T_{(s(q)-1)\cdot(q)}\mathrm{Div}^{\,s(q)-1}(U_{q,1})\oplus T_{(s(q)-1)\cdot(q)}\mathrm{Div}^{\,s(q)-1}(U_{q,2})$, where $s(q)$ is the contact order of f along the singular locus of $Y_{[k]}$ at q.

Definition 5.3.1.3 [admissible subspace]. A subspace V in $W^{1,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ is called *admissible* if there exists such an U_{Λ^+} so that $V|_{U_q}\subset(D_f\bar\partial_J)^{-1}(\mathcal{H}_q)$ for all $q\in\Lambda$.

As all the maps $D_f div_q$, $jet_q^{s(q)}$, $D_f ev_q$, $D_f ev_{p_i}$, and $D_f ev_{q_{ij}}$ depend only on a jet at the specified point in Λ^+, they extend canonically to maps on an admissible subspace of $W^{1,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ by pre-composition with the restriction-to-U_{Λ^+} map.

Definition 5.3.1.4 [saturated/pre-deformable subspace]. A subspace V in $W^{1,p}(\Sigma,\partial\Sigma;$ $f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ is said to be *saturated* if

(1) V is admissible;

(2) the map

$$(\oplus_{q\in\Lambda}D_f div_q)\bigoplus(\oplus_{p_i}D_f ev_{p_i})\bigoplus\left(\oplus_{q_{ij}}D_f ev_{q_{ij}}\right)\;:\;V\;\longrightarrow$$
$$\left(\bigoplus_{q\in\Lambda}\left(T_{(s(q)-1)\cdot(q)}\mathrm{Div}^{\,s(q)-1}(U_{q,1})\oplus T_{(s(q)-1)\cdot(q)}\mathrm{Div}^{\,s(q)-1}(U_{q,2})\right)\right)$$
$$\bigoplus\left(\oplus_{p_i}T_{f(p_i)}Y_{[k]}\right)\bigoplus\left(\oplus_{q_{ij}}T_{f(q_{ij})}L\right)$$

is surjective;

(3) let V^{pd} be the subspace $(\oplus_{q\in\Lambda} D_f div_q)^{-1}(\mathbf{0})$ in V, then the linear map

$$\oplus_{q\in\Lambda}\,(D_f ev_q \oplus jet_q^{s(q)}) \;:\; V^{\mathrm{pd}} \;\longrightarrow\; \oplus_{q\in\Lambda}\,(T_{f(q)}D \oplus \mathbb{C}^2)$$

is surjective, where we have identified D_i, $i = 0, \ldots, k$, canonically with D.

In the above statement, V^{pd} is called the *pre-deformable subspace* of V.

A subspace E of $L^p(\Sigma; \Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})$ is said to be *saturated* if $(D_f\bar\partial_J)^{-1}(E) \subset W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ is saturated.

Definition/Lemma 5.3.1.5 [saturated obstruction space]. Denote by $Im\,(D_f\bar\partial_J)$ the image of $D_f\bar\partial_J$, $(D_f\bar\partial_J)(W^{1,\,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}))$, in $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$. Then *there exists a subspace E_ρ of $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$ such that*

(1) *$Im\,(D_f\bar\partial_J) + E_\rho = L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$,*

(2) *E_ρ is finite-dimensional, complex linear, and $Aut(\rho)$-invariant,*

(3) *E_ρ consists of smooth sections supported in a compact subset of Σ disjoint from the set of all (three types of) nodes on Σ,*

(4) *$(D_f\bar\partial_J)^{-1}(E_\rho)$ is a saturated subspace of $W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}))$.*

E_ρ is called a *saturated obstruction space* of $\overline{\mathcal{M}}_{(g,h),(n,\vec m)}(W/B, L\,|\,[\beta], \vec\gamma, \mu)$ at ρ.

Proof. Since J is strongly locally regular and the vector spaces above is constructed from a gluing of the ordinary case for smooth target spaces, the existence of E'_ρ with Properties (1), (2), and (3) follows the same argument as in [F-O: 12.7] and [Liu(C): Lemma 6.18]. It remains now to enlarge E'_ρ to incorporate Property (4).

As J is locally strongly regular, there exist finite-dimensional subspaces V_{U_q} (resp. $V_{U_{p_i}}$, $V_{U_{q_{ij}}}$) in $(D_{f|_{U_q}}\circ\bar\partial_J)^{-1}(\mathcal{H}_q)$ (resp. $C^\infty(U_p, (f|_{U_p})^*T_*Y_{[k]})$, $C^\infty(U_{q_{ij}}\partial_0 U_{ij}; (f|_{U_{q_{ij}}})^*T_*Y_{[k]}$, $(f|_{\partial_0 U_{q_{ij}}})^*T_*L)$) such that, for all q, p_i, $q_{ij} \in \Lambda^+$, (a) the restriction of $D_f div_q$ (resp. $D_f ev_{p_i}$, $D_f ev_{q_{ij}}$) thereon is surjective; (b) the restriction of $D_f ev_q \oplus jet_q^{s(q)}$ on the local pre-deformable subspace ${V_{U_q}}^{\mathrm{pd}}$ is surjective, and (c) $D_{f|_{U_q}}(V_{U_q})$ (resp. $D_{f|_{U_{p_i}}}(V_{U_{p_i}})$, $D_{f|_{U_{q_{ij}}}}(V_{U_{q_{ij}}})$) is supported in the complement of a small neighborhood of q (resp. p_i, q_{ij}). One can extend V_{U_q}, $V_{U_{p_i}}$, $V_{U_{q_{ij}}}$ to subspaces V_q, V_{p_i}, $V_{q_{ij}}$ in $C^\infty(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}$, $(f|_{\partial\Sigma})^*T_*L_{[k]})$ so that the summation $V := (\sum_{q\in\Lambda} V_q) + (\sum_{p_i} V_{p_i}) + (\sum_{q_{ij}} V_{q_{ij}})$ in $C^\infty(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ is a direct sum. Then the image $(D_f\bar\partial_J)(V)$ is a finite-dimensional saturated subspace of $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$ that satisfies Condition (3). Let E_ρ be the span of $E'_\rho + D_f\bar\partial_J(V)$ and its image under the complex rotation and the $Aut(\rho)$-action. Then E_ρ satisfies Properties (1), (2), (3), (4).

□

Let E_ρ be a such obstruction space at ρ. Property (4) of E_ρ implies that $(D_f\bar\partial_J)^{-1}(E_\rho)^{\mathrm{pd}}$ has (real) codimension $4\sum_{i=0}^k (deg\,\vec s_i - l(\vec s_i))$ in $(D_f\bar\partial_J)^{-1}(E_\rho)$. It follows thus from Lemma 5.3.1.1 that:

Corollary 5.3.1.6 [pre-deformable subspace of $(D_f\bar\partial_J)^{-1}(E_\rho)$].

$$\begin{aligned} dim\,(D_f\bar\partial_J)^{-1}(E_\rho)^{\mathrm{pd}} &= \mu(f) + dim\,Y\cdot(1-\tilde g) + 2\sum_{i=0}^k l(\vec s_i) + dim\,E_\rho \\ &= \mu(f) + dim\,Y\cdot(1-\tilde g) + 2\,|\Lambda| + dim\,E_\rho \qquad . \end{aligned}$$

Definition 5.3.1.7 [pre-deformable index]. *We define the pre-deformable index of $D_f\bar\partial_J$ to be*

$$ind^{\rm pd}(D_f\bar\partial_J) \;:=\; \mu(f) \,+\, dim Y\cdot(1-\tilde g) \,+\, 2\,|\Lambda|\,.$$

Remark 5.3.1.8 [*fixed vs. non-fixed* (*domain, target*)]. While there is no local obstruction to extending stable map f from a fixed nodal curve Σ to a fixed transverse nodal target $Y_{[k]}$, there remain obstructions when extending such maps to a partial smoothing of $Y_{[k]}$, enforcing a deformation of the domain as well. In algebro-geometric/holomorphic setting, such obstructions are encoded in the cohomology $H^0(\Sigma, f^*\mathcal{E}xt^1(\Omega_{Y_{[k]}}, \mathcal{O}_{Y_{[k]}}))$. The existence of such obstructions is reflected in the dropping of $ind^{\rm pd}(D_\bullet\bar\partial_J)$ when f is deformed to a nearby stable map to $Y_{[k-1]}$ that smoothes D_i, cf. Definition 5.3.1.7,

Definition 5.3.1.9 [(J,E_ρ)-stable map]. Given E_ρ in Definition/Lemma 5.3.1.5, a map $h:(\Sigma,\partial\Sigma)\to(Y_{[k]},L)$ is called (J,E_ρ)-*stable* if it satisfies the *perturbed J-holomorphy equations* $\bar\partial_J h\in E_\rho$, is *pre-deformable* at the distinguished nodes, and has a finite $Aut(h)$.

For later use, we introduce the quotient map

$$\pi_{E_\rho}\;:\; L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]}) \;\longrightarrow\; L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})/E_\rho$$

and denote $(D_f\bar\partial_J)^{-1}(E_\rho)^{\rm pd}$ also as $Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$. With respect to the holomorphic coordinates around Λ in Σ and normal coordinates to $\cup_{i=0}^k D_i$ around $f(\Lambda)$ in $Y_{[k]}$ that defines $jet^{s(q)}_q$, $q\in\Lambda$, one thus has the linear map

$$\begin{array}{rcccc} jet^{\mathbf{s}}_\Lambda & : & Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd} & \longrightarrow & \mathbb{C}^{\,2\,|\Lambda|} \\ & & \xi & \longmapsto & \Big(jet^{s(q)}_q\left((\xi|_{U_q})^n\right)\Big)_{q\in\Lambda}\,. \end{array}$$

For $q\in\Lambda$, suppose that with respect to the fixed local coordinates $f|_{U_q}$ is given by

$$f(z_{q,i}) = \Big(\,f(q)+O\left(|z_{q,i}|\right)\,,\,a_{q,i}z_{q,i}^{s(q)}+O\left(|z_{q,i}|^{s(q)+1}\right)\Big)\,,\quad i=1,\,2\,.$$

Define the *shift-product map* $sp_q:\mathbb{C}^2\to\mathbb{C}$, $(\,\bullet_1\,,\,\bullet_2)\mapsto(\,a_{q,1}+\bullet_1)(\,a_{q,2}+\bullet_2)$. Then, the image of a small enough neighborhood of 0 in $Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ under the composition $sp_q\circ jet^s_q$ lies in a simply-connected neighborhood of $a_{q,1}a_{q,2}$ in $\mathbb{C}-\{0\}$. For $f(q)\in D_i$, $sp_q\circ jet^{s(q)}_q$ is a (nonlinear) map from $Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ to $\big((T^*_q\Sigma_{(i)})^{\otimes s(q)}\otimes T_{f(q)}\Delta_i\big)\bigotimes\big((T^*_q\Sigma_{(i+1)})^{\otimes s(q)}\otimes T_{f(q)}\Delta_{i+1}\big)$. Define the nonlinear map

$$\begin{array}{rcccc} sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda & : & Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd} & \longrightarrow & \mathbb{C}^{\,|\Lambda|} \\ & & \xi & \longmapsto & \Big(sp_q\circ jet^{s(q)}_q\left((\xi|_{U_q})^n\right)\Big)_{q\in\Lambda}\,. \end{array}$$

Property (4) of E_ρ in Definition/Lemma 5.3.1.5 implies that the map $sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda$ is a bundle map over a small enough neighborhood of $sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda(0)$ in $(\mathbb{C}-\{0\})^{|\Lambda|}$ with fiber of (real) dimension $\mu(f)\,+\,dim Y\cdot(1-\tilde g)\,+\,dim E_\rho$.

5.3.2 The algebraic subset $\widetilde{V}_\rho$ in $Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$.

A family Kuranishi neighborhood V_ρ of $\rho=[f]\in\overline{\mathcal{M}}_{(g,h),(n,\vec m)}(W/B,L\,|\,[\beta],\vec\gamma,\mu)$ over B is to be obtained from an enlarged deformation theory of the underlying moduli problem. Corollary 5.3.1.6 implies that $ind^{\rm pd}D_h\bar\partial_J$ is piecewise-constant and upper semi-continuous with respect to the stratification of $B[k]$ when h runs over J-stable maps of the given combinatorial type from deformed Σ to fibers of $W[k]/B[k]$. This hints that the product space $Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ is large enough to accommodate all the new maps to appear in a candidate Kuranishi neighborhood V_ρ of ρ. We describe in this subsubsection an algebraic subset $\widetilde{V}_\rho$ in $Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$, characterized by the deformation theory of maps at the distinguished nodes, that will finally give V_ρ.

As observed in [I-P2] (see also [Li1] and [Gr-V] in algebro-geometric category), when $Y_{[k]}$ is partially smoothed to some $Y_{[k']}$ with D_i being smoothed, all the nodes in Σ_i in Σ have to be simultaneously smoothed in order for there to exist a J-holomorphic map f' from the new Σ' to $Y_{[k']}$ that is close to f. Thus, V_ρ should only come from a subset of a locus $\widetilde{V}_\rho$ in $Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ that is characterized by such target-space-driven deformations of the domain. With the higher-order terms omitted, the germ of a target-space-driven deformation of Σ at a distinguished node is modelled on the family of maps from $\mathbb{C}^2/\mathbb{C}$ to $\mathbb{C}^2/\mathbb{C}$ given by

$$\begin{array}{ccc}\mathbb{C}^2 & \longrightarrow & \mathbb{C}^2\\ (z_1,z_2) & & (w_1,w_2)=(a_1\,z_1{}^s\,,\,a_2\,z_2{}^s)\\ \downarrow & & \downarrow\\ \mathbb{C} & \longrightarrow & \mathbb{C}\\ \mu=z_1z_2 & & \lambda\,=\,w_1w_2\,=\,a_1a_2\,\mu^s\end{array}\qquad a_1,\,a_2\,\in\mathbb{C}-\{0\}\,,\quad ,$$

where the local target (resp. domain) deformations are parameterized by λ (resp. μ). Such constraints from deformation theory at distinguished nodes select a subset $\widetilde{V}_\rho$ in $Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ described as follows.

Fix a factorization

$$Def(\Sigma)\;=\;Def(\Sigma;\Lambda)\;\times\;H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}}\,,$$

where $H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}}$ is the space of local smoothing of distinguished nodes in Λ and $Def(\Sigma;\Lambda)$ consists of deformations of Σ that keep Λ as nodes. $H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}}$ is a neighborhood of $0\in\mathbb{C}^{\,|\Lambda|}$, with coordinates $(\vec\lambda_0,\cdots,\vec\lambda_k)$ with 0 corresponding to no smoothing of nodes in Λ. Let $H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}\subset\mathbb{C}^{\,|\Lambda|}$ be a neighborhood of $sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda(0)$ in $(\mathbb{C}-\{0\})^{|\Lambda|}$, with coordinates $\vec a=(\vec a_0,\cdots,\vec a_k)$, over which the map $sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda: Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}\to\mathbb{C}^{\,|\Lambda|}$ is a bundle map (of fiber dimension $\mu(f)\,+\,dim\,Y\cdot(1-\tilde g)\,+\,dim\,E_\rho$).

> **[Choice].** From now on in the construction, we will assume that the local chart around $q\in\Lambda$ is chosen so that $a_{q,1}=a_{q.2}$ in the normal form expression of f_ρ around q, cf. Sec. 5.2. Fix a section from $H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}$ to $Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ so that the condition $a_{q,1}=a_{q.2}$ is preserved for all $q\in\Lambda$. The value in $Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}$ of this section for $\vec a\in H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}$ will be denoted $\xi_{\vec a}$.

This gives a trivialization

$$Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\rm pd}_\Lambda\;:=\;(sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda)^{-1}(H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}})\;\simeq\;H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}\times H^{(0,\Lambda)}_{\rho,\,\text{map}}\,.$$

(By convention we fix coordinates on $H^{(0,\Lambda)}_{\rho,\,\text{map}}$ so that the afore-mentioned section has image $H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}\times\{0\}$ in $H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}\times H^{(0,\Lambda)}_{\rho,\,\text{map}}$.) Combining the two, one has a decomposition of the relevant

open neighborhood of the origin of $Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$:

$$\begin{aligned} & Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}_\Lambda \\ & \quad \simeq \; Def(\Sigma;\Lambda) \times \Big(H^{(\rm smooth,\Lambda)}_{\rho,\rm domain} \times B[k] \times H^{(\rm loc,\Lambda)}_{\rho,\,\rm map} \Big) \times H^{(0,\Lambda)}_{\rho,\,\rm map} \\ & \quad \subset \; Def(\Sigma;\Lambda) \times \Big(\mathbb{C}^{\,|\Lambda|} \times \mathbb{C}^{\,k+1} \times \mathbb{C}^{\,|\Lambda|} \Big) \times H^{(0,\Lambda)}_{\rho,\,\rm map} \,. \end{aligned}$$

The product $\mathbb{C}^{\,|\Lambda|} \times \mathbb{C}^{\,k+1} \times \mathbb{C}^{\,|\Lambda|}$ has coordinates $(\,\vec\mu_0, \cdots, \vec\mu_k\,;\,\vec\lambda\,;\,\vec a_0, \cdots, \vec a_k\,)$ with

$$\vec\mu_i = (\mu_{i1}, \cdots, \mu_{i,|\Lambda_i|})\,, \quad \vec\lambda = (\lambda_0, \cdots, \lambda_k)\,, \quad \text{and} \quad \vec a_i = (a_{i1}, \cdots, a_{i,|\Lambda_i|})$$

that correspond to the deformations of domain, target, and maps respectively around Λ.

Compare this with the basic deformation model above, one concludes that in terms of these coordinates, the subset $\widetilde V_\rho$ of $Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}_\Lambda$ is described by a system of algebraic equations on the $\big(H^{(\rm smooth,\Lambda)}_{\rho,\rm domain} \times B[k] \times H^{(\rm loc,\Lambda)}_{\rho,\,\rm map}\big)$-factor:

$$\begin{aligned} \widetilde V_\rho \;\; &= \;\; \left\{ (\cdots\,;\,\vec\mu_0, \cdots, \vec\mu_k\,;\,\vec\lambda\,;\,\vec a_0, \cdots, \vec a_k\,;\,\cdots) \;\middle|\; \begin{array}{l} {\mu_{ij}}^{s_{ij}} \;=\; \lambda_i/a_{ij}\,, \\ i \,=\, 0,\,\ldots,\,k\,;\; j \,=\, 1,\,\ldots,\,|\Lambda_i| \end{array} \right\} \\ &=: \;\; Def(\Sigma;\Lambda) \times \overline{V}_\rho \times H^{(0,\Lambda)}_{\rho,\,\rm map}\,. \end{aligned}$$

As each a_{ij} takes values in a simply-connected domain in $\mathbb{C} - \{0\}$,

$$\widetilde V_\rho \;\simeq\; Def(\Sigma;\Lambda) \times \Xi_{\mathbf{s}} \times H^{(\rm loc,\Lambda)}_{\rho,\,\rm map} \times H^{(0,\Lambda)}_{\rho,\,\rm map} \;=\; Def(\Sigma;\Lambda) \times \Xi_{\mathbf{s}} \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}_\Lambda$$

in the category of piecewise-smooth stratified spaces, where $\Xi_{\mathbf{s}}$ is defined in Sec. 5.1.

The projection map from $Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$ to $B[k]$ restricts to a morphism $\pi_{B[k]} : \widetilde V_\rho \to B[k]$ of constant fiber dimension $\mu(f) + dim\, Y \cdot (1-\tilde g) + dim\, Def(\Sigma) + dim\, E_\rho$. The restriction of $\widetilde V_\rho$ over each stratum of $B[k]$ can be made a trivial bundle under $\pi_{B[k]}$. On the other hand, the projection map from $Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$ to $Def(\Sigma) \times B[k]$ restricts to a morphism $\pi_{Def(\Sigma)\times B[k]} : \widetilde V_\rho \to Def(\Sigma) \times B[k]$ whose image is only a constructible subset in a neighborhood of $0 \in Def(\Sigma) \times B[k]$ and whose fiber dimensions is given by the upper semi-continuous function $ind^{\rm pd}(D_\bullet\bar\partial_J) + dim\, E_\rho$.

Definition/Convention 5.3.2.1 [linear/nonlinear coordinates on $Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$]. Coordinates of $Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$ as a subset of a vector space will be called *linear coordinates* on $Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$. Those from the isomorphism with $H^{(\rm loc,\Lambda)}_{\rho,\,\rm map} \times H^{(0,\Lambda)}_{\rho,\,\rm map}$ will be called *nonlinear coordinates*. Unless otherwise mentioned, we adopt by convention the nonlinear coordinates for $Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$ (particularly when written as coordinates from the factorization) except the *origin* $0 \in Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$.

Finally, $Aut(\rho)$ acts on $Def(\Sigma) \times B[k] \times Ker(\pi_{E_\rho} \circ D_f\bar\partial_J)^{\rm pd}$. Shrinking if necessary, we take $\widetilde V_\rho$ to be $Aut(\rho)$-invariant in the above construction.

5.3.3 A $\widetilde{V}_\rho$-family of approximate-J-stable C^∞ maps to fibers of $W[k]/B[k]$.

We construct in this subsubsection an $Aut(\rho)$-invariant $\widetilde{V}_\rho$-family of approximate-J-holomorphic C^∞ maps $h_{\text{approx},\bullet}$ from deformed Σ to fibers of $W[k]/B[k]$ by gluing maps around nodes of Σ. Such construction is given in [MD-S1: Appendix A] and in [F-O], [Liu(C)], [Liu(G)], [R-T1], [R-T2], [Sal], and [I-P2], [L-R] for various extensions.

To separate the effect from various types of deformations involved, the factorization $Def(\Sigma) = Def(\Sigma;\Lambda) \times H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}}$ is refined to

$$\begin{aligned} & Def(\Sigma) \quad (=:\ H_{\rho,\text{domain}}\,) \\ & \quad = \ \Big(H^{(\text{deform},\Sigma)}_{\rho,\text{domain}} \times H^{(\text{smooth,o.i.n})}_{\rho,\text{domain}} \times H^{(\text{smooth,b.n.})}_{\rho,\text{domain}} \Big) \times H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}} \,, \end{aligned}$$

where $H^{(\text{deform},\Sigma)}_{\rho,\text{domain}}$ consists of deformations of the *complex structure* on Σ (as a bordered Riemann surface with marked points) without changing the topology of Σ, $H^{(\text{smooth,o.i.n})}_{\rho,\text{domain}}$ consists of local deformations of Σ that smooth some *ordinary interior nodes* of Σ, and $H^{(\text{smooth,b.n.})}_{\rho,\text{domain}}$ consists of local deformations of Σ that smooth some *boundary nodes* of Σ. For Σ of genus g, h holes, n_{oin} ordinary interior nodes, $|\Lambda|$ distinguished interior nodes, n_{bn} boundary nodes, n ordinary marked points, and $|\vec{m}|$ boundary marked points, $H_{\rho,\text{domain}}$ is parameterized by a neighborhood of $\mathbf{0}$ in the 4-factor product space (with coordinates $(\zeta,\vec{t},\vec{t}',\vec{\mu})$)

$$(\,\mathbb{C}^{3g-3+h-n_{oin}-|\Lambda|+n'+d_c} \times \overline{\mathbb{H}}^{n''} \times \mathbb{R}^{h-n_{bn}+|\vec{m}|+d_b}\,) \times \mathbb{C}^{n_{oin}} \times {\mathbb{R}_{\geq 0}}^{n_{bn}} \times \mathbb{C}^{|\Lambda|}\,, \quad n \doteq n'+n''\,,$$

with respect to the above decomposition. Let $\mathcal{C}/Def(\Sigma)$ be the universal curve over $Def(\Sigma)$, with the fiber labelled-bordered Riemann surface-with-marked-points over $(\zeta,\vec{t},\vec{t}',\vec{\mu}) \in Def(\Sigma)$ denoted by $\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$. With a fixed local model chart at each node of Σ, cf. Definiton 2.1, a fixed $\varepsilon > 0$ small, and the assumption that $\|(\vec{t},\vec{t}',\vec{\mu})\| \ll \varepsilon$, following the same construction as in the case of $W[k]/B[k]$, there is a *ε-neck-trunk decomposition*[22] of $\mathcal{C}/Def(\Sigma)$ and gluing maps[23]:

$$\begin{aligned} I_{(0,\vec{t},\vec{t}',\vec{\mu})} \quad &: \quad \Sigma - \cup_{q:\,\text{node}}\, N_{\sqrt{|t_q|}}(q) \quad \longrightarrow \quad \Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}\,, \\ I_{(0,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} \quad &: \quad \Sigma - \cup_{q:\,\text{node}}\, N_{|t_q|/\varepsilon}(q) \quad \longrightarrow \quad \Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}\,, \end{aligned}$$

where t_q is the entry of $(\vec{t},\vec{t}',\vec{\mu})$ associated to the node q, and $N_{(\cdots)}(q)$ is the $(\cdots)$-neighborhood of q in the local model of node q. We also have a fixed family of diffeomorphisms $\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \simeq \Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}$. The combination of the two defines the gluing maps

$$\begin{aligned} I_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \quad &: \quad \Sigma - \cup_{q:\,\text{node}}\, N_{\sqrt{|t_q|}}(q) \quad \longrightarrow \quad \Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}\,, \\ I_{(\zeta,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} \quad &: \quad \Sigma - \cup_{q:\,\text{node}}\, N_{|t_q|/\varepsilon}(q) \quad \longrightarrow \quad \Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}\,. \end{aligned}$$

These maps satisfy the $Aut(\rho)$-conjugation property that

$$\begin{aligned} \alpha \circ I_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \circ \alpha^{-1} \quad &= \quad I_{\alpha\cdot(\zeta,\vec{t},\vec{t}',\vec{\mu})}\,, \\ \alpha \circ I_{(\zeta,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} \circ \alpha^{-1} \quad &= \quad I_{\alpha\cdot(\zeta,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} \end{aligned}$$

for $\alpha \in Aut(\rho)^{\text{domain}}$ acting on $\mathcal{C}/Def(\Sigma)$. The ε-neck region of the fiber $\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$ of $\mathcal{C}/Def(\Sigma)$ will be denoted by $Neck_{\varepsilon\,,\,(\zeta,\vec{t},\vec{t}',\vec{\mu})}$. It is a disjoint union of annuli/strips, of the form

$$\{(z_1,z_2) \in \mathbb{C}^2 \ :\ z_1 z_2 = t_q\,,\ |z_1| < \varepsilon\,,\ |z_2| < \varepsilon\}$$

[22] Cf. the thick-thin decompsoition in terms of hyperbolic geometry.

[23] Cf. the maps $I_{\vec{\lambda}} : Y_{[k]} - \cup_{i=0}^k N_{\sqrt{|\lambda_i|}}(D_i) \to W[k]_{\vec{\lambda}}$ and $I_{\vec{\lambda},\varepsilon} : Y_{[k]} - \cup_{i=0}^k N_{|\lambda_i|/\varepsilon}(D_i) \to W[k]_{\vec{\lambda}}$ defined in Sec. 1.1.1 by cut-and-glue.

(resp.

$$\{(z_1,z_2)\ :\ z_1z_2=t_q\,,\ |z_1|<\varepsilon\,,\ |z_2|<\varepsilon\}/(z_1,z_2)\sim(\overline{z_2},\overline{z_1})\,,$$
$$\{(z_1,z_2)\ :\ z_1z_2=t_q\,,\ |z_1|<\varepsilon\,,\ |z_2|<\varepsilon\}/(z_1,z_2)\sim(\overline{z_1},\overline{z_2})\ )$$

in $\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$, associated to smoothed interior (resp. type-E boundary, type-H boundary) nodes q of Σ.

To homogenize the notation, we write interchangeably $H_{\rho,\text{target}} := B[k]$ for the deformations of the target $Y_{[k]}$, and $H_{\rho,\,\text{map}} := Ker(\pi_{E_\rho}\circ D_f\bar{\partial}_J)^{\text{pd}}_\Lambda$ as the deformation space of f with the fixed domain Σ and rigid target $Y_{[k]}$. The coordinates for $H_{\rho,\text{target}}\times H_{\rho,\,\text{map}}$ will be denoted by $(\vec{\lambda},\vec{a},\xi)$ with respect to its decomposition as $B[k]\times H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}\times H^{(0,\Lambda)}_{\rho,\,\text{map}}$, (cf. Definition/Convention 5.3.2.1). Recall then

$$\begin{array}{ccccc}
\widetilde{V}_\rho & & \subset & H_\rho := H_{\rho,\text{domain}}\times H_{\rho,\text{target}}\times H_{\rho,\,\text{map}} & \\
{\scriptstyle \pi_{Def(\Sigma)\,\times\,B[k]}}\downarrow & & & \downarrow{\scriptstyle \pi_{Def(\Sigma)\,\times\,B[k]}} & \\
\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho) & & \subset & Def(\Sigma)\times B[k] & .
\end{array}$$

We will use the product coordinates $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\xi)$ of H_ρ for the algebraic subset $\widetilde{V}_\rho$, with $\vec{\lambda}$ being the redundant coordinates expressible in terms of $(\vec{\mu},\vec{a})$, (cf. Sec. 5.3.2).

The intersections

$$\begin{aligned}
\Theta_{\rho,0} &:= (H_{\rho,\text{domain}}\times H_{\rho,\text{target}}\times\{sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda(0)\}\times\{0\})\cap\widetilde{V}_\rho\,,\\
\Theta_\rho &:= (\ H_{\rho,\text{domain}}\times H_{\rho,\text{target}}\times H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}\times\{0\}\ )\cap\widetilde{V}_\rho\\
&\simeq Def(\Sigma;\Lambda)\times\overline{V}_\rho
\end{aligned}$$

in H_ρ are both connected constructible subsets of H_ρ. $\Theta_{\rho,0}$ is a deformation retract of Θ_ρ and, hence, $\pi_{Def(\Sigma)\times B[k]}(\Theta_{\rho,0})$ is a deformation retract of $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$. The restriction of $\pi_{Def(\Sigma)\times B[k]}$ to $\Theta_{\rho,0}$ is one-to-one. Its inverse defines a (continuous) section

$$S_0\ :\ \pi_{Def(\Sigma)\times B[k]}(\Theta_{\rho,0})\ \longrightarrow\ \widetilde{V}_\rho|_{\pi_{Def(\Sigma)\times B[k]}(\Theta_{\rho,0})}$$

with image $\Theta_{\rho,0}$. The restriction of $\pi_{Def(\Sigma)\times B[k]}$ on Θ_ρ is one-to-one only on an open dense subset (i.e. the subset described by $\lambda_i\neq 0$, $i=0,\ldots,k$). It follows that S_0 extends uniquely to a *piecewise-continuous* section

$$S\ :\ \pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)\ \longrightarrow\ \widetilde{V}_\rho\,,$$

whose image has closure Θ_ρ in $\widetilde{V}_\rho$. Both S_0 and S are $Aut(\rho)$-equivariant. As $\widetilde{V}_\rho$ is a bundle over Θ_ρ with fiber $H^{(0,\Lambda)}_{\rho,\,\text{map}}$, this says in particular that, while it is not possible to make all the ingredients in the relative construction (of $\widetilde{V}_\rho$-family of maps) continuous with respect to $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$ in $Def(\Sigma)\times B[k]$, we have to ensure their extendibility and continuity over Θ_ρ. We now proceed to construct a piecewise-continuous-$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of approximate-$J$-holomorphic maps that extends to a continuous-Θ_ρ-family of approximate-J-holomorphic maps.

Fix a rotation-invariant smooth cutoff function $\beta_1:\mathbb{C}\to[0,1]$ such that

$$\beta_1(z)\ =\ \begin{cases} 1 & \text{if } |z|\geq 2\,,\\ 0 & \text{if } |z|\leq 1\,,\end{cases}\qquad\text{and}\quad |\nabla\beta_1\,|\leq 2\,,$$

([MD-S1: Lemma A.1.1]). Then the local model of our approximate-J-stable maps around a smoothed node is given as follows for a fixed $\varepsilon>0$ small and $|t|,|t'|,|\mu|,|\lambda|\ll\varepsilon$. (Cf. [MD-S: Sec. A.2], [F-O: (12.13)], [Liu(C): Sec. 6.4.1], and [L-R: Sec. 4.1].)

(*a*) $H^{(\text{smooth,o.i.n})}_{\rho,\text{domain}}$: The local model of the deformation/smoothing of an ordinary interior node q of Σ is given by

$$\begin{array}{ccc} B_\varepsilon := \{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_1|,|z_2|\le\varepsilon\} & \longrightarrow & \mathbb{C} \\ (z_1,z_2) & \longmapsto & z_1 z_2 \quad . \end{array}$$

Let A_t be the fiber over $t\in\mathbb{C}$ and $f_\rho|_{A_0} = f_1\cup f_2$; then, for $t\neq 0$, define $h_t : A_t \to Y_{[k]}$ by

$$h_t(z,\frac{t}{z}) \;=\; \exp_{f(q)}\left(\beta_1\left(\frac{z}{|t|^{1/4}}\right)\exp^{-1}_{f(q)}(f_1(z)) \;+\; \beta_1\left(\frac{|t|^{3/4}}{z}\right)\exp^{-1}_{f(q)}\left(f_2\left(\frac{t}{z}\right)\right)\right) .$$

(*b*) $H^{(\text{smooth,b.n.})}_{\rho,\text{domain}}$: The local model of the deformation/smoothing of the two types of boundary node q of Σ is given respectively by

$$\text{(type E)}\qquad \begin{array}{ccc} B_\varepsilon/\!\sim_{\text{E}} := \{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_1|,|z_2|\le\varepsilon\}/(z_1,z_2)\sim(\overline{z_2},\overline{z_1}) & \longrightarrow & \mathbb{R}_{\ge 0} \\ (z_1,z_2) & \longmapsto & z_1 z_2 \quad , \end{array}$$

$$\text{(type H)}\qquad \begin{array}{ccc} B_\varepsilon/\!\sim_{\text{H}} := \{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_1|,|z_2|\le\varepsilon\}/(z_1,z_2)\sim(\overline{z_1},\overline{z_2}) & \longrightarrow & \mathbb{R}_{\ge 0} \\ (z_1,z_2) & \longmapsto & z_1 z_2 \quad . \end{array}$$

For q of type E, let $A'_{t'}$ be the fiber over $t'\in\mathbb{R}_{\ge 0}$ and $f_\rho|_{A'_0} = f$; then, for $t'>0$, define $h_{t'} : A'_{t'} \to Y_{[k]}$ by

$$h_{t'}(z,\frac{t'}{z}) \;=\; \exp_{f(q)}\left(\beta_1\left(\frac{z}{|t'|^{1/4}}\right)\exp^{-1}_{f(q)}(f(z))\right) .$$

For q of type H, let $A'_{t'}$ be the fiber over $t'\in\mathbb{R}_{\ge 0}$ and $f_\rho|_{A'_0} = f_1\cup f_2$; then, for $t'>0$, define $h_{t'} : A'_{t'} \to Y_{[k]}$ by

$$h_{t'}(z,\frac{t'}{z}) \;=\; \exp_{f(q)}\left(\beta_1\left(\frac{z}{|t'|^{1/4}}\right)\exp^{-1}_{f(q)}(f_1(z)) \;+\; \beta_1\left(\frac{|t'|^{3/4}}{z}\right)\exp^{-1}_{f(q)}\left(f_2\left(\frac{t'}{z}\right)\right)\right) .$$

(*c*) $\overline{V}_\rho \subset H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}} \times H_{\rho,\text{target}} \times H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}$: Let $q\in\Lambda$ be a distinguished node of contact order s. Recall the fixed local coordinates around $f(q)$. Denote by $(f_1^D, f_1^N)\cup(f_2^D, f_2^N)$ the restriction of f around q with the expression in terms of the coordinates on D and the normal coordinate to D around $f(q)$. Recall also the local model in Sec. 5.3.2 (cf. [I-P2])

$$\begin{array}{ccccc} B_\varepsilon & & \longrightarrow & \mathbb{C}^2 & \\ (z_1,z_2) & & & (w_1,w_2) = (a_1\, {z_1}^s\,,\, a_2\, {z_2}^s) & \\ \downarrow & & \downarrow & & a_1,\, a_2 \;\in \mathbb{C}-\{0\}\,, \\ \mathbb{C} & & \longrightarrow & \mathbb{C} & \\ & \mu = z_1 z_2 & & \lambda \;=\; w_1 w_2 \;=\; a_1 a_2\, \mu^s \;=\; a\,\mu^s \quad , & \end{array}$$

that links the deformation/smoothing (here parameterized by μ) of the node q, the deformation (here parameterized by λ) of $Y_{[k]}$ along the D_i that contains $f(q)$, and the product (here parameterized by a) of the lowest-order pre-deformable deformations of the normal-to-D component of the germ of f on the two branches of Σ at q.

Let (μ,λ,a) be the relevant coordinates in the coordinates of $H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}} \times H_{\rho,\text{target}} \times H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}$ with $\lambda = a\mu^s$. Define $h_{(\mu,\lambda,a)} = (h^D_{(\mu,\lambda,a)}, h^N_{(\mu,\lambda,a)})$ as follows:

· For $\lambda = 0$: recall the ξ_a in $Ker(\pi_{E_\rho} \circ D_f \bar\partial_J)^{\rm pd}$ associated to a and define

$$h_{(0,0,a)}(\,\cdot\,) \;=\; \exp_{f(\,\cdot\,)}(\xi_a(\,\cdot\,))\,.$$

· For $\lambda \neq 0$: express $h_a := h_{(0,0,a)}$ above as $h_{a,1} \cup h_{a,2} = (h^D_{a,1}, h^N_{a,1}) \cup (h^D_{a,2}, h^N_{a,2})$ and define

$$h^D_{(\mu,\lambda,a)}(z, \tfrac{\mu}{z}) \;=\; \exp_{h_a(q)}\Big(\beta_1\Big(\tfrac{z}{|\mu|^{1/4}}\Big) \exp^{-1}_{h_a(q)}(h^D_{a,1}(z)) \,+\, \beta_1\Big(\tfrac{|\mu|^{3/4}}{z}\Big) \exp^{-1}_{h_a(q)}\big(h^D_{a,2}\big(\tfrac{\mu}{z}\big)\big)\Big)\,,$$

$$h^N_{(\mu,\lambda,a)}(z, \tfrac{\mu}{z}) \;=\; \begin{cases} \beta_1\Big(\tfrac{z}{|\mu|^{1/4}}\Big)\, h^N_{a,1}(z) \,+\, \beta_1\Big(\tfrac{|\mu|^{3/4}}{z}\Big)\, \sqrt{a}\, z^s & \text{for } |\mu|^{1/2} \leq |z| \leq \varepsilon\,, \\ \beta_1\Big(\tfrac{z}{|\mu|^{1/4}}\Big)\, \sqrt{a}\, \big(\tfrac{\mu}{z}\big)^s \,+\, \beta_1\Big(\tfrac{|\mu|^{3/4}}{z}\Big)\, h^N_{a,2}(\tfrac{\mu}{z}) & \text{for } |\mu|^{1/2} \leq |\mu/z| \leq \varepsilon\,, \end{cases}$$

where $\sqrt{a}$ is chosen so that $\sqrt{a_f}$ fits the normal-form expression of f at q.

This describes what happens on a smoothed neighborhood of q with all irrelevant indices of the coordinates of $H^{(\text{smooth},\Lambda)}_{\rho,\text{domain}} \times H_{\rho,\text{target}} \times H^{(\text{loc},\Lambda)}_{\rho,\,\text{map}}$ suppressed. The substitutions $\mu \to \mu_{ij}$, $s \to s_{ij}$, $\lambda \to \lambda_i$, $a \to a_{ij}$, for $i = 0, \ldots, k$, $j = 1, \ldots, |\Lambda_i|$ to the above expression recover the complete $\overline{V}_\rho$-family of maps from $\Lambda_{0,\vec t,\vec t'}$ to the fiber $W[k]_{\vec\lambda}$ of $W[k]/B[k]$.

By construction, these maps are defined on disjoint subsets of $\Sigma_{(0,\vec t,\vec t',\vec\mu)}$ and coincide with f on their intersection with a compact subset $K_{\varepsilon_{\varepsilon_-}}$ of Σ by removing a small ε_--neighborhood of all the nodes, with ε_- slightly less than ε. As $W[k]_{\vec\lambda}$ are obtained from gluing truncated $Y_{[k]}$ around $\vec\lambda$-specified D_i's (cf. Sec. 1.1.1), they can be combined with and extended by $f|_{K_{\varepsilon_-}}$ to a map from $\Sigma_{(0,\vec t,\vec t',\vec\mu)}$ to $W[k]_{\vec\lambda}$. In this way, one obtains a (continuous-)$H^{(\text{smooth,o.i.n})}_{\rho,\text{domain}} \times H^{(\text{smooth,b.n.})}_{\rho,\text{domain}} \times \overline{V}_\rho$-family of maps

$$h_{\text{approx},(0,\vec t,\vec t',\vec\mu,\vec\lambda,\vec a,\vec 0)} \;:\; \Sigma_{(0,\vec t,\vec t',\vec\mu)} \;\longrightarrow\; W[k]_{\vec\lambda}\,.$$

For $\zeta \in H^{(\text{deform},\Sigma)}_{\rho,\text{domain}}$, one defines $h_{\text{approx},(\zeta,\vec t,\vec t',\vec\mu,\vec\lambda,\vec a,\vec 0)} : \Sigma_{(\zeta,\vec t,\vec t',\vec\mu)} \to W[k]_{\vec\lambda}$ by setting

$$h_{\text{approx},(\zeta,\vec t,\vec t',\vec\mu,\vec\lambda,\vec a,\vec 0)} \;=\; h_{\text{approx},(0,\vec t,\vec t',\vec\mu,\vec\lambda,\vec a,\vec 0)}.$$

(In all the discussion, though the $\vec\lambda$-label is determined uniquely by $(\vec\mu,\vec a)$, we keep it in the notation to remind us of the change of the target.) To summarize:

Lemma 5.3.3.1 [pre-deformable Θ_ρ-family]. *$h_{\text{approx},(\zeta,\vec t,\vec t',\vec\mu,\vec\lambda,\vec a,\vec 0)}$, $(\zeta,\vec t,\vec t',\vec\mu,\vec\lambda,\vec a) \in \Theta_\rho$, defines a (continuous-)Θ_ρ-family of C^∞ maps of the same contact order and pre-deformability behavior as f at un-smoothed distinguished nodes of Σ.*

To keep the relative-to-(domain, target)-construction picture manifest, one should think of this Θ_ρ-family of maps as an extension/completion-at-$[f]$ of the corresponding (piecewise-continuous-)$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of maps via the open-dense embedding $S : \pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho) \hookrightarrow \Theta_\rho$. The Θ_ρ-family of maps can be extended further to a (continuous-)$\widetilde{V}_\rho$-family of maps by defining first

$$h_{\text{approx},(0,\vec 0,\vec 0',\vec 0,\vec 0,\vec a,\vec b)}(\,\cdot\,) \;=\; \exp_{f(\,\cdot\,)}\xi_{(\vec a,\vec b)}(\,\cdot\,)\,.$$

This is a $H_{\rho,\,\text{map}}$-family of C^∞ maps from Σ to $Y_{[k]}$ for which pre-deformability at each distinguished node remains hold with the same order. Repeating then the above construction

that deforms the map at the three types of nodes with f replaced by $h_{\text{approx},(0,\vec{0},\vec{0}',\vec{0},\vec{0},\vec{a},\vec{b})}$. For $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})\in\Theta_\rho$ this gives $h_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0}))}$ as constructed above.

Lemma 5.3.3.2 [$\widetilde{V}_\rho$-family of pre-deformable approximate-J-holomorphic maps]. *Assume that $\|\zeta\|$, $\|\vec{t}\|$, $\|\vec{t}'\|$, $\|\vec{\mu}\|$, $\|\vec{a}-\vec{a}_f\|$, $\|\vec{b}\|$ are all sufficiently small (say, bounded above uniformly by an $\varepsilon\ll 1$), then*

$$\left\| \bar\partial_J h_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \right\|_{L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})})} \;\le\; C\,\left(\|\zeta\| + \|\vec{t}\| + \|\vec{t}'\| + \|\vec{\mu}\| + \|\vec{a}-\vec{a}_f\| + \|\vec{b}\| \right)^{\frac{1}{2p}} ,$$

where C is a constant that depends only on ε, f, ∇f, J, ∇J, the norm of the differential of $sp_\Lambda\circ jet^{\mathbf{s}}_\Lambda$, and the norm of the differential of the exponential map and its inverse along f. Thus, $h_{\text{approx},(\,\cdot\,)}$ gives a (continuous) $Aut(\rho)$-invariant $\widetilde{V}_\rho$-family of pre-deformable approximate-J-holomorphic maps.

Proof. The approximate J-holomorphy property follows from [MD-S1: Lemma A.4.3], [F-O: Lemma 12.14, Lemma 12.15], [Liu(C): Lemma 6.22], and [L-R: Lemma 4.6]. Here we have assumed that $\|\zeta\|$, $\|\vec{t}\|$, $\|\vec{t}'\|$, $\|\vec{\mu}\|$, $\|\vec{a}-\vec{a}_f\|$, $\|\vec{b}\|$ are all sufficiently small so that the combination of all the estimates in ibidem is bounded above by the right-hand side of the inequality above. The $Aut(\rho)^{\text{domain}}$-invariance of the domain decomposition involved, the $Aut(\rho)^{\text{target}}$-invariance of the metric on $W[k]$, and the cutoff function chosen imply that the gluing construction is $Aut(\rho)$-invariant. This implies that the $\widetilde{V}_\rho$-family of maps as constructed is $Aut(\rho)$-invariant. □

Notation 5.3.3.3. We will assume that $\|\zeta\|$, $\|\vec{t}\|$, $\|\vec{t}'\|$, $\|\vec{\mu}\|$, $\|\vec{a}-\vec{a}_f\|$, $\|\vec{b}\|$ are all sufficiently small so that $h_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(\cdot)=\exp_{h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\,\cdot\,)}\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(\,\cdot\,)$ for a unique

$$\begin{aligned}&\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\\ &\in\; W^{1,p}\left(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})},\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})};h^*_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}}),(h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}})^*T_*L\right).\end{aligned}$$

This expression renders the $\widetilde{V}_\rho$-family of maps a continuous extension of the Θ_ρ-family of maps by the exponential-map construction along the $H^{(0,\Lambda)}_{\rho,\,\text{map}}$-factor directions; this helps making the later relative construction over $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$ manifest.

5.3.4 The $\widetilde{V}_\rho$-family of (exact) $(J,E_\bullet)$-stable maps f. to fibers of $W[k]/B[k]$.

In this subsubsection, we extend the $Aut(\rho)$-invariant obstruction space E_ρ at ρ step by step to trivialized $Aut(\rho)$-equivariant auxiliary obstruction bundles $E^{\text{aux}}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$ over $S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))$, E_{Θ_ρ} over Θ_ρ, and $E_{\widetilde{V}_\rho}$ over $\widetilde{V}_\rho$. We then deform the $Aut(\rho)$-invariant $\widetilde{V}_\rho$-family of approximate-J-stable C^∞ maps in Sec. 5.3.3 to a (continuous) $Aut(\rho)$-invariant $\widetilde{V}_\rho$-family of $(J,E_\bullet)$-stable maps. The major step is a construction of a $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of right inverses (of $\pi_{E^{\text{aux}}_\bullet}\circ D_{h_{\text{approx},\,\cdot}}\bar\partial_J$)

$$\begin{aligned}Q_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}\;:\;& L^p\left(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})};\Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}\otimes_J h^*_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*W[k]_{\vec{\lambda}}\right)\Big/E^{\text{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}\;\longrightarrow\\ & W^{1,p}\left(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})},\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})};h^*_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*W[k]_{\vec{\lambda}},(h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}})^*T_*L\right)\end{aligned}$$

to deform the $\widetilde{V}_\rho$-family of approximate-J-stable C^∞ maps

$$h_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \ : \ (\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}) \ \longrightarrow \ (W[k]_{\vec{\lambda}}, L)$$

recursively to a $\widetilde{V}_\rho$-family of $(J, E_\bullet)$-stable maps

$$f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \ : \ (\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}) \ \to \ (W[k]_{\vec{\lambda}}, L) \, .$$

Such construction is provided by [MD-S1: Sec. 3.3 and A.4] and its extensions to various situations in [F-O], [Liu(C)], and [L-R]. The discussion below follows these four works with mild necessary modifications to fit our overall presentation and notations.

Throughout the discussion, we assume that $\|\zeta\|$, $\|\vec{t}\|$, $\|\vec{t}'\|$, $\|\vec{\mu}\|$, $\|\vec{a} - \vec{a}_f\|$, $\|\vec{b}\|$, and, hence, $\widetilde{V}_\rho$ are all sufficiently small so that statements in the construction hold.

The $Aut(\rho)$-equivariant auxiliary bundle $E^{\text{aux}}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$ over $S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))$.

Introduce first the following operations. Let $P_{\bullet,\bullet'}$ be the parallel transport from point $\bullet$ to point $\bullet'$ along the minimal geodesic on a $W[k]_{\vec{\lambda}}$ for $\bullet, \bullet' \in W[k]_{\vec{\lambda}}$ of distance $<$ the injective radius of $W[k]_{\vec{\lambda}}$ and $P'_{\bullet,\bullet'}$ be its J-linear part. For $\eta \in L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}))$, define

$$P'_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\eta \ \in \ L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} T_*(W[k]_{\vec{\lambda}}))$$

by

$$(P'_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\eta)(x) \ = \ P'_{h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x)\, , h_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(x)}\, \eta(x) \, , \quad x \in \Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \, .$$

This is the J-linear parallel transport along the geodesic determined by $\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(x)$ in Notation 5.3.3.3. Recall also the gluing maps for domain curves and targets spaces: (with $\varepsilon > 0$ small and fixed, and $\|(\vec{t}, \vec{t}', \vec{\lambda})\| \ll \varepsilon$)

$$\begin{array}{lcll}
I_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} & : & \Sigma - \cup_{q:\,\text{node}}\, N_{\sqrt{|t_q|}}(q) & \longrightarrow \ \Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \, , \\
I_{(\zeta,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} & : & \Sigma - \cup_{q:\,\text{node}}\, N_{|t_q|/\varepsilon}(q) & \longrightarrow \ \Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \, , \\
I_{\vec{\lambda}} & : & Y_{[k]} - \cup_{i=0}^k\, N_{\sqrt{|\lambda_i|}}(D_i) & \longrightarrow \ W[k]_{\vec{\lambda}} \, , \\
I_{\vec{\lambda},\varepsilon} & : & Y_{[k]} - \cup_{i=0}^k\, N_{|\lambda_i|/\varepsilon}(D_i) & \longrightarrow \ W[k]_{\vec{\lambda}}
\end{array}$$

and conjugation properties:

$$\begin{array}{rcl}
\alpha \circ I_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \circ \alpha^{-1} & = & I_{\alpha\cdot(\zeta,\vec{t},\vec{t}',\vec{\mu})} \, , \\
\alpha \circ I_{(\zeta,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} \circ \alpha^{-1} & = & I_{\alpha\cdot(\zeta,\vec{t},\vec{t}',\vec{\mu}),\varepsilon} \, , \\
\beta \circ I_{\vec{\lambda}} \circ \beta^{-1} & = & I_{\beta\cdot\vec{\lambda}} \, , \\
\beta \circ I_{\vec{\lambda},\varepsilon} \circ \beta^{-1} & = & I_{\beta\cdot\vec{\lambda},\varepsilon}
\end{array}$$

for $\alpha \in Aut(\rho)^{\text{domain}}$ acting on $\mathcal{C}/Def(\Sigma)$ and $\beta \in Aut(\rho)^{\text{target}}$ action on $W[k]/B[k]$.

Since E_ρ is supported in a compact subset in the complement of all three types of nodes of Σ, it can be canonically realized as a subspace in $L^p(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\text{approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*($

$W[k]_{\vec{\lambda}})$) via the composition $I_{\vec{\lambda}\,*} \circ I^{-1\,*}_{(0,\vec{t},\vec{t}',\vec{\mu})} \circ P'_{(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ on E_ρ. Define $E^{\rm aux}_{S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ to be this subspace in $L^p(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}))$. To extend the above along the $H^{({\rm deform},\Sigma)}_{\rho,{\rm domain}}$-factor, note that $E^{\rm aux}_{S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ is canonically a subspace of $L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Omega^1_{\mathbb{C}}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}))$ via the composition $\Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \hookrightarrow \Omega^1_{\mathbb{C}}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \stackrel{\sim}{\to} \Omega^1_{\mathbb{C}}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$ of the canonical inclusion and the fixed isomorphism from the fixed $\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \simeq \Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$. The restriction to $E^{\rm aux}_{S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ of the following projection map

$$\begin{aligned} P^{0,1}_\zeta \;:\; & L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Omega^1_{\mathbb{C}}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}})) \\ & \longrightarrow\; L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}})) \end{aligned}$$

induced by the projection map $\Omega^1_{\mathbb{C}}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \to \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$ is injective for $\|\zeta\|$ sufficiently small. Define $E^{\rm aux}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ to be the image of $E^{\rm aux}_{S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ in $L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}))$ under this projection. This gives a trivialized vector bundle $E^{\rm aux}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$ over $S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))$. One can further define

$$E^{\rm aux}_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;:=\; \left\{ P'_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\,\eta \;:\; \eta \in E^{\rm aux}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \right\}$$

to extend $E^{\rm aux}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$ to a trivialized vector bundle $E^{\rm aux}_{\widetilde{V}_\rho}$ over $\widetilde{V}_\rho$, with specified isomorphisms of fibers to E_ρ. In particular, $E^{\rm aux}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$ extends to $E_{\Theta_\rho} := E^{\rm aux}_{\widetilde{V}_\rho}|_{\Theta_\rho}$ over Θ_ρ. The various group-invariance and conjugation properties of the objects and maps used in the construction implies that these trivialized bundles are $Aut(\rho)$-equivariant.

Definition 5.3.4.1 [auxiliary obstruction bundle]. We will call the $Aut(\rho)$-equivariant trivialized bundle $E^{\rm aux}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$ (resp. $E^{\rm aux}_{\Theta_\rho}$, $E^{\rm aux}_{\widetilde{V}_\rho}$) as constructed above the *auxiliary obstruction bundle* over $S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))$ (resp. Θ_ρ, $\widetilde{V}_\rho$) induced by E_ρ at ρ.

$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of right inverse $Q_\bullet$ of $\pi_{E^{\rm aux}_\bullet} \circ D_\bullet\bar{\partial}_J$ from approximate one.

Let $Ker(\pi_{E_\rho} \circ D_f\bar{\partial}_J)^\perp$ be the L^2-orthogonal complement of $Ker(\pi_{E_\rho} \circ D_f\bar{\partial}_J)$ in $W^{1,p}(\Sigma, \partial\Sigma;$ $f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L)$. This space is $Aut(\rho)$-invariant, as the metric on Σ and $W[k]$ are respectively $Aut(\rho)^{\rm domain}$- and $Aut(\rho)^{\rm target}$-invariant. Then

$$\pi_{E_\rho} \circ D_f\bar{\partial}_J \;:\; Ker(\pi_{E_\rho} \circ D_f\bar{\partial}_J)^\perp \;\longrightarrow\; L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\big/E_\rho$$

is an isomorphism and its inverse

$$Q_\rho \;:\; L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\big/E_\rho \;\longrightarrow\; Ker(\pi_{E_\rho} \circ D_f\bar{\partial}_J)^\perp$$

is a bounded operator. This defines Q_ρ as a right inverse of

$$\pi_{E_\rho} \circ D_f\bar{\partial}_J \;:\; W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \;\longrightarrow\; L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\big/E_\rho\,.$$

We now proceed to construct first a suitable $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of approximate right inverse $Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ of

$$\begin{aligned}
&\pi_{E^{\rm aux}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ D_{h_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}\bar{\partial}_J \ : \\
&\quad W^{1,p}(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}}), (h_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}})^*T_*L) \\
&\qquad \longrightarrow \ L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}}))\Big/ E^{\rm aux}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}
\end{aligned}$$

by passing to Q_ρ at ρ.

The combination of $I_{(0,\vec{t},\vec{t}',\vec{\mu})}$ and $I_{\vec{\lambda}}$ on domains and targets induces a map

$$\begin{aligned}
I^*_{(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \ &: L^p(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}})) \\
&\longrightarrow \ L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})
\end{aligned}$$

by first using $I_{\vec{\lambda}}$ to turn an

$$\eta \ \in \ L^p(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}}))$$

to an element

$$\eta' \ = \ I^*_{\vec{\lambda}}\eta \ \in \ L^p(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J (I^{-1}_{\vec{\lambda}} \circ h_{{\rm approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})})^*T_*Y_{[k]})$$

and then using parallel transport $P_{(I^{-1}_{\vec{\lambda}}\circ h_{{\rm approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(I_{(0,\vec{t},\vec{t}',\vec{\mu})}(x)),\, f(x)}$ on $Y_{[k]}$ for $x \in I^{-1}_{(0,\vec{t},\vec{t}',\vec{\mu})}(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}) \subset \Sigma$ to move η' to an element

$$\eta'' \ = \ P_{\bullet,\bullet}(\eta') \ =: \ I^*_{(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\eta) \ \in \ L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\,.$$

The composition of

$$\begin{aligned}
& L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}})) \\
\stackrel{P_0^{0,1}}{\longrightarrow} \quad & L^p(\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(0,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}})) \\
\stackrel{I^*_{(0,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}{\longrightarrow} \quad & L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})
\end{aligned}$$

gives the map

$$\begin{aligned}
IP_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \ &: \ L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{{\rm approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}})) \\
&\longrightarrow \ L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})\,.
\end{aligned}$$

Fix a C^∞ rotation-invariant cutoff function $\beta_\delta : \mathbb{C} \to \mathbb{R}$ as in [MD-S1: Lemma A.1.1] with the following properties: (mixed with the presentation of [F-O])

$$\beta_\delta(z) \ = \ \begin{cases} 1 & \text{if } |z| \ \le \ \delta \ (< 1) \\ 0 & \text{if } |z| \ge 1 - o \\ & \quad \text{for some } 0 < o \ll 1 - \delta \end{cases} \qquad \text{and} \quad \int_{|z|\le 1} |\nabla\beta_\delta(z)|^2 \ \le \ \frac{4\,\pi}{|\log\delta|}\,;$$

and recall that $\varepsilon > 0$ is small and fixed, and $\|(\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})\| \ll \varepsilon$. Define the map.

$$Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} : W^{1,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L) \longrightarrow$$
$$W^{1,p}(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; h^*_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}), (h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}})^*T_*L)$$

by gluing locally defined bundle-valued fields to a continuous field as follows. Let $\xi \in W^{1,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L)$ and recall the gluing construction of the maps $h_{\mathrm{approx},\bullet}$ in Sec. 5.3.3.

(o) For x in $\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} - Neck_{\varepsilon,(\zeta,\vec{t},\vec{t}',\vec{\mu})}$, define

$$(Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\xi))(x) \;=\; \left(I_{\vec{\lambda}*} \circ P_{f(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(I_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}{}^{-1}(x)) \,.$$

(a) For x in the *annulus A_t from a smoothed ordinary interior node q*, let $x = (z, \frac{t}{z}) \in \mathbb{C}^2$ in the local model in Sec. 5.3.3, where t is an entry of $\vec{t}$ involved and $\xi = \xi_1 \cup \xi_2$ on the two irreducible components the neighborhood of $q = (0,0)$ in $\{(z_1,z_2) : z_1 z_2 = 0, |z_1| < \varepsilon, |z_2| < \varepsilon\} \subset \Sigma$, with $\xi_1 = \xi_1(z_1)$ and $\xi_2 = \xi_2(z_2)$. Define

$$(Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\xi))(x)$$
$$= \begin{cases} \left(I_{\vec{\lambda}*} \circ P_{f_1(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_1(z) \\ \quad + \left(1 - \beta_\delta(\frac{|t|^{1/2}}{z})\right) \Big(\left(I_{\vec{\lambda}*} \circ P_{f_2(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_2(\frac{t}{z}) \\ \qquad\qquad - \left(I_{\vec{\lambda}*} \circ P_{f(q),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \Big) \\ \qquad\qquad\qquad\qquad \text{for } \; |t|^{1/2} \le |z| \le \varepsilon \,, \\ \left(I_{\vec{\lambda}*} \circ P_{f_2(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_2(\frac{t}{z}) \\ \quad + \left(1 - \beta_\delta(\frac{z}{|t|^{1/2}})\right) \Big(\left(I_{\vec{\lambda}*} \circ P_{f_1(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_1(z) \\ \qquad\qquad - \left(I_{\vec{\lambda}*} \circ P_{f(q),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \Big) \\ \qquad\qquad\qquad\qquad \text{for } \; |t|^{1/2} \le |t/z| \le \varepsilon \,. \end{cases}$$

(b) For x in the *annulus $A'_{t'}$, $t' > 0$, from smoothing a type E boundary node q*, let $x = (z, \frac{t'}{z})$ in the local model in Sec. 5.3.3, where t' is an entry of $\vec{t}'$ involved, and define

$$(Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\xi))(x)$$
$$= \left(I_{\vec{\lambda}*} \circ P_{f(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(z)$$
$$- \left(1 - \beta_\delta(\frac{|t'|^{1/2}}{z})\right) \left(I_{\vec{\lambda}*} \circ P_{f(q),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q)$$
$$\text{for } \; |t'|^{1/2} \le |z| \le \varepsilon \,.$$

For x in the *band* $A'_{t'}$, $t' > 0$, *from smoothing a type H boundary node* q, let $x = (z, \frac{t'}{z})$ in the local model in Sec. 5.3.3, where t' is an entry of $\vec{t}'$ involved, $\xi = \xi_1 \cup \xi_2$ on the two irreducible components the neighborhood of $q = (0,0)$ in $\{(z_1, z_2) : z_1 z_2 = 0, |z_1| < \varepsilon, |z_2| < \varepsilon\}/(z_1, z_2) \sim (\overline{z_1}, \overline{z_2}) \subset \Sigma$, with $\xi_1 = \xi_1(z_1)$ and $\xi_2 = \xi_2(z_2)$, and define

$$
\begin{aligned}
&(Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\xi))(x) \\
&= \begin{cases}
\left(I_{\vec{\lambda}*} \circ P_{f_1(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_1(z) \\
\quad + \left(1 - \beta_\delta(\frac{|t'|^{1/2}}{z})\right) \Big(\left(I_{\vec{\lambda}*} \circ P_{f_2(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_2(\frac{t'}{z}) \\
\qquad\qquad - \left(I_{\vec{\lambda}*} \circ P_{f(q),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \Big) \\
\qquad\qquad\qquad \text{for } |t'|^{1/2} \le |z| \le \varepsilon\,, \\
\left(I_{\vec{\lambda}*} \circ P_{f_2(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_2(\frac{t'}{z}) \\
\quad + \left(1 - \beta_\delta(\frac{z}{|t'|^{1/2}})\right) \Big(\left(I_{\vec{\lambda}*} \circ P_{f_1(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(z)),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_1(z) \\
\qquad\qquad - \left(I_{\vec{\lambda}*} \circ P_{f(q),\, I^{-1}_{\vec{\lambda}}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \Big) \\
\qquad\qquad\qquad \text{for } |t'|^{1/2} \le |t/z| \le \varepsilon\,.
\end{cases}
\end{aligned}
$$

(*c*) For x in the *annulus* A_μ *from a smoothed distinguished interior node* q, let $x = (z, \frac{\mu}{z})$ in the local model in Sec. 5.3.3, where μ is an entry of $\vec{\mu}$ involved. Suppose that $f(q) \in D_i \subset Y_{[k],sing}$; then denote the restriction of $I_{\vec{\lambda},\varepsilon}$ to Δ_i (resp. Δ_{i+1}) by $I^{f(q),1}_{\vec{\lambda},\varepsilon}$ (resp. $I^{f(q),2}_{\vec{\lambda},\varepsilon}$). Define

$$
\begin{aligned}
&(Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\xi))(x) \\
&= \begin{cases}
\left({I^{f(q),1}_{\vec{\lambda},\varepsilon}}_* \circ P_{f_1(I^{-1}_{(\zeta,\varepsilon,\vec{t},\vec{t}',\vec{\mu})}(x)),\, (I^{f(q),1}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_1(z) \\
\quad + \left(1 - \beta_\delta(\frac{|\mu|^{1/2}}{z})\right) \Big(-\tfrac{1}{2} \left({I^{f(q),1}_{\vec{\lambda},\varepsilon}}_* \circ P_{f(q),\, (I^{f(q),1}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \\
\qquad\qquad + \left({I^{f(q),2}_{\vec{\lambda},\varepsilon}}_* \circ P_{f_2(I^{-1}_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}(x)),\, (I^{f(q),2}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_2(\tfrac{\mu}{z}) \\
\qquad\qquad - \tfrac{1}{2} \left({I^{f(q),2}_{\vec{\lambda},\varepsilon}}_* \circ P^2_{f(q),\, (I^2_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \Big) \\
\qquad\qquad\qquad \text{for } |\mu|^{1/2} \le |z| \le \varepsilon_1\,, \\
\left({I^{f(q),2}_{\vec{\lambda},\varepsilon}}_* \circ P_{f_2(I^{-1}_{(\vec{\zeta},\vec{t},\vec{t}',\vec{\mu})}(x)),\, (I^{f(q),2}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_2(\tfrac{\mu}{z}) \\
\quad + \left(1 - \beta_\delta(\frac{z}{|\mu|^{1/2}})\right) \Big(-\tfrac{1}{2} \left({I^{f(q),2}_{\vec{\lambda},\varepsilon}}_* \circ P_{f(q),\, (I^{f(q),2}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \\
\qquad\qquad + \left({I^{f(q),1}_{\vec{\lambda},\varepsilon}}_* \circ P_{f_1(I^{-1}_{(\vec{\zeta},\vec{t},\vec{t}',\vec{\mu})}(x)),\, (I^{f(q),1}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi_1(z) \\
\qquad\qquad - \tfrac{1}{2} \left({I^{f(q),1}_{\vec{\lambda},\varepsilon}}_* \circ P_{f(q),\, (I^{f(q),1}_{\vec{\lambda},\varepsilon})^{-1}(h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(x))} \right) \xi(q) \Big) \\
\qquad\qquad\qquad \text{for } |\mu|^{1/2} \le |\mu/z| \le \varepsilon_1\,.
\end{cases}
\end{aligned}
$$

Then the composition

$$
\begin{aligned}
Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} &:= \mathit{Glue}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \circ Q_\rho \circ \mathit{IP}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \\
&: L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}})) \Big/ E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \longrightarrow \\
&\quad W^{1,p}(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; h^*_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(T_*W[k]_{\vec{\lambda}}), (h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu}}})^* T_* L)\,,
\end{aligned}
$$

where we regard Q_ρ as a linear map on $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$ that is 0 on E_ρ, has the following property:

Lemma 5.3.4.2 [approximate right inverse]. *$Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ is an approximate right inverse of $\pi_{E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ D_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \bar{\partial}_J$ in the sense that*

$$
\left\| \left(\pi_{E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ D_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \bar{\partial}_J \right) \circ Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\eta) \, - \, \eta \right\|_{L^p} \, \leq \, \frac{1}{2} \|\eta\|_{L^p}
$$

for $\|\zeta\|$, $\|\vec{t}\|$, $\|\vec{t}'\|$, $\|\vec{\mu}\|$ small enough.

Proof. See [MD-S1: Lemma A.4.2], [F-O: Lemma 13.11], [Liu(C): Proposition 6.30], [L-R: proof of Lemma 4.8].

□

Recall the universal approximate-J-holomorphic map $h_{\mathrm{approx}} : \mathcal{C}/\Theta_\rho \to W[k]/B[k]$ associated to the family $h_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}$, $(\zeta, \vec{t}, \vec{t}', \vec{\mu}, \vec{\lambda}, \vec{a}, \vec{0}) \in \Theta_\rho$. The following definition is inspired by the built-in family-treatment in the study of moduli problems in algebraic geometry and the fact that a $W^{k,p}$ Sobolev space is the completion of the related C^∞ space with the $W^{k,p}$ norm:

Definition 5.3.4.3 [continuous-$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of operators]. A collection of linear operators

$$
\begin{aligned}
O_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} &: L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}})) \Big/ E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \longrightarrow \\
&\quad W^{1,p}(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; h^*_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(T_*W[k]_{\vec{\lambda}}), (h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu}}})^* T_* L)\,,
\end{aligned}
$$

over $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$ are said to form a *continuous-$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family* of operators if the collection can be enlarged to a collection of linear operators

$$
\begin{aligned}
O_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})} &: L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})} T_*(W[k]_{\vec{\lambda}})) \Big/ E^{\mathrm{aux}}_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})} \longrightarrow \\
&\quad W^{1,p}(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; h^*_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}(T_*W[k]_{\vec{\lambda}}), (h_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu}}})^* T_* L)\,,
\end{aligned}
$$

over Θ_ρ such that, for all $\eta \in C^\infty(\mathcal{C}; \Lambda^{0,1}_{\mathcal{C}/\Theta_\rho} \otimes h^*_{\mathrm{approx}} T_{W[k]/B[k]}/C^\infty(E^{\mathrm{aux}}_{\Theta_\rho})$ with $\eta|_{C_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}}$ $\in C^\infty(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})} T_*(W[k]_{\vec{\lambda}})) \Big/ E^{\mathrm{aux}}_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}$, there exists a $\xi \in C^0(\mathcal{C}; h^*_{\mathrm{approx}} T_{W[k]/B[k]})$ such that $\xi|_{C_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}} = O_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}(\eta|_{C_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}})$.

Proposition 5.3.4.4 [continuous-$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family of right inverse]. *For $\|\zeta\|$, $\|\vec{t}\|$, $\|\vec{t}'\|$, $\|\vec{\mu}\|$ small enough, there exist a constant c and right inverses $Q_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ of $\pi_{E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ$*

$D_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}\bar{\partial}_J$ *such that their operator norm is uniformly bounded by* c *and that they form a continuous-*$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$*-family of linear operators in the sense of Definition 5.3.4.3.*

Proof. Lemma 5.3.4.2 implies that $\left(\ \pi_{E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ D_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}\bar{\partial}_J\ \right)\circ Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ is invertible. A right inverse of $\pi_{E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ D_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}\bar{\partial}_J$ is thus given by

$$Q_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \ = \ Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \circ \left(\left(\ \pi_{E^{\mathrm{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ D_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}\bar{\partial}_J\ \right)\circ Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}\right)^{-1},$$

(cf. [MD-S1: Sec. 3.3], [F-O: (13.2)], [Liu(C): Corollary 6.31], and [L-R: (4.31)]).

To see that they form a continuous-$\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$-family, recall from Notation 5.3.3.3 that $h_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(\cdot) = \exp_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}(\,\cdot\,)}\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(\,\cdot\,)$ for a unique

$$\begin{aligned}&\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\\ &\in\ W^{1,p}\left(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; h^*_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}T_*(W[k]_{\vec{\lambda}}), (h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}|_{\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}})^*T_*L\right).\end{aligned}$$

Using the parallel transport along the geodesic determined by $\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}(x)$, one can extend the collections of operators $IP_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$, $Glue_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ to the collections of operators $IP_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$, $Glue_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ over $\widetilde{V}_\rho$, and, hence, in particular over Θ_ρ. The collections of operators

$$\begin{aligned}Q'_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})} \ &:= \ Glue_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}\circ Q_\rho \circ IP_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})},\\ Q_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})} \ &:= \ Q'_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}\circ\left(\left(\ \pi_{E^{\mathrm{aux}}_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}}\circ D_{h_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}}\bar{\partial}_J\ \right)\circ Q'_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}\right)^{-1}\end{aligned}$$

extend the collections $Q'_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$, $Q_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$. The explicit expressions of $IP_\bullet$ and $Glue_\bullet$ imply that these operators over Θ_ρ together satisfy the continuous-family behavior required in Definition 5.3.4.3.

□

Newton-Picard iteration: deforming $h_{\mathrm{approx},\bullet}$ to a $(J,E_\bullet)$-holomorphic map $f_\bullet$.

Once one realized that the continuity of the relative construction has to be made over Θ_ρ, not directly over $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)\subset Def(\Sigma)\times B[k]$, and has constructed the ingredients accordingly, the rest of the discussion is similar to those in [MD-S1: proof of Theorem 3.3.4], [F-O: pp. 987-988] (directly on the maps), and [Liu(C): proof of Proposition 6.32]; see also [I-P2: Sec. 9] and [L-R: proof of Proposition 4.10]. We give a sketch below to conclude the discussion.

Beginning with the $\widetilde{V}_\rho$-family of maps $h_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$, define a sequence of $\widetilde{V}_\rho$-family of maps as follows:

· Set

$$\begin{aligned}h_{1,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \ &= \ h_{\mathrm{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \ = \ \exp_{h_{\mathrm{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}}\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})},\\ \xi_{1,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \ &= \ \xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}.\end{aligned}$$

· Suppose that $h_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} = \exp_{h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \xi_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ is defined, let

$$\xi_{n+1,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;=\; \xi_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu}\vec{\lambda},\vec{a},\vec{b})} \;-\; Q_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} \circ \pi_{E^{\text{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ P_n \circ (\bar{\partial}_J h_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})})\,,$$

where

$$\begin{aligned} P_n \;:\; & L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} T_*(W[k]_{\vec{\lambda}})) \\ & \longrightarrow\; L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}})) \end{aligned}$$

is the map induced by the parallel transport along the geodesics determined by $\xi_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu}\vec{\lambda},\vec{a},\vec{b})}$, and define

$$h_{n+1,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;=\; \exp_{h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \xi_{n+1,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\,.$$

The series

$$-\sum_{n=1}^{\infty} \pi_{E^{\text{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \circ P_n \circ (\bar{\partial}_J h_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})})$$

converges to an

$$\eta_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;\in\; L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J h^*_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}))\Big/ E^{\text{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$$

and the sequence of maps $h_{n,(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$, $n = 1, \ldots, \infty$, converge both uniformly and with respect to the $W^{1,p}$-topology (as the $W^{1,p}$-norm dominates the C^0-norm for $p \gg 0$) to

$$f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;=\; \exp_{h_{\text{approx},S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}} \left(\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;+\; Q_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}\eta_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}\right)\,.$$

This gives rise to a continuous-$\widetilde{V}_\rho$-family of maps.

Define the trivialized *obstruction bundle* $E_{\widetilde{V}_\rho}$ *over* $\widetilde{V}_\rho$ by setting its fiber

$$E_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \;\subset\; L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J f^*_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})} T_*(W[k]_{\vec{\lambda}}))$$

at $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})$ to be the parallel transport of $E^{\text{aux}}_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}$ along the geodesics determined by $\xi_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} + Q_{S(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda})}\eta_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$. Let

$$\begin{aligned} \pi_{E_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}} \;:\; & L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J f^*_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} T_* W[k]_{\vec{\lambda}}) \\ & \longrightarrow\; L^p(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}; \Lambda^{0,1}\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})} \otimes_J f^*_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} T_* W[k]_{\vec{\lambda}})\Big/ E_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \end{aligned}$$

be the quotient map; then, by construction,

$$\pi_{E_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}} \circ \bar{\partial}_J f_{\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b}} \;=\; 0\,.$$

In other words, the collection of maps $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ form a continuous-$\widetilde{V}_\rho$-family of $(J, E_\bullet)$-holomorphic maps.

Proposition 5.3.4.5 [*Aut*(ρ)**-equivariant pre-deformable family**]. *The bundle* $\widetilde{E}_{\widetilde{V}_\rho}$ *is* *Aut*(ρ)*-equivariant over* $\widetilde{V}_\rho$ *and the collection of maps* $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ *form a* *Aut*(ρ)*-equivariant continuous-*$\widetilde{V}_\rho$*-family of pre-deformable* $(J, E_\bullet)$*-holomorphic maps.*

Proof. The $Aut(\rho)$-equivariance of the family $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ follows from the $Aut(\rho)$-invariance of the family of maps $h_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}$, the bundle $E^{\text{aux}}_{S(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))}$, and the family of operators $Q_{\text{approx},(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{0})}$. The $Aut(\rho)$-equivariance of $E_{\widetilde{V}_\rho}$ follows then from the $Aut(\rho)$-equivariant of the family of maps $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$. It remains to prove the pre-deformability of $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$.

Note first that, by construction, elements of $E_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ are supported in a compact subset in the complement of the union of ε-neighborhood of nodes and the annuli or bands on $\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}$ from smoothing related nodes of Σ. This implies in particular that $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ is J-holomorphic in the ε-neighborhood of nodes and hence it makes sense to talk about pre-deformability of $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ at distinguished nodes. Furthermore, as the universal map $F : \mathcal{C}/\widetilde{V}_\rho \to W[k]/B[k]$ associated to the family of maps $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ is continuous with the central f pre-deformable, there can be no mass falling into the locus $(W[k]/B[k])_{sing}$ of singularities of the fibers of $W[k]/B[k]$. In other words, F is a family of flat maps in the sense of [I-P2: Definition 3.1] as long as $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a}-\vec{a}_f,\vec{b})$ is sufficiently small, a condition that is already incorporated implicitly into the definition of $\widetilde{V}_\rho$. As the fibers of F over an open-dense subset of $\widetilde{V}_\rho$ are maps from smooth domains(-with-boundary) to smooth fibers of $W[k]/B[k]$, it follows from [I-P2: Lemma 3.3] that this above flatness property implies that the fibers $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ of F over $\widetilde{V}_\rho$ must be all pre-deformable for $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a}-\vec{a}_f,\vec{b}) \in \widetilde{V}_\rho$). This concludes the proof. □

5.3.5 Rigidification: a Kuranishi neighborhood-in-$\mathcal{C}_{\text{spsccw}}$ V_ρ/B of ρ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)\,/B$.

How the $\widetilde{V}_\rho$-family of $(J,E_\bullet)$-holomorphic maps $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$ gives rise to a Kuranishi neighborhood V_ρ/B of ρ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)\,/B$ is explained in this subsubsection.

A stratified subset V_ρ/B of $\widetilde{V}_\rho/B$ from the rigidification of $Aut(\Sigma)\times\mathbb{G}_m[k]$.

$(\widetilde{V}_\rho, E_{\widetilde{V}_\rho})$ and the associated $\widetilde{V}_\rho$-family of $(J,E_\bullet)$-holomorphic maps from deformed Σ to fibers of $W[k]/B[k]$ are only $Aut(\rho)$-equivariant. However, the equivariant approximate product pseudo-action of $Aut(\Sigma)\times\mathbb{G}_m[k]$ on $(\mathcal{C}/Def(\Sigma))\times W[k]/B[k]$ remains to induce an equivalence relation on $\widetilde{V}_\rho$, defined by setting $(\zeta_1,\vec{t}_1,\vec{t}'_1,\vec{\mu}_1,\vec{\lambda}_1,\vec{a}_1,\vec{b}_1) \sim (\zeta_2,\vec{t}_2,\vec{t}'_2,\vec{\mu}_2,\vec{\lambda}_2,\vec{a}_2,\vec{b}_2)$ if there exists a pair $(\alpha,\beta)\in Aut(\Sigma)\times\mathbb{G}_m[k]$ such that $\beta\circ f_{(\zeta_1,\vec{t}_1,\vec{t}'_1,\vec{\mu}_1,\vec{\lambda}_1,\vec{a}_1,\vec{b}_1)}\circ\alpha^{-1} = f_{(\zeta_2,\vec{t}_2,\vec{t}'_2,\vec{\mu}_2,\vec{\lambda}_2,\vec{a}_2,\vec{b}_2)}$. Denote the $\sim$-equivalence class of ρ by O_ρ; then one has:

Lemma 5.3.5.1 [O_ρ maximal]. *O_ρ is a maximal equivalence class at ρ in the sense that a small enough neighborhood of ρ in O_ρ is homeomorphic to a neighborhood of the identity element in $Aut(\Sigma)\times\mathbb{G}_m[k]$.*

Proof. This is a consequence of transversality at ρ. The fiber $\widetilde{V}_0$ of $\widetilde{V}_\rho/(Def(\Sigma)\times B[k])$ over $(\vec{0},\vec{0})$ is embedded in the Banach manifold $W^{1,p}(\Sigma, Y_{[k]})$ of $W^{1,p}$-maps from Σ to (the rigid) $Y_{[k]}$. The latter is (approximate-pseudo-)acted upon by $Aut(\Sigma)\times\mathbb{G}_m[k]$. Under this embedding, $E_{\widetilde{V}_0} := \widetilde{E}_{\widetilde{V}_\rho}|_{\widetilde{V}_0}$ is embedded in the L^p -obstruction bundle $T^2_{W^{1,p}(\Sigma,Y_{[k]})}$ of $W^{1,p}(\Sigma,Y_{[k]})$, whose fiber at ρ is precisely $L^p(\Sigma;\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})$. The operator $\bar\partial_J$ defines a section $s_{\bar\partial_J}$ of $T^2_{W^{1,p}(\Sigma,Y_{[k]})}$, whose linearization at ρ gives precisely the map

$$D_f\bar\partial_J \,:\, W^{l,p}(\Sigma,\partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}) \;\longrightarrow\; W^{l-1,p}(\Sigma,\Lambda^{0,1}\Sigma\otimes_J f^*T_*Y_{[k]})\,,$$

where $W^{l,p}(\Sigma,\partial\Sigma;f^*T_*Y_{[k]},(f|_{\partial\Sigma})^*T_*L_{[k]})$ is now regarded as the fiber of the tangent bundle $T^1_{W^{1,p}(\Sigma,Y_{[k]})}$ of $W^{1,p}(\Sigma,Y_{[k]})$ at ρ. Extend $E_{\widetilde{V}_0}$ to a subbundle E_U of $T^2_{W^{1,p}(\Sigma,Y_{[k]})}$ over a neighborhood U of ρ in $W^{1,p}(\Sigma,Y_{[k]})$ and let π_{E_U} be the quotient map $\pi_{E_U}: T^2_U \to T^2_U/E_U$ over U. Then the saturatedness of E_ρ, Lemma 5.3.1.1, Corollary 5.3.1.6, and the Implicit Function Theorem (Theorem 5.3.0.2) together imply that the pre-deformability condition on (J,E_U)-holomorphic $W^{1,p}$-maps is a transverse condition on $(\pi_{E_\bullet}\circ s_{\bar\partial_J})^{-1}(0)$ at ρ and that the space of pre-deformable (J,E_U)-holomorphic $W^{1,p}$-maps near ρ coincides with a neighborhood of ρ in $\widetilde{V}_0$. The equivalence relation $\sim$ on $\widetilde{V}_\rho$ is the restriction of the equivalence relation on $W^{1,p}(\Sigma,Y_{[k]})$ defined by the $(Aut(\Sigma)\times\mathbb{G}_m[k])$-orbits on $W^{1,p}(\Sigma,Y_{[k]})$. All these together imply that the intersection $((Aut(\Sigma)\times\mathbb{G}_m[k])\cdot\rho)\cap\widetilde{V}_0$ in $W^{1,p}(\Sigma,Y_{[k]})$ coincides with $O_\rho\subset\widetilde{V}_0$ around ρ. This concludes the lemma.

□

With respect to the embedding $\widetilde{V}_\rho\subset Def(\Sigma)\times B[k]\times Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\mathrm{pd}}$ in Sec. 5.3.2, T_fO_ρ lies in the subspace $\{0\}\times\{0\}\times\big(Ker(D_f\bar\partial_J)\cap Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\mathrm{pd}}\big)$. We will denote the quotient space $Ker(\pi_{E_\rho}\circ D_f\bar\partial_J)^{\mathrm{pd}}/T_fO_\rho$ by $H^{\mathrm{rigidified}}_{\rho,\mathrm{map}}$.

By a combination of the same center-of-mass construction in [Sie1: Sec. 5.3] that rigidifies the approximate pseudo-$Aut(\Sigma)$-action and the same construction in Sec. 4.2 that rigidifies the $\mathbb{G}_m[k]$-action, there exists a $Aut(\rho)$-equivariant rigidifying map

$$R_\rho\ :\ \widetilde{V}_\rho\ \longrightarrow\ \mathbb{R}^{a+a'}\times\mathbb{C}^{b+b'+k}\,,$$

where a (resp. b) is the total number of unstable disc-components (resp. sphere-components) of Σ and a' (resp. b') is the total number of unstable disc-components (resp. sphere-components) of Σ that has only one special point. Let

$$V_\rho\ =\ \text{a small enough } Aut(\rho)\text{-invariant open neighborhood of } \rho \text{ in } R_\rho^{-1}(R_\rho(0))\,,$$

then $Aut(\rho)$ acts on V_ρ effectively. The composition of the standard fibrations $\widetilde{V}_\rho/B[k]$ and $B[k]/B$ induces a standard fibration V_ρ/B. The stratified space $\Xi_{\mathbf{s}}$ induces a stratification on $\widetilde{V}_\rho$ via the projection map $\widetilde{V}_\rho\to\Xi_{\mathbf{s}}$. The latter stratification restricts to a stratification on V_ρ.

Lemma 5.3.5.2 [piecewise-transverse slice at ρ]. *As a fibered stratified space, V_ρ/B is isomorphic to $(Def(\Sigma;\Lambda)\times\Xi_{\mathbf{s}}\times H^{\mathrm{rigidified}}_{\rho,\mathrm{map};\Lambda})/B$.*

Proof. Embed $\widetilde{V}_\rho$ in a singular *un-rigidified* chart $\widetilde{V}_\rho{}'^\sharp$ in Siebert's construction (cf. [Sie1: Sec. 5.2, Sec. 5.3]) for ρ regarded as a point in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec m)}(W[k],L[k]\,|\,[\beta],\vec\gamma,\mu)^{W[k]/B[k]}$; then $Aut(\Sigma)\times\mathbb{G}_m[k]$ now does approximate-pseudo-act on $\widetilde{V}_\rho{}'^\sharp$. The rigidifying map $R_\rho:\widetilde{V}_\rho\to\mathbb{R}^{a+a'}\times\mathbb{C}^{b+b'+k}$ extends canonically to $R'_\rho:\widetilde{V}'_\rho\to\mathbb{R}^{a+a'}\times\mathbb{C}^{b+b'+k}$ since the average-weight functions in [Sie1: Sec. 5.3] and Sec. 4.2 that constitutes R_ρ are well-defined for $\check{W}^{1,p}$-maps from deformed Σ to fibers of $W[k]/B[k]$. In particular, after shrinking $\widetilde{V}_\rho$ if necessary, $V_\rho=\widetilde{V}_\rho\cap{R'_\rho}^{-1}(R_\rho(0))$. Recall the stratification of $\widetilde{V}_\rho/B[k]$ and $\widetilde{V}_\rho{}'^\sharp/B[k]$ induced from the coordinate-subspace stratification of $B[k]$. It follows from the three facts: (1) R_ρ, R'_ρ are continuous, and are continuously differentiable when restricted to each stratum, (2) the pseudo-action on $\widetilde{V}_\rho^{\prime\sharp}$ of a small enough neighborhood of the identity element of $Aut(\Sigma)\times\mathbb{G}_m[k]$ is free, and (3) $\widetilde{V}_\rho$ contains a whole orbit O_ρ (cf. Lemma 5.3.5.1), that, for $\widetilde{V}_\rho$ small enough, V_ρ can be interpreted as a stratified space

through 0 (i.e. ρ) in $\widetilde{V}_\rho$ that, in each strata, is transverse to the span of the $(a+a'+2b+2b'+2k)$-many gradient-flow directions from the component weight functions that constitute R_ρ. The lemma then follows.

□

V_ρ/B as a Kuranishi neighborhood of $\rho \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$.

To recapitulate, we have constructed

- $\mathcal{C}_{V_\rho}/V_\rho$: an $Aut(\rho)$-equivariant family $\mathcal{C}_{V_\rho}/V_\rho$ of labelled-bordered Riemann surfaces with marked points over V_ρ;
- $F_{V_\rho} : (\mathcal{C}_{V_\rho}, \dot{\partial}\mathcal{C}_{V_\rho})/V_\rho \to (W[k], L[k])/B[k]$: a map over $V_\rho \to B[k]$ that satisfies $\beta \circ f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} \circ \alpha^{-1} = f_{(\alpha,\beta)\cdot(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$.

Here $\mathcal{C}_{V_\rho}/V_\rho$ is the pull-back of the family $\mathcal{C}/Def(\Sigma)$ to V_ρ via $V_\rho \to Def(\Sigma)$ from the construction, $\dot{\partial}\mathcal{C}_{V_\rho}$ is the labelled boundary of $\mathcal{C}_{V_\rho}$ relative to V_ρ, and $F_{V_\rho}|_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})} = f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$. Through the construction, V_ρ is equipped with the following data:

- $\Gamma_{V_\rho} = Aut(\rho)$ that acts on V_ρ,
- E_{V_ρ}, the Γ_{V_ρ}-equivariant bundle on $(V_\rho, \Gamma_{V_\rho})$ from the restriction of $E_{\widetilde{V}_\rho}$ to V_ρ,
- $s_\rho : V_\rho \to E_\rho$ from the operator $\bar{\partial}_J$, and
- $\psi_\rho : s_\rho^{-1}(0) \to \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$, the map over B that sends each pre-deformable J-holomorphic map $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$, $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b}) \in s_\rho^{-1}(0) \subset V_\rho$, to its isomorphism class $[f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}]$ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$.

Proposition 5.3.5.3 [V_ρ Kuranishi neighborhood]. *The 5-tuple $(V_\rho, \Gamma_{V_\rho}, E_{V_\rho}; s_\rho, \psi_\rho)$ forms a Kuranishi neighborhood-in-$\mathcal{C}_{\text{spsccw}}$ of $\rho \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$.*

Proof. That V_ρ/B is an object in the category $\mathcal{C}_{\text{spsccw}}$ follows from Lemma 5.3.5.2. Injectivity of the ψ_ρ-induced map $s_\rho^{-1}(0)/\Gamma_{V_\rho} \to \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ follows from rigidification. To show that the image of ψ_ρ contains a neighborhood of ρ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$, let $\tilde{s}_\rho$ be the section of $E_{\widetilde{V}_\rho}$ associated to the operator $\bar{\partial}_J$. By construction, $s_\rho^{-1}(0)$ is the rigidification of $\tilde{s}_\rho^{-1}(0)$ by R_ρ with respect to the approximate pseudo-$(Aut(\Sigma) \times \mathbb{G}_m[k])$-action on $\widetilde{s}^{-1}(0)$, and there is a (continuous) map $\tilde{\psi}_\rho : \tilde{s}_\rho^{-1}(0) \to \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ over B that sends each $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$, $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b}) \in \tilde{s}_\rho^{-1}(0) \subset \widetilde{V}_\rho$, to its isomorphism class $[f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b}}]$ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$. We will show that the image of $\tilde{\psi}_\rho$ contains a neighborhood of ρ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$. This then implies the same for ψ_ρ.

Recall the standard piecewise-continuous section $S : \pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho) \to \widetilde{V}_\rho$ of the fibration $\pi_{Def(\Sigma)\times B[k]} : \widetilde{V}_\rho \to Def(\Sigma) \times B[k]$. For a fixed $(\zeta,\vec{t},\vec{t}',\vec{\mu};\vec{\lambda}) \in \pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho) \subset Def(\Sigma) \times B[k]$, let

$$W^{1,p}\left((\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}),\ (W[k]_{\vec{\lambda}}, L)\right)$$

be the Banach manifold of $W^{1,p}$-maps from $(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})}, \partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})})$ to (the *rigid*) $(W[k]_{\vec{\lambda}}, L)$. Then the same transversality argument as in the proof of Lemma 5.3.5.1 implies that $\widetilde{V}_\rho$ contains all

pre-deformable $(J,E_\bullet)$-holomorphic $W^{1,p}$-maps from $(\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})},\partial\Sigma_{(\zeta,\vec{t},\vec{t}',\vec{\mu})})$ to $(W[k]_{\vec{\lambda}},L)$ that are close to $f_{S(\zeta,\vec{t},\vec{t}',\vec{\mu};\vec{\lambda})}$.

Let $(\zeta,\vec{t},\vec{t}',\vec{\mu};\vec{\lambda})$ now vary in $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)\subset Def(\Sigma)\times B[k]$. Note that the fiber-dimension of $\widetilde{V}_\rho$ over $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$ is upper semi-continuous and that there is a well-defined flattening stratification on $\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$ so that the restriction of the fibration $\widetilde{V}_\rho/\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho)$ to each stratum is a bundle whose fibers do not shrink or get pinched when moving toward the boundary of the stratum. Together with the conclusion of the previous paragraph, these imply that $\widetilde{V}_\rho$ contains all pre-deformable $(J,E_\bullet)$-holomorphic $W^{1,p}$-maps that are close to some $f_{S(\zeta,\vec{t},\vec{t}',\vec{\mu};\vec{\lambda})}$, $(\zeta,\vec{t},\vec{t}',\vec{\mu};\vec{\lambda})\in(\pi_{Def(\Sigma)\times B[k]}(\widetilde{V}_\rho))$. In particular, $\widetilde{V}_\rho$ contains all (J-holomorphic, pre-deformable) stable maps near f. This concludes the proof.

□

Remark 5.3.5.4 [*E_ρ-dependence of V_ρ*]. Different choices of E_ρ in Definition/Lemma 5.3.1.5 give rise to different but equivalent family Kuranishi neighborhoods of ρ in the sense of Definition 5.1.1. E.g. taking $E_{1,\rho}+E_{2,\rho}$ creates a third family Kuranishi neighborhood of ρ that dominates both $V_{1,\rho}$ and $V_{2,\rho}$, as in [Liu(C): Remark 6.34].

5.4 Construction of a family Kuranishi structure.

We now proceed to construct a family Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ by relating Kuranishi neighborhoods on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ with sub-fibrations of the $\check{L}^p$-obstruction-space fibration $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$. The construction connects Fukaya-Ono's construction in [F-O: Sec. 15] with Siebert's construction in [Sie1: Sec. 5 - Sec. 6].

The following remark should be kept in mind as it is everywhere behind the discussion.

Remark 5.4.1 [*isomorphism class vs. representative*]. A point ρ in the moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}($ $W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ represents an *isomorphism class* of maps while a family Kuranishi neighborhood $(V_\rho,\Gamma_{V_\rho},E_{V_\rho};s_\rho,\psi_\rho)$ of ρ, as constructed in Sec. 5.3, parameterizes a collection of *maps* that contains a sub-collection, namely $s_\rho^{-1}(0)$, of *representatives* $f_{(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})}$, $(\zeta,\vec{t},\vec{t}',\vec{\mu},\vec{\lambda},\vec{a},\vec{b})\in s_\rho^{-1}(0)$, as in Sec. 5.3.5, whose corresponding set of isomorphism classes covers a neighborhood, namely $U_\rho:=\psi_\rho(s_\rho^{-1}(0))$, of ρ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ via ψ_ρ. In particular, every $p\in V_\rho$ goes with a unique representative $h_p:\Sigma_p/pt\to W[k_\rho]/B[k_\rho]$. The set of isomorphisms from a representative f_1 to another representative f_2 of ρ (which may come from two different Kuranishi neighborhoods V_{ρ_1} and V_{ρ_2} that cover ρ) is parameterized by $Aut(f_1)$ up to a right multiplication and by $Aut(f_2)$ up to a left multiplication. By definition, $Aut(f_1)\simeq Aut(f_2)\simeq Aut(\rho)=\Gamma_\rho$. The same distinction holds between points on the moduli space $\check{W}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ and points on its (singular) orbifold local charts.

Kuranishi neighborhoods in terms of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$.

Note that the same construction in Sec. 5.3 works also with $W^{1,p}$ replaced by $\check{W}^{1,p}$ and L^p replaced by $\check{L}^p$. Let $\rho\in\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$ be represented by $f:\Sigma\to(Y_{[k]},L_{[k]})$, Then, in terms of the fibration $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$, the construction of a Kuranishi neighborhood $(V_\rho,\Gamma_{V_\rho},E_{V_\rho};s_\rho,\psi_\rho)$ of ρ in Sec. 5.3 can be deformed and rephrased as follows:

(1) Choose a saturated obstruction space $E_\rho \subset C^\infty(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$ at ρ. Regard ρ as a point in $\check{W}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ that is represented also by f gives an embedding $E_\rho \hookrightarrow \check{L}^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$.

· Extend E_ρ at ρ to a trivialized $Aut(\rho)$-equivariant trivial bundle $E_{\check{V}_\rho}$ over a sufficiently small orbifold local chart $\check{V}_\rho$ of ρ in $\check{W}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ such that, for all $p \in \check{V}_\rho$ with its corresponding representative h_p is J-holomorphic, the fiber $E_{\check{V}_\rho}|_p$ is a saturated obstruction space $\subset C^\infty(\Sigma_p; \Lambda^{0,1}\Sigma_p \otimes_J h^*T_*W[k_\rho]_{\vec{\lambda}_p})$. By construction there is a map

$$E_{\check{V}_\rho} \longrightarrow T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$$

as an orbifold sub-fibration; we shall think of $(E_{\check{V}_\rho}, Aut(\rho))$ equally as a sub-orbifold of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$.

(2) Recall the global section

$$s_{\bar\partial_J} : \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu) \longrightarrow T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$$

of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ as a morphism of orbifolds. Denote its image sub-orbifold in $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ by $Im(s_{\bar\partial_J})$. Let

$$\pi^2 : T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet} \longrightarrow \check{W}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$$

be the fibration orbifold-map. Then, on the orbifold local chart $\check{V}_\rho$,

$$V_\rho := \pi^2\left(Im(s_{\bar\partial_J}) \cap E_{\check{V}_\rho}\right)$$

is $Aut(\rho)$-invariant. Furthermore, V_ρ defines a Kuranishi neighborhood-in-$\mathcal{C}_{\text{spsccw}}$ of $\rho \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ with $E_{V_\rho} = E_{\check{V}_\rho}|_{V_\rho}$, $\Gamma_{V_\rho} = Aut(\rho)$ now acting on E_{V_ρ}/V_ρ equivariantly, $s_\rho = s_{\bar\partial_J}|_{V_\rho}$, and $\psi_\rho : s_\rho^{-1}(0) \to \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ by sending $p \in s_\rho^{-1}(0)$ to $[h_p]$.

· On $\check{V}_\rho$ it follows by construction that $\bar\partial_J h_p \in E_{V_\rho}|_p$ if and only if $p \in V_\rho$. Thus, V_ρ parameterizes all the (J,E)-holomorphic $\check{W}^{1,p}$-maps near ρ. Indeed it parameterizes also all the (J,E)-holomorphic $W^{1,p}$-maps near ρ.

Deformations of the bundle $E_{\check{V}_\rho}$ in $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ as orbifold sub-fibrations without violating the C^∞-class and the saturatedness condition on the locus $(s_{\bar\partial})^{-1}(0)$ give rise to Kuranishi neighborhoods-in-$\mathcal{C}_{\text{spsccw}}$ of ρ, all of the same actual dimension and the same virtual dimension.[24]

[24] This deformation freedom is crucial in the construction of a Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$. In Sec. 5.3, $E_{\check{V}_\rho}$ is constructed via parallel transport from the trivialized trivial bundle on $S_0(\pi_{Def(\Sigma)\times B[k]}(\Theta_{\rho,0}))$. Such parallel transport construction depends on the metric on the fibers of $W[k_\rho]/B[k_\rho]$. The curvature of the metric makes the bundle $E_{\check{V}_{\rho'}}$ constructed from nearby $\rho' \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ distinct on $Im(\psi_\rho) \cap Im(\psi_{\rho'})$. The corresponding V_ρ and V'_ρ for such $E_{\check{V}_\rho}$ and $E_{\check{V}_{\rho'}}$ cannot be glued at the level of the universal map on the universal curve. Furthermore, while the almost-complex structure on fibers of $\widehat{W}/\widehat{B}$ is well-defined, the metric is not. So a deformation to the construction in Sec. 5.3 that preserves the C^∞-class and the saturatedness condition is indispensable.

Definition 5.4.2 [saturated obstruction local bundle]. The $Aut(\rho)$-equivariant bundle $E_{\check{V}_\rho}$ on $\check{V}_\rho$ in the above rephrasing, with the prescribed properties and the orbifold sub-fibration map $E_{\check{V}_\rho} \to T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$, is called a *saturated obstruction local bundle* on $\check{W}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$. The local orbifold chart $\check{V}_\rho$ is called the *support* of $E_{\check{V}_\rho}$. The tuple

$$V_\rho(E_{\check{V}_\rho}) \;:=\; (V_\rho, \Gamma_{V_\rho}, E_{V_\rho}; s_\rho, \psi_\rho)\,,$$

(also denoted by V_ρ in shorthand), in the rephrasing is called *the Kuranishi neighborhood of* $\rho \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ *determined by* $E_{\check{V}_\rho}$.

Kuranishi structures associated to a fine system of local bundles.

Definition 5.4.3 [direct-sum/fine system of local bundles]. A collection $\{E_{\check{V}_{\rho_i}}\}_{i\in I}$, $\rho_i \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$, of saturated obstruction local bundles is said to form a *direct-sum system* for $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ if the following two conditions are satisfied:

(1) $\{Im\,(\psi_{\rho_i})\}_{i\in I}$ is a locally finite (open) cover of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ that is finite over a compact subset of B, (here ψ_{ρ_i} is from the Kuranishi neighborhood data associated to $E_{\check{V}_\rho}$);

(2) the span of $\{E_{\check{V}_{\rho_i}}\}_{i\in I}$ in each vector-space fiber of a fibration local chart of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ is a direct sum of the related fibers of $E_{\check{V}_{\rho_i}}$'s.

$\{E_{\check{V}_{\rho_i}}\}_{i\in I}$ is said to be *fine* if, in addition,

(3) there exists an open cover $\{U^\flat_{\rho_i}\}_{i\in I}$ of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ such that $U^\flat_{\rho_i}$ is an open neighborhood of ρ_i with the closure $\overline{U^\flat_{\rho_i}}$ a compact subset of $Im\,(\psi_{\rho_i})$.

Lemma 5.4.4 [existence of fine system]. *A fine system of saturated obstruction local bundles for* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ *exists.*

Proof. Let E'_ρ be a saturated obstruction local bundle at $\rho \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ and $U^\flat_\rho$ be an open neighborhood of ρ with the closure $\overline{U^\flat_\rho}$ a compact subset of $Im\,(\psi'_\rho)$. Since $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ is compact over a compact subset of B, one can choose a subcover $\{U_{\rho_i}\}_{i\in I}$ of $\{U_\rho\}_\rho$ that is locally finite and is finite over a compact subset of B. We may assume that each $E'_{\check{V}_\rho}$, and hence $E'_{\check{V}_{\rho_i}}$, is constructed as in Sec. 5.3 so that elements in the fiber of $E'_{\check{V}_\rho}$ are sections of sheaves supported away from the nodes of bordered Riemann surfaces. As this is a locally finite system of trivial bundles, the direct-sum condition can be achieved by deforming $E'_{\check{V}_{\rho_i}}$ inductively to another equivariant $E_{\check{V}_{\rho_i}}$ that satisfies also the C^∞-class and the saturatedness conditions, and with the same support, as sub-fibrations in $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$. This makes the ψ_{ρ_i} from $E_{\check{V}_{\rho_i}}$ coincides with the ψ'_{ρ_i} from $E'_{\check{V}_\rho}$. Thus the cover $\{Im\,(\psi_{\rho_i})\}_{i\in I}$ of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ from the new system $\{E_{\check{V}_{\rho_i}}\}_{i\in I}$ coincides with $\{Im\,(\psi'_{\rho_i})\}_{i\in I}$ and, hence, Condition (2) and Condition (3) in Definition 5.4.3 are also satisfied.

□

Recall the canonical orbifold-embedding

$$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu) \;\hookrightarrow\; \check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)\,.$$

Let $\mathcal{E} := \{E_{\check{V}_{\rho_i}}\}_{i\in I}$ be a fine system of saturated obstruction local bundles for $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$. Let $\mathbf{F}(\mathcal{E})$ be the fiberwise linear span of the union of the image set of $E_{\rho_i} \to T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ with the induced subset topology. The orbifold structure on $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ induces an orbifold structure on $\mathbf{F}(\mathcal{E})$ that fibers over $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}|\,[\beta],\vec{\gamma},\mu)$. This realizes $\mathbf{F}(\mathcal{E})$ as an orbifold sub-fibration of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$ that is mapped to a neighborhood of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)$ in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ under

$$\pi^2 \;:\; T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet} \;\longrightarrow\; \check{W}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)\,.$$

The map

$$\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu) \;\longrightarrow\; \mathbb{Z}_{\geq 0}\,, \qquad p \;\longmapsto\; dim\,(\mathbf{F}(\mathcal{E})|_p)$$

defines the flattening stratification of $\mathbf{F}(\mathcal{E})$ on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ by its preimage subsets. Over each stratum, $\mathbf{F}(\mathcal{E})|_{\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet)}$ is an orbi-bundle. For any $I' \subset I$, the same construction applied to $\mathcal{E}_{I'} = \{E_{\check{V}_{\rho_i}}\}_{i\in I'}$ gives an orbifold sub-fibration $\mathbf{F}(\mathcal{E}_{I'})$ in $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$. The flattening stratification of $\mathbf{F}(\mathcal{E}_{I'})$ is defined similarly. By construction, $\mathbf{F}(\mathcal{E}_{I'})$ is an orbifold sub-fibration of $\mathbf{F}(\mathcal{E})$.

Recall the locally finite cover $\{U^\flat_{\rho_i}\}_{i\in I}$ of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)\,/B$. This induces a stratification $\mathcal{S} := \{S_{I'}\}_{I'\subset I}$ of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)\,/B$ by setting

$$S_{I'} \;=\; (\cap_{i\in I'}\,\overline{U^\flat_{\rho_i}}) \;-\; (\cup_{i\in I-I'}\,\overline{U^\flat_{\rho_i}})\,.$$

Define also the subset $S'_{I'} = (\cap_{i\in I'}\, Im\,(\psi_{\rho_i}) \;-\; (\cup_{i\in I-I'}\,\overline{U^\flat_{\rho_i}})$, $I' \subset I$. For $\rho \in S_{I'}$, let $\check{V}_\rho$ be an orbifold local chart of ρ in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ such that

- the image of $\check{V}_\rho$ in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}|\,[\beta],\vec{\gamma},\mu)$ is covered by the union of the image of $\check{V}_{\rho_i}$, $i \in I'$,
- the J-holomorphy locus of $\check{V}_\rho$ is mapped to $S'_{I'}$.

Denote the image of $\check{V}_\rho$ in $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ by $\underline{\check{V}_\rho}$. Then, $\mathbf{F}(\mathcal{E}_{I'})$ is an orbi-bundle when restricted to $\underline{\check{V}_\rho}$. Let $E_{\check{V}_\rho}$ be the associated orbi-bundle local chart of $\mathbf{F}(\mathcal{E}_{I'})|_{\underline{\check{V}_\rho}}$; then, by construction, $E_{\check{V}_\rho}$ is a saturated obstruction local bundle on $\check{\mathcal{W}}^{1,p}_{(g,h),(n,\vec{m})}((\widehat{W},\widehat{L})/\widehat{B}\,|\,[\beta],\vec{\gamma},\mu)$ in the sense of Definition 5.4.2.

In this way, one recovers a family $\{E_{\check{V}_\rho}\}_{\rho\in\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet)}$ of saturated obstruction local bundles from the orbifold sub-fibration $\mathbf{F}(\mathcal{E})$ of $T^2_{\check{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$. Define

$$\mathfrak{N}^{(0)}_{\text{Kuranishi}}(\mathcal{E}) \;:=\; \Big\{\, V_\rho(E_{\check{V}_\rho}) \;=\; (V_\rho, \Gamma_\rho, E_\rho; s_\rho, \psi_\rho) \,\Big\}_{\rho\in\overline{\mathcal{M}}_\bullet(W/B,L\,|\,\bullet)}$$

from Definition 5.4.2. This gives the set of family Kuranishi neighborhoods-in-$\mathcal{C}_{\text{spsccw}}$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$. The orbifold fibration transition data of $T^2_{\tilde{\mathcal{W}}^{1,p}_\bullet((\widehat{W},\widehat{L})/\widehat{B}\,|\,\bullet)/\widetilde{\mathcal{M}}_\bullet}$, or of $\mathbf{F}(\mathcal{E})$, induces a collection of 4-tuples

$$\mathfrak{N}^{(1)}_{\text{Kuranishi}}(\mathcal{E}) \;:=\; \Big\{(V_\rho, h_{\rho'\rho}, \phi_{\rho'\rho}, \hat{\phi}_{\rho'\rho}) \;:\; \rho \in \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)\,/B\,,\; \rho' \in \psi_\rho\left(s_\rho^{-1}(0)\right)\Big\}$$

that gives the set of transition functions between elements in $\mathfrak{N}^{(0)}_{\text{Kuranishi}}(\mathcal{E})$ in the sense of Definition 5.1.2. We shall call the pair

$$\mathcal{K}(\mathcal{E}) \;=\; \left(\mathfrak{N}^{(0)}_{\text{Kuranishi}}(\mathcal{E})\,,\, \mathfrak{N}^{(1)}_{\text{Kuranishi}}(\mathcal{E})\right)$$

a *Kuranishi structure* associated to the fine system $\mathcal{E}$ of saturated obstruction local bundles. We remark that the gluing thus constructed is at the level of the universal map on the universal curve and that different choices of $\{U^\flat_{\rho_i}\}_{i\in I}$ give equivalent Kuranishi structures.

To summarize:

Proposition 5.4.5 [Kuranishi structure from fine system]. *A fine system of saturated obstruction local bundles for $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$ determine a unique equivalence class of Kuranishi structures-in-$\mathcal{C}_{\text{spsccw}}$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L\,|\,[\beta],\vec{\gamma},\mu)/B$.*

6 The moduli space $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ of relative stable maps and its Kuranishi structure.

We apply and extend the construction in Sec. 1 - Sec. 3 to a relative pair $(Z, L; D)$ and its expansions to define the moduli space $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ of relative stable maps of type $((g,h),(n+l(\vec{s}),\vec{m})\,|\,\beta',\vec{\gamma},\mu';\vec{s})$, from labelled-bordered Riemann surfaces with marked points to the fibers of the expanded relative pairs $(\widehat{Z}, \widehat{L}; \widehat{D})/\widehat{A}$ associated to $(Z, L; D)$; (Sec. 6.1). The same technique in Sec. 4 and Sec. 5 is used to construct a Kuranishi structure thereupon; (Sec. 6.2). See also [I-P1], [L-R] for the symplecto-analytic setting in different formats and [Li1: Sec. 4], [Li2: Sec. 2], [Gr-V] for the algebro-geometric setting.

6.1 The moduli space $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ of relative stable maps.

Let $(Z, L; D)$ be a symplectic pair $(Z; D)$, with a compatible almost-complex structure, together with a Lagrangian/almost-complex submanifold L that is disjoint from D. Recall the space $(\widehat{Z}; \widehat{D})/\widehat{A}$ of expanded relative pairs associated to $(Z; D)$ with the quotient topology, its standard local charts $\varphi[k] : (Z[k]; D[k])/A[k] \to (\widehat{Z}; \widehat{D})/\widehat{A}$ with $k \in \mathbb{Z}_{\geq 0}$, and the equivariant pseudo-$\mathbb{G}_m[k]$-action on $(Z[k]; D[k])/A[k]$ from Sec. 1.2. Let $L[k]$ be the submanifold $\tilde{\mathbf{p}}[k]^{-1}(L)$ of $Z[k]$ from the map $\tilde{\mathbf{p}}[k] : (Z[k]; D[k])/A[k] \to (Z; D)/pt$. Over $A[k]$, $L[k] = A[k] \times L$. Sec. 1.2 can be made to incorporate $L[k]$. This gives the space $(\widehat{Z}, \widehat{L}; \widehat{D})/\widehat{A}$. The central fiber of $(Z[k], L[k]; D[k])/A[k]$ is almost-complex isomorphic to the pair-with-a-totally-real-submanifold $(Z_{[k]}, L_{[k]}; D_{[k]})$.

Recall the definition of the relative Maslov index $\mu^{rel}(h)$ of a smooth map $h : (\Sigma, \partial\Sigma) \to (Z_{[k]}, L_{[k]}; D_{[k]})$ from Sec. 3.1. Note also that the monodromies of $(Z[k], L[k]; D[k])/A[k]$, $k \in \mathbb{Z}_{\geq 0}$, on a smooth fiber, which is almost-complex isomorphic to $(Z, L; D)$, are relatively isotopic

to the identity map with respect to $(L;D)$; thus, the monodromy $(\widehat{Z}[k],\widehat{L}[k];\widehat{D})/\widehat{A}$-action on $H_1(L;\mathbb{Z})$, $H_2(Z,L;\mathbb{Z})$, and $H_2(Z,L\cup D;\mathbb{Z})$ are all trivial.

Moduli space of relative stable maps to fibers of $(\widehat{Z},\widehat{L};\widehat{D})/\widehat{A}$.

Definition 6.1.1 [relative stable map to fibers of $(Z[k],L[k];D[k])/A[k]$]. Let $\beta'\in H_2(Z,L;\mathbb{Z})$, $\vec{\gamma}=(\gamma_1,\ldots,\gamma_h)\in H_1(L;\mathbb{Z})^{\oplus h}$ such that $\partial\beta=\gamma_1+\cdots\gamma_h$, $\mu'\in\mathbb{Z}$, and $\vec{s}=(s_1,\ldots,s_l)\in(\mathbb{Z}_{\geq 0})^l$.[25] A relative map $f:(\Sigma,\partial\Sigma)/pt\to(Z[k],L[k];D[k])/A[k]$ from a bordered Riemann surface Σ to a fiber of $(Z[k],L[k];D[k])/A[k]$ is called *prestable* of (*combinatorial*) *type* $((g,h),(n+l(\vec{s}),\vec{m})\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ if

- f is prestable of type $((g,h),(n+l(\vec{s}),\vec{m})\,|\,\beta',\vec{\gamma},\mu'+2\,deg\,(\vec{s}))$ as a map to a fiber of $(Z[k],L[k])/A[k]$, cf. Definition 3.3.1 (with $[\beta]=\{\beta'\}$); the last $l(\vec{s})$ free marked points on Σ shall be called the *distinguished marked points*;
- (f is non-degenerate with respect to $D[k]$;) $f^{-1}(D[k])=s_1\,p_{n+1}+\cdots+s_l\,p_{n+l(\vec{s})}$, where $p_{n+1},\ldots,p_{n+l(\vec{s})}$ are the distinguished marked points on Σ; (in particular, $\mu^{rel}(f)=\mu'$ and all distinguished marked points are smooth interior points on Σ).

An *isomorphism* between two relative prestable maps $f_1:\Sigma_1/pt\to(Z[k],L[k];D[k])/A[k]$, $f_2:\Sigma_2/pt\to(Z[k],L[k];D[k])/A[k]$ of the same type is a pair (α,β), where $\alpha:\Sigma_1\to\Sigma_2$ is an isomorphism of prestable labelled-bordered Riemann surfaces with marked points and $\beta\in\mathbb{G}_m[k]$ such that $f_1\circ\beta=f_2\circ\alpha$. The isomorphism class of f is denoted by $[f]$. The notion of *non-degenerate* (resp. *pre-deformable*) relative prestable maps, *distinguished nodes* q, and the *contact order* at q are defined exactly the same as in Definition 3.3.1.

A relative prestable map $f:\Sigma/pt\to(Z[k].L[k];D[k])/A[k]$ is called *stable* if f is pre-deformable and its group $Aut(f)$ of automorphisms is finite. The moduli space of isomorphism classes of stable maps to fibers of $(Z[k],L[k];D[k])/A[k]$ of type $((g,h),(n+l(\vec{s}),\vec{m})\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ is denoted by $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})$; it is equipped with the C^∞-*topology*, defined similarly as in Sec. 3.3.

The pseudo-embedding $\varphi'_{k',k;I}:(Z[k'],L[k];D[k])/A[k']\hookrightarrow(Z[k],L[k];D[k])/A[k]$, $k'<k$ and $I\subset\{0,\ldots,k-1\}$, from Sec. 1.2 induces a *pseudo-embedding*

$$\begin{aligned}\varphi'_{k',k;I}\;:\;&\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k'],L[k'];D[k'])/A[k']\,|\,\beta',\vec{\gamma},\mu';\vec{s})\\ &\hookrightarrow\;\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})\,.\end{aligned}$$

Define the set of isomorphism classes of relative stable maps to fibers of $(\widehat{Z},\widehat{L};\widehat{D})/\widehat{A}$:

$$\begin{aligned}&\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})\\ &:=\;\Big(\amalg_{k=0}^{\infty}\mathcal{M}^{\text{non-rigid}}_{(g,h),(n,\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})\Big)\Big/\sim,\end{aligned}$$

where the equivalence relation $\sim$ is generated by $[f]\sim\varphi'_{k',k;I}([f'])$ for $[f]\in\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ and $[f']\in$ the defining domain of $\varphi_{k',k;I}$ on $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k'],L[k'];D[k'])/A[k']\,|\,\beta',\vec{\gamma},\mu';\vec{s})$. By construction, there are embeddings of sets

$$\begin{aligned}\varphi'_{(k)}\;:\;&\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})\\ &\hookrightarrow\;\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})\,,\quad k\in\mathbb{Z}_{\geq 0}\,.\end{aligned}$$

[25] For $\vec{s}$, we define its *length* $l(\vec{s}):=l$, *degree* $deg\,(\vec{s}):=s_1+\cdots+s_l$, and *multiplicity* $m(\vec{s}):=s_1\cdots s_l$.

A subset U of $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ is said to be *open* if $U=\cup_\alpha U_\alpha$ such that U_α is contained in the image of some $\varphi'_{(k)}$ and ${\varphi'_{(k)}}^{-1}(U_\alpha)$ is open in $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})$. This defines the C^∞-*topology* on the moduli space $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ of relative stable maps to fibers of $(\widehat{Z},\widehat{L};\widehat{D})/\widehat{A}$.

Definition 6.1.2 [tautological cover]. By construction,

$$\Big\{\,\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})\,\Big\}_{k\in\mathbb{Z}_{\geq 0}}$$

is an open cover of $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m};\vec{s})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$. We will call it the *tautological cover* of $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$.

Indeed, there exists k_0 depending $(Z,L;D)$ and $((g,h),(n+l(\vec{s}),\vec{m})|\beta',\vec{\gamma},\mu';\vec{s})$ such that

$$\begin{aligned}&\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k_0],L[k_0];D[k_0])/A[k_0]\,|\,\beta',\vec{\gamma},\mu',\vec{s})\\ &\quad\supset\ \mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k_0+1],L[k_0+1];D[k_0+1])/A[k_0+1]\,|\,\beta',\vec{\gamma},\mu',\vec{s})\\ &\quad\supset\ \mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k_0+2],L[k_0+2];D[k_0+2])/A[k_0+2]\,|\,\beta',\vec{\gamma},\mu',\vec{s})\ \supset\ \cdots.\end{aligned}$$

Thus, the tautological cover of $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ is finite in effect, cf. Theorem 6.1.3. The universal maps on the universal curve over each $\mathcal{M}^{\text{non-rigid}}_{(g,h),(n+l(\vec{s}),\vec{m})}((Z[k],L[k];D[k])/A[k]\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ are glued to give the universal map (between spaces with charts)

$$F\ :\ \mathcal{C}/\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})\ \longrightarrow\ (\widehat{Z},\widehat{L};\widehat{D})/\widehat{A}\,.$$

Hausdorffness, finite stratification, and compactness.

Recall the notion of *weighted layered* $(A_2\to A_1)$*-graphs* from Definition 3.3.5, the category $\mathfrak{G}(A_2\to A_1)$ of graphs, and how a stable map f to fibers of $(W[k-1],L[k-1])/B[k-1]$ (now $=(Z[k],L[k])/A[k]$) corresponds a such graph $\tau_{[f]}$. To encode the contact-order data $\vec{s}$ of relative maps with $D[k]$, we add to the objects τ in $\mathfrak{G}(A_2\to A_1)$ the following data:

- an ordered set $R(\tau)$ of l-many *roots* r_i, $i=1,\ldots,l$, that are attached to vertices of the largest layer-value;
- an additional *weight function* $ord':R(\tau)\to\mathbb{Z}_{\geq 0}$, $r_i\mapsto s_i$;
- replace $\mu(\tau)$ in Definition 3.3.5 by $\mu'(\tau)$, called the *relative index* of τ.[26]

Denote a such graph still by τ with the same name: *weighted layered* $(A_1\to A_1)$*-graph*. An *isomorphism* $\alpha:\tau_1\to\tau_2$ between two such graphs is defined the same as in Definition 3.3.5 with the index replaced by relative index and the additional requirement that α induces an isomorphism $R(\tau_1)\xrightarrow{\sim}R(\tau_2)$ as ordered weighted sets. The corresponding new *category of graphs* enlarges the previous one and will be denoted still by $\mathfrak{G}(A_2\to A_1)$ (or simply $\mathfrak{G}$ when $(A_2\to A_1)$ is understood).

The notion of *genus*, *b-weight*, *contraction*, and (*red-to-blue*) *color change* of weighted layered $(A_2\to A_1)$-graphs extend to the new $\mathfrak{G}(A_2\to A_1)$. The correspondence of a point $[f:\Sigma/pt\to$

[26] Let $\vec{s}=(ord'(r_1),\cdots,ord'(r_l))$. Then, we will call the quantity $\mu'+2\,deg\,(\vec{s})$ the (absolute) *index* of τ and denote it by $\mu(\tau)$. When $l=0$, $\mu'(\tau)=\mu(\tau)$.

$(\widehat{Z}, \widehat{L}; \widehat{D})/\widehat{A}] \in \overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu; \vec{s})$, with target isomorphic to $(Z_{[k]}, L_{[k]}; D_{[k]})$, to an element $\tau_{[f]} \in \mathfrak{G}(A_2 \to A_1)$ is the same as in Sec. 3.3 with the following addition/modification:

$f : \Sigma \to (Z_{[k]}, L; D_{[k]})$	$(H_2(Z, L; \mathbb{Z}) \xrightarrow{\partial} H_1(L; \mathbb{Z}))$-graph τ
.....	.....
i-th distinguished marked point p_{n+i}	*root* $r_i \in R(\tau)$ attached to $v \in V(\tau)$ with *layer*$(v) = k$
contact order s_i of f with $D_{[k]}$ at p_{n+i}	$ord'(r_i)$, $r_i \in R(\tau)$
relative Maslov index $\mu^{\rm rel}(f)$	*relative index* μ'.

Two relative stable maps $f_i : \Sigma_i/pt \to (Z[k_i], L[k_i]; D[k_i])/A[k_i]$, $i = 1, 2$, are said to be *of the same topological type* if $\tau_{[f_1]}$ is isomorphic to $\tau_{[f_2]}$ in the category $\mathfrak{G}$. Degenerations of relative stable maps to fibers of $(\widehat{Z}, \widehat{L}; \widehat{D})/\widehat{A}$ are reflected contravariantly by compositions of contractions and color-changes of their dual graphs.

Same reasons that give Proposition 3.3.4, Lemma 3.3.7, and Theorem 3.3.8 now imply:

Theorem 6.1.3 [Hausdorffness and compactness]. *The classification of relative stable maps by their topological types gives rise to a finite stratification of* $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu; \vec{s})$*, with each stratum* S_τ *labelled by a weighted layered* $(H_2(Y, L; \mathbb{Z}), H_1(L; \mathbb{Z}))$*-graph* $\tau \in \mathfrak{G}$*. The moduli space* $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu; \vec{s})$ *of relative stable maps to fibers of* $(\widehat{Z}, \widehat{L}; \widehat{D})/\widehat{A}$ *of combinatorial type* $((g, h), (n + l(\vec{s}), \vec{m}) \,|\, \beta', \vec{\gamma}, \mu'; \vec{s})$*, with the* C^∞*-topology, is Hausdorff and compact.*

Cf. [L-R: Sec. 3.3], [I-P1: Theorem 7.4]; [Li1: Theorem 4.10].

6.2 A Kuranishi structure for $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu'; \vec{s})$.

Introduce first the following category of topological spaces, which is closely related to $\mathcal{C}_{\rm spsccw}$:

Definition 6.2.1 [category $\mathcal{C}'_{\rm spsccw}$]. We define $\mathcal{C}'_{\rm spsccw}$ to be the category that has the same objects as $\mathcal{C}_{\rm spsccw}$ but with the fibrations over the complex line $\mathbb{C}$ removed. A morphism in $\mathcal{C}'_{\rm spsccw}$ is a continuous map as stratified spaces.

The same construction in Sec. 4 - Sec. 5 gives a Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu'; \vec{s})$ that is modelled in the category $\mathcal{C}'_{\rm spsccw}$. There are only two major modifications in the discussion:

- (the *non-rigidity* of target): while treating $(Z[k], L[k]; D[k])/A[k]$ as the $(k-1)$-th expanded degeneration of the degeneration $(Z[1], L[1]; D[1])/A[1]$, it is $\mathbb{G}_m[k]$ – rather than $\mathbb{G}_m[k-1]$ – that acts equivariantly on $(Z[k], L[k]; D[k])/A[k]$ and that corresponds to choices of the renormalization in removing degeneracy/falling-into-D;

(T4) (additional *transversality*): local transversality of the contact-order-s_i condition along $D_{[k]}$ at the distinguished marked point p_{n+i}, for $i = 1, \ldots, l$; cf. Conditions (T1) - (T3) in Sec. 5.2.

Let $(\Sigma, \dot\partial\Sigma; \vec{p}, \vec{p}_1, \ldots, \vec{p}_h; f)$ be a relative stable map to the central fiber $(Z_{[k]}, L_{[k]}; D_{[k]})$ of $(Z[k], L[k]; D[k])/A[k]$ that represents $\rho \in \overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu; \vec{s})$. Recall that $Z_{[k]} = Z \cup_{D=D_{1,\infty}} \Delta_1 \cup_{D_{1,0}=D_{2,\infty}} \cdots \cup_{D_{k-1,0}=D_{k,\infty}} \Delta_k$ and $D_i := \Delta_i \cap \Delta_{i+1}$ in $Z_{[k]}$ for $i = 1, \ldots, k-1$. Here we set $\Delta_0 = Z$ by convention. Let $\Lambda_i = f^{-1}(D_i)$ and $\Lambda = \amalg_{i=0}^{k-1} \Lambda_i$ be

the set of distinguished nodes on Σ under f. Let $\mathbf{s} = (\vec{s}_0, \cdots, \vec{s}_{k-1}\,;\, \vec{s}_k)$, with $\vec{s}_k = \vec{s}$, be the tuple of contact orders of f at $\Lambda \cup \{p_{n+1}\,, \cdots, p_{n+l(\vec{s})}\}$. Recall the discussion and notations in Sec. 5.2. The notion of a saturated subspace in $W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Z_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ from Definition 5.3.1.4 now has to be revised to incorporate Condition (T4) as well:

Definition 6.2.2 [saturated/relative pre-deformable subspace]. A subspace V in $W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Z_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ is said to be *saturated* if

(1) V is admissible;

(2) the map

$$\begin{aligned}(\oplus_{q\in\Lambda} D_f\, div_q) \bigoplus \left(\oplus_{i=1}^{n+l(\vec{s})}\, D_f\, ev_{p_i}\right) \bigoplus (\oplus_{q_{ij}} D_f\, ev_{q_{ij}}) \bigoplus \left(\oplus_{i=1}^{l(\vec{s})}\, D_f\, div_{p_{n+i}}\right)\ :\ V\ \longrightarrow \\ \left(\bigoplus_{q\in\Lambda} \left(T_{(s(q)-1)\cdot(q)}\mathrm{Div}^{\,s(q)-1}(U_{q,1}) \oplus T_{(s(q)-1)\cdot(q)}\mathrm{Div}^{\,s(q)-1}(U_{q,2})\right)\right) \\ \bigoplus (\oplus_{p_i} T_{f(p_i)}Y_{[k]}) \bigoplus \left(\oplus_{q_{ij}} T_{f(q_{ij})}L\right) \bigoplus_{i=1}^{l(\vec{s})} T_{(s_i-1)\cdot(p_{n+i})}\mathrm{Div}^{\,s_i-1}(U_{p_{n+i}})\end{aligned}$$

is surjective;

(3) let $V^{rel\text{-}\mathrm{pd}}$ be the subspace $((\oplus_{q\in\Lambda} D_f\, div_q)\oplus(\oplus_{q\in\Lambda} D_f\, div_q))^{-1}(\mathbf{0})$ in V, then the linear map

$$\begin{aligned}\left(\oplus_{q\in\Lambda}\,(D_f\, ev_q \oplus jet_q^{s(q)})\right) \bigoplus \left(\oplus_{i=1}^{l(\vec{s})}\,(D_f\, ev_{p_{n+i}} \oplus jet_{p_{n+i}}^{s_i})\right)\ : \\ V^{rel\text{-}\mathrm{pd}}\ \longrightarrow\ \left(\oplus_{q\in\Lambda}\,(T_{f(q)}D \oplus \mathbb{C}^2)\right) \bigoplus \left(\oplus_{i=1}^{l(\vec{s})}(T_{f(p_{n+i})}D \oplus \mathbb{C})\right)\end{aligned}$$

is surjective, where we have identified D_i, $i = 0, \ldots, k-1$, canonically with D.

In the above statement, $V^{rel\text{-}\mathrm{pd}}$ is called the *relative pre-deformable subspace* of V.

A subspace E of $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Y_{[k]})$ is said to be *saturated* if $(D_f\bar{\partial}_J)^{-1}(E) \subset W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Y_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})$ is saturated.

The notion of a saturated obstruction space E_ρ in $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Z_{[k]})$ for $\rho = [f] \in \overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu'; \vec{s})$ from Definition/Lemma 5.3.1.5 is converted accordingly:

Definition/Lemma 6.2.3 [saturated obstruction space]. Denote by $Im\,(D_f\bar{\partial}_J)$ the image of $D_f\bar{\partial}_J$, $(D_f\bar{\partial}_J)(W^{1,\,p}(\Sigma, \partial\Sigma; f^*T_*Z_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}))$, in $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Z_{[k]})$. Then *there exists a subspace E_ρ of $L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Z_{[k]})$ such that*

(1) *$Im\,(D_f\bar{\partial}_J) + E_\rho = L^p(\Sigma; \Lambda^{0,1}\Sigma \otimes_J f^*T_*Z_{[k]})$,*

(2) *E_ρ is finite-dimensional, complex linear, and $Aut\,(\rho)$-invariant,*

(3) *E_ρ consists of smooth sections supported in a compact subset of Σ disjoint from the union of the set of all (three types of) nodes and the set $\{p_{n+1}\,, \cdots, p_{n+l(\vec{s})}\}$ of all distinguished marked points on Σ,*

(4) *$(D_f\bar{\partial}_J)^{-1}(E_\rho)$ is a saturated subspace of $W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Z_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]}))$.*

E_ρ is called a *saturated obstruction space* of $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z, L; D \,|\, \beta', \vec{\gamma}, \mu; \vec{s})$ at ρ.

The index of

$$D_f\bar{\partial}_J\ :\ W^{1,p}(\Sigma, \partial\Sigma; f^*T_*Z_{[k]}, (f|_{\partial\Sigma})^*T_*L_{[k]})\ \longrightarrow\ L^p(\Sigma, \Lambda^{0,1}\Sigma \otimes_J f^*T_*Z_{[k]})$$

is given by

$$\begin{aligned} ind\,(D_f\bar\partial_J) &= \mu(f) + dim\,Z\cdot(1-\tilde g) - 2\sum_{i=0}^{k-1} l(\vec s_i) + 4\sum_{i=0}^{k-1} deg\,\vec s_i \\ &= \mu^{rel}(f) + dim\,Z\cdot(1-\tilde g) - 2\sum_{i=0}^{k-1} l(\vec s_i) + 4\sum_{i=0}^{k-1} deg\,\vec s_i + 2\,deg\,\vec s\,, \end{aligned}$$

where $\tilde g$ is the arithmetic genus of $\Sigma_{\mathbb{C}}$.

Definition 6.2.4 [relative pre-deformable index]. We define the *relative pre-deformable index* of $D_f\bar\partial_J$ to be

$$ind^{rel\text{-}\mathrm{pd}}(D_f\bar\partial_J) := \mu^{rel}(f) + dim\,Z\cdot(1-\tilde g) + 2\,|\Lambda|\,.$$

Note that

$$dim\,(D_f\bar\partial_J)^{-1}(E_\rho)^{rel\text{-}\mathrm{pd}} = \mu^{rel}(f) + dim\,Z\cdot(1-\tilde g) + 2\,|\Lambda| + dim\,E_\rho\,.$$

The same routine of Sec. 5.3 - Sec. 5.4 now proves that:

Theorem 6.2.5 [Kuranishi structure on $\overline{\mathcal{M}}_\bullet(Z,L;D\,|\,\bullet\,)$]. *The moduli space $\overline{\mathcal{M}}_{(g,h),(n+l(\vec s),\vec m)}(Z,L;D\,|\,\beta',\vec\gamma,\mu';\vec s)$ of relative stable maps to fibers of $(\widehat Z,\widehat L;\widehat D)/\widehat A$ admits a Kuranishi structure $\mathcal{K}'$ modelled in $\mathcal{C}'_{\mathrm{spsccw}}$. $\mathcal{K}'$ has the expected dimension*

$$vdim\,\overline{\mathcal{M}}_\bullet(Z,L;D\,|\,\bullet\,) := \mu' + (N-3)(2-2g-h) + 2\,(n+l(\vec s)) + (m_1+\cdots+m_h)\,,$$

where $2N$ is the dimension of Z. The Kuranishi neighborhood-in-$\mathcal{C}'_{\mathrm{spsccw}}$ $(V_\rho,\Gamma_{V_\rho},E_{V_\rho};s_\rho,\psi_\rho)$ at $\rho=[f:(\Sigma,\partial\Sigma)\to(Z_{[k]},L_{[k]};D_{[k]})]$ has V_ρ isomorphic to a neighborhood of the origin of

$$\Xi_{(\vec s_0,\ldots,\vec s_{k-1})}\times\mathbb{R}^{n_1}\times(\mathbb{R}_{\geq 0})^{n_2}$$

where

- *$\vec s_i$ is the contact order of f along D_i at the ordered set of distinguished nodes in $f^{-1}(D_i)$, $i=0,\ldots,k-1$, (and recall that $dim\,\Xi_{(\vec s_0,\ldots,\vec s_{k-1})}=2k$);*
- *$n_1 = vdim\,\overline{\mathcal{M}}_\bullet(Z,L;D\,|\,\bullet\,) + dim\,E_\rho - (2k+n_2)$; and*
- *n_2 = the total number of boundary nodes and free marked points that land on $\partial\Sigma$.*

The homeomorphism-type $\{Z_{[k']}\}_{0\leq k'\leq k}$ of the targets of maps gives a Γ_{V_ρ}-invariant stratification $\{S_{k'}\}_{0\leq k'\leq k}$ on V_ρ; each connected component of $S_{k'}$ is a manifold of codimension $2k'$ in V_ρ. This stratification coincides with the induced stratification on V_ρ from the stratification[27] of $\Xi_{(\vec s_0,\ldots,\vec s_{k-1})}$.

[27] Which, recall that, is induced by the map $\Xi_{(\vec s_0,\ldots,\vec s_{k-1})}\to\mathbb{C}^k$ and the stratification of $\mathbb{C}^k$ by the coordinate subspaces.

7 Degeneration and gluing of Kuranishi structures and axioms of open Gromov-Witten invariants under a symplectic cut.

In this last section of the current work, we derive a degeneration-gluing relation of the Kuranishi structure of the moduli space of stable maps to (X,L) with the Kuranishi structure of the moduli spaces of relative stable maps to $(Y_1,L_1;D)$, $(Y_2,L_2;D)$ that occur in a symplectic cut $\xi:(X,L)\to(Y,L)=(Y_1,L_1)\cup_D(Y_2,L_2)$. This degeneration-gluing relation is insensitive to the real codimension-1 boundary of the Kuranishi structures involved when L is non-empty. Taking this formula as the foundation, together with (a) its reduction to the degeneration/gluing formula of virtual fundamental classes and Gromov-Witten invariants in closed Gromov-Witten theory when L is empty and (b) the deformation-invariance requirement of Gromov-Witten invariants, we propose a degeneration axiom and a gluing axiom under a symplectic cut for open Gromov-Witten invariants of a symplectic/almost-complex manifold with a decorated Lagrangian/totally-real submanifold.

7.1 The degeneration-gluing relations of Kuranishi structures.

Central fiber, layer-structure stratification, and descendent Kuranishi structure.

Definition 7.1.1 [category $\mathcal{C}_{\text{spsccw},0}$ and its descendants $\mathcal{C}^{(i)}_{\text{spsccw},0}$]. We define $\mathcal{C}_{\text{spsccw},0}$ to be the *category of weighted stratified spaces* Q_0 that occur in the central fiber of objects $Q/\mathbb{C}$ in $\mathcal{C}_{\text{spsccw}}$. Here the *weight* to an irreducible component of Q_0 is given by the *multiplicity* of that component in terms of the associated flat affine fibrations $\Xi_{\mathbf{s}}/Spec\,\mathbb{C}[t]$ of schemes, cf. footnote 19. Define also the *depth-i descendant* $\mathcal{C}^{(i)}_{\text{spsccw},0}$ of $\mathcal{C}_{\text{spsccw},0}$ to be the category of stratified spaces locally modelled on the central fiber of the fibration $(\Xi_{(\vec{s}_0,\ldots,\vec{s}_i)}\times\mathbb{R}^{n_1}\times(\mathbb{R}_{\geq 0})^{n_2})/\mathbb{C}^{i+1}$ for some n_1, n_2. Note that $\mathcal{C}^{(0)}_{\text{spsccw},0}=\mathcal{C}_{\text{spsccw},0}$.

Definition 7.1.2 [descendants $\mathcal{C}'^{,(i)}_{\text{spsccw}}$ of $\mathcal{C}'_{\text{spsccw}}$]. We define the *depth-i descendants* $\mathcal{C}'^{,(i)}_{\text{spsccw}}$ *of* $\mathcal{C}'_{\text{spsccw}}$ *to be the category of stratified spaces locally modelled on the central fiber of the fibration* $(\Xi_{(\vec{s}_0,\ldots,\vec{s}_{i-1})}\times\mathbb{R}^{n_1}\times(\mathbb{R}_{\geq 0})^{n_2})/\mathbb{C}^i$ *for some* n_1, n_2. *Note that* $\mathcal{C}'^{,(0)}_{\text{spsccw}}=\mathcal{C}'_{\text{spsccw}}$.

Definition 7.1.3 [standard Kuranishi structure]. *We will call a Kuranishi structure* $\mathcal{K}$ *on a moduli space* $\mathcal{M}$ *of stable maps standard if* $\mathcal{K}$ *is constructed via the routine in Sec. 4 - Sec. 5.*

In particular, a standard Kuranishi structure-in-$\mathcal{C}_{\text{spsccw}}$ $\mathcal{K}/B$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ is flat over B in the sense that each $\lambda\in B$ has a neighborhood U_λ over which $\mathcal{K}$ is equivalent to a standard Kuranishi structure $\hat{\mathcal{K}}/B$ with the Kuranishi neighborhoods and obstruction bundles from $\hat{K}/B$ flat over U_λ. Indeed, a standard $\mathcal{K}/B$ constructed through Sec. 4 - Sec. 5 is already flat over a neighborhood of $0\in B$. This motivates/implies the following definiton/theorem, which is a corollary of Proposition 3.3.4, Theorem 3.3.8, and Theorem 5.1.6:

Definition/Theorem 7.1.4 [stable maps to (Y,L)]. Recall the symplectic cut $\xi:X\to Y$ and let $\underline{\beta}=\xi_*([\beta])\in H_2(Y,L;\mathbb{Z})$. Define the *moduli space* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ of stable maps of type $((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)$ from labelled-bordered Riemann surfaces to (Y,L) to be the central fiber of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ over $0\in B$, with the induced C^∞-*topology*. Then $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ *is Hausdorff and compact. The correspondence*

$$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\ \longrightarrow\ \mathfrak{G}(H_2(Y,L;\mathbb{Z})\xrightarrow{\partial}H_1(L;\mathbb{Z}))\,,\qquad [f]\ \longmapsto\ \tau_{[f]}$$

gives a finite stratification of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ *by the topological type of maps. The central fiber* $\mathcal{K}_0$ *of a standard Kuranishi structure-in-*$\mathcal{C}_{\rm spsccw}$ $\mathcal{K}/B$ *on* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)$ $/B$ *gives a Kuranishi structure-in-*$\mathcal{C}_{\rm spsccw,0}$ *on* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. We will call a Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ thus obtained a *standard Kuranishi structure* on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. The virtual dimension of $\mathcal{K}_0$ is the same as the virtual dimension of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(X,L\,|\,[\beta],\vec{\gamma},\mu)$.

For a $\tau\in\mathfrak{G}:=\mathfrak{G}(H_2(Y,L;\mathbb{Z})\xrightarrow{\partial}H_1(L;\mathbb{Z}))$, denote the layer map $V(\tau)\to\mathbb{Z}_{\geq 0}$ by $layer_\tau$. Then the composition

$$[f]\ \longmapsto\ \tau_{[f]}\ \longmapsto\ \max\{\,0\,,\,|Im(layer_{\tau_{[f]}})|-2\,\}$$

gives a correspondence

$$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\ \longrightarrow\ \mathbb{Z}_{\geq 0}\,.$$

Definition 7.1.5 [layer-structure stratification]. The (finite) collection of the pre-image of the elements of $\mathbb{Z}_{\geq 0}$ under the above correspondence gives, by definition, the *layer-structure stratification* of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. The stratum $\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ associated to $i\in\mathbb{Z}_{\geq 0}$ is called the stratum of *depth* i. A standard Kuranishi structure $\mathcal{K}_0$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ restricts to a Kuranishi structure-in-$\mathcal{C}^{(i)}_{\rm spsccw,0}$ $\mathcal{K}^{(i)}_0$ on $\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ as follows:

- Let $\mathcal{K}/B$ be a standard Kuranishi structure-in-$\mathcal{C}_{\rm spsccw}$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ that gives the Kuranishi structure-in-$\mathcal{C}_{\rm spsccw,0}$ $\mathcal{K}_0$, $\rho\in\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$, $(V_\rho,E_{V_\rho},\Gamma_{V_\rho};s_{V_\rho},\psi_{V_\rho})/B$ be a Kuranishi neighborhood of ρ from $\mathcal{K}$ with ρ treated as a point in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$. Then, by definition, the Kuranishi neighborhood-in-$\mathcal{C}_{\rm spsccw,0}$ of ρ from $\mathcal{K}_0$ is given by

  $$(V_{\rho,0},\Gamma_{V_{\rho,0}}=\Gamma_{V_\rho},E_{V_{\rho,0}}=E_{V_\rho}|_{V_{\rho,0}};s_{V_{\rho,0}}=s_{V_\rho}|_{V_{\rho,0}},\psi_{V_{\rho,0}}=\psi_{V_\rho}|_{V_{\rho,0}})\,,$$

  where $V_{\rho,0}$ is the central fiber of V_ρ/B, which is invariant under Γ_{V_ρ}, and $\Gamma_{V_{\rho,0}}$ is Γ_{V_ρ} that acts on $V_{\rho,0}$.

- By construction V_ρ also fibers over $B[i]$. Let $V^{(i)}_{\rho,0}$ be the central fiber of $V_\rho/B[i]$; then $V^{(i)}_{\rho,0}$ is Γ_{V_ρ}-invariant and the restriction

  $$(\,V^{(i)}_{\rho,0}\,,\,\Gamma_{V^{(i)}_{\rho,0}}=\Gamma_{V_\rho}\,,\,E_{V^{(i)}_{\rho,0}}=E_{V_\rho}|_{V^{(i)}_{\rho,0}}\,;\,s_{V^{(i)}_{\rho,0}}=s_{V_\rho}|_{V^{(i)}_{\rho,0}}\,,\,\psi_{V^{(i)}_{\rho,0}}=\psi_{V_\rho}|_{V^{(i)}_{\rho}}\,)\,.$$

  define a Kuranishi neighborhood-in-$\mathcal{C}^{(i)}_{\rm spsccw,0}$ of $\rho\in\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. The system of transition data in $\mathcal{K}$ restricts to a system of transition data for such system of Kuranishi neighborhoods-in-$\mathcal{C}^{(i)}_{\rm spsccw,0}$ for $\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. This defines $\mathcal{K}^{(i)}_0$.

We shall call such $\mathcal{K}^{(i)}_0$ a *standard Kuranishi structure* on $\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. Note that in the above description, $V^{(i)}_\rho$ has codimension $2i$ in $V_{\rho,0}$; thus $vdim\,\mathcal{K}^{(i)}_0=vdim\,\mathcal{K}_0-2i$. We say that the stratum $\mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ has *virtual codimension* $2i$ (everywhere) in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ Note also that $\mathcal{K}^{(0)}_0=\mathcal{K}_0$.

Similarly, the composition

$$[f] \;\longmapsto\; \tau_{[f]} \;\longmapsto\; |Im\,(layer_{\tau_{[f]}})|$$

gives a correspondence

$$\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu;\vec{s}) \;\longrightarrow\; \mathbb{Z}_{\geq 0}\,.$$

This defines a *layer-structure stratification*

$$\Big\{\,\mathcal{M}^{(i)}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})\,\Big\}_{i\in\mathbb{Z}_{\geq 0}}$$

of $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$. Given a standard Kuranishi structure-in-$\mathcal{C}'_{\rm spsccw}$ $\mathcal{K}'$ for $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$, the same *take-central-fiber-then-restrict* construction in Definition 7.1.5 gives a *standard Kuranishi structure*-in-$\mathcal{C}'^{,(i)}_{\rm spsccw}$ $\mathcal{K}'^{,(i)}$ on $\mathcal{M}^{(i)}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$. The *depth-i stratum* $\mathcal{M}^{(i)}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$ has *virtual codimension* $2i$ in $\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})$.

Lemma 7.1.6 [unique equivalence class]. *Any two standard Kuranishi structures on $\mathcal{M}$ are equivalent, where $\mathcal{M}$ is any of the following moduli spaces:*

$$\begin{aligned}
&\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B\,, && \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W_\lambda,L\,|\,[\beta],\vec{\gamma},\mu)\,,\ \lambda\in B-\{0\}\,,\\
&\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,, && \mathcal{M}^{(i)}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,,\\
&\overline{\mathcal{M}}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})\,, && \mathcal{M}^{(i')}_{(g,h),(n+l(\vec{s}),\vec{m})}(Z,L;D\,|\,\beta',\vec{\gamma},\mu';\vec{s})\,.
\end{aligned}$$

Proof. For $\mathcal{M}=\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$, let $\mathcal{K}_i$ be the Kuranishi structure associated to a fine system of saturated obstruction local bundles $\mathcal{E}_i$ $i=1,\,2$; cf. Sec. 5.4. Then there exists another fine system $\mathcal{E}_3$ of saturated obstruction local bundles so that both $\mathbf{F}(\mathcal{E}_1)$ and $\mathbf{F}(\mathcal{E}_2)$ are orbifold sub-fibrations of $\mathbf{F}(\mathcal{E}_3)$. The lemma for $\mathcal{M}=\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)/B$ follows then from the construction in Sec. 5.4. Similarly for all other choices of $\mathcal{M}$ in the list.

□

Topological spaces with a Kuranishi structure: morphisms and fibered products.[28]

We digress here to define two fundamental notions that we did not truly need until now: *morphisms* and *fibered products* of topological spaces with a Kuranishi structure. These two notions are fundamental in any category of spaces/geometries.

Definition 7.1.7 [Kuranishi structure: morphism]. Let X_i be a topological space with a Kuranishi structure $\mathcal{K}_{i,0}$ modelled in a category $\mathcal{C}$, $i=1,\,2$. A *morphisms* from $(X_1,\mathcal{K}_{1,0})$ to $(X_2,\mathcal{K}_{2,0})$ is a *continuous map* $\varphi: X_1\to X_2$ together with a *tuple of systems of morphisms* $\varphi^\sharp := (\varphi_{V.}\,,\,\varphi_{\Gamma.}\,,\,\varphi_{E.})\,:\,\mathcal{K}_1\to\mathcal{K}_2$, where $\mathcal{K}_1\sim\mathcal{K}_{1,0}$ and $\mathcal{K}_2\sim\mathcal{K}_{2,0}$, consisting of a system of continuous maps $\varphi_{V.}: V_{x_1}\to V_{\varphi(x_1)}$, group homomorphisms $\varphi_{\Gamma.}:\Gamma_{V_{x_1}}\to\Gamma_{V_{\varphi(x_1)}}$, and $\varphi_{\Gamma.}$-equivariant bundle maps $\varphi_{E.}: E_{V_{x_1}}\to E_{V_{\varphi(x_1)}}$ that covers $\varphi_{V.}$, such that[29]

[28] Our definitions here are tailored to what we have explicitly, what we are allowed to do in these cases, and what we are aiming for. There is still room for further polishments/generalizations of these notions/definitions.

[29] The necessity of passing to equivalent Kuranishi structures to define morphisms is enforced on us when one considers the simplest case: the notion of *embeddings* of a topological space-with-Kuranishi-structure to another. This also makes the definition ring more compatibly with its parallel in algebraic geometry. There one has the notion of two-term locally-free resolutions of a perfect tangent-obstruction complex on the moduli stack in question. Morphisms between such complexes are at the level of derived categories of coherent sheaves on the moduli stacks. In particular, they have to pass to quasi-isomorphisms of chain complexes, rather than directly on the two chain complexes one wants to compare.

(1) [compatibility on each Kuranishi neighborhood]:

$$\varphi_{E_\cdot} \circ s_p = s_{\varphi(p)} \circ \varphi_{V_\cdot} \text{ on } V_p\,, \qquad \varphi \circ \psi_p = \psi_{\varphi(p)} \circ \varphi_{V_\cdot} \text{ on } s_p^{-1}(0) \subset V_p \qquad \text{for } p \in X_1;$$

(2) [gluability: compatibility with transition data]:

$$\varphi_{V_\cdot}(V_{qp}) \subset V_{\varphi(q)\varphi(p)}\,, \qquad \varphi_{\Gamma_\cdot} \circ h_{qp} = h_{\varphi(q)\varphi(p)} \circ \varphi_{\Gamma_\cdot}\,,$$
$$\varphi_{V_\cdot} \circ \phi_{qp} = \phi_{\varphi(q)\varphi(p)} \circ \varphi_{V_\cdot}\,, \qquad \varphi_{E_\cdot} \circ \hat{\phi}_{qp} = \hat{\phi}_{\varphi(q)\varphi(p)} \circ \varphi_{E_\cdot}\,.$$

For convenience, we will denote a morphism as $(\varphi, \varphi^\sharp) : (X_1, \mathcal{K}_{1,0}) \to (X_2, \mathcal{K}_{2,0})$ with it understood that $\varphi^\sharp$ is defined subject to passing to an equivalent Kuranishi structure.

Definition/Example 7.1.8 [embedding]. A morphism $(\varphi, \varphi^\sharp) : (X_1, \mathcal{K}_1) \to (X_2, \mathcal{K}_2)$ is called an *embedding* if both φ and $\varphi^\sharp$ are embeddings.

Definition/Example 7.1.9 [covering map]. A morphism $(\varphi, \varphi^\sharp) : (X_1, \mathcal{K}_1) \to (X_2, \mathcal{K}_2)$ is called a *covering map* if φ is a covering map and $\varphi^\sharp$ is an isomorphism[30]. In this case, $vdim\,\mathcal{K}_1 = vdim\,\mathcal{K}_2$.

Definition/Example 7.1.10 [virtual bundle map]. Given a topological space S, we shall regard it also as a topological space with the *trivial Kuranishi structure* $\mathcal{K}^{\text{trivial}}$ that consists of exactly one Kuranishi neighborhood $(S, \{e\}, \mathbf{0}_S := S \times \{0\}; 0, Id_S)$. A morphism $(\varphi, \varphi^\sharp) : (X, \mathcal{K}) \to S = (S, \mathcal{K}^{\text{trivial}})$, is called a *virtual bundle map* if $\varphi : X \to S$ is continuous and $\varphi^\sharp : \mathcal{K} \to \mathcal{K}^{\text{trivial}}$ is a bundle map[31]. Note that, in this case, the Γ_{V_p}-action on V_p leaves each fiber of $V_p \to S$ invariant.

Definition 7.1.11 [Kuranishi structure: fibered product]. Let S be a topological space with the trivial Kuranishi structure $\mathcal{K}^{\text{trivial}}$. Given two virtual bundle maps

$$(X_1, \mathcal{K}_1) \overset{(\varphi_1, \varphi_1^\sharp)}{\longrightarrow} S \overset{(\varphi_2, \varphi_2^\sharp)}{\longleftarrow} (X_2, \mathcal{K}_2)\,,$$

define the *fibered product* $(X_1 \times_S X_2, \mathcal{K}_1 \times_S \mathcal{K}_2)$ of $(X_1, \mathcal{K}_1)$ and $(X_2, \mathcal{K})$ *over* S to be the topological space

$$X_1 \times_S X_2 \;:=\; (\varphi_1 \times \varphi_2)^{-1}(\Delta_S) \subset X_1 \times X_2\,,$$

where $\varphi_1 \times \varphi_2 : X_1 \times X_2 \to S \times S$ and $\Delta_S \subset S \times S$ is the diagonal, equipped with the following Kuranishi structure:

(1) [the induced Kuranishi neighborhood at $(p_1, p_2) \in X_1 \times_S X_2$]:

- define $V_{(p_1,p_2)} := V_{p_1} \times_S V_{p_2}$ and let $V_{p_1} \overset{\pi_1}{\leftarrow} V_{p_1} \times_S V_{p_2} \overset{\pi_2}{\to} V_{p_2}$ be the projection maps;
- the diagonal action of $\Gamma_{V_{p_1}} \times \Gamma_{V_{p_2}}$ on $V_{p_1} \times V_{p_2}$ leaves $V_{(p_1,p_2)} = V_{p_1} \times_S V_{p_2}$ invariant, define $\Gamma_{V_{(p_1,p_2)}} = \Gamma_{V_{p_1}} \times \Gamma_{V_{p_2}}$ now acting on $V_{(p_1,p_2)}$;
- let $E_{V_{(p_1,p_2)}} := \pi_1^* E_{V_{p_1}} \oplus \pi_2^* E_{V_{p_2}}$ on $V_{(p_1,p_2)}$, then the induced $\Gamma_{V_{(p_1,p_2)}}$-action on $E_{V_{(p_1,p_2)}}$ is equivariant;
- let $s_{(p_1,p_2)} = (\pi_1^* s_{p_1}\,,\, \pi_2^* s_{p_2})$, then $s_{(p_1,p_2)}$ is a $\Gamma_{V_{(p_1,p_2)}}$-invariant section of $E_{V_{(p_1,p_2)}}$;

[30] By this we mean that all maps in $\varphi^\sharp$ are isomorphisms. Note that $\varphi^\sharp$ alone sees only the local properties of the topology. That maps in $\varphi^\sharp$ are all isomorphisms implies only that $\varphi : X_1 \to X_2$ is a local isomorphism.

[31] By this we mean that each $\varphi_{V_\cdot} : V_p \to S$, $p \in X$, in $\varphi^\sharp$ is a *bundle map* (i.e. locally trivial fibration) *over a non-empty open subset* of S.

· let $\psi_{(p_1,p_2)} = (\,\psi_{p_1} \times \psi_{p_2} \,:\, s_{p_1}^{-1}(0) \times s_{p_2}^{-1}(0) \to X_1 \times X_2\,)|_{V_{p_1} \times_S V_{p_2}}$, then $\psi_{(p_1,p_2)}$ is a map from $s_{(p_1,p_2)}^{-1}(0)$ to $X_1 \times_S X_2$.

The 5-tuple $(V_{(p_1,p_2)}\,,\,\Gamma_{V_{(p_1,p_2)}}\,,\,E_{V_{(p_1,p_2)}}\,;\,s_{(p_1,p_2)}\,,\,\psi_{(p_1,p_2)})$ defined above is called the *induced Kuranishi neighborhood* of $(p_1,p_2) \in X_1 \times_S X_2$ from $\mathcal{K}_1$ and $\mathcal{K}_2$. Define $\mathfrak{N}^{(0)}$ to be the system of Kuranishi neighborhoods of $X_1 \times_S X_2$ thus constructed.

(2) [transition data]:

· the diagonal product construction defines a canonical Kuranishi structure $\mathcal{K}_1 \times \mathcal{K}_2$ on $X_1 \times X_2$; the Kuranishi neighborhoods for $X_1 \times_S X_2$, as constructed above, are embedded in the Kuranishi neighborhoods in $\mathcal{K}_1 \times \mathcal{K}_2$; the canonical transition data in $\mathcal{K}_1 \times \mathcal{K}_2$ restricts to a system $\mathfrak{N}^{(1)}$ of transition data for $\mathfrak{N}^{(0)}$.

Define $\mathcal{K}_1 \times_S \mathcal{K}_2 = (\mathfrak{N}^{(0)}\,,\,\mathfrak{N}^{(1)})$. When $S = \{pt\}$, we call $(X_1 \times X_2, \mathcal{K}_1 \times \mathcal{K}_2)$ the *direct product*, or simply the *product*, of $(X_1, \mathcal{K}_1)$ and $(X_2, \mathcal{K}_2)$.

By construction, there are a *tautological virtual bundle map*

$$(\varphi_1 \times_S \varphi_2\,,\,\varphi_1^\sharp \times_S \varphi_2^\sharp)\;:(X_1 \times_S X_2\,,\,\mathcal{K}_1 \times_S \mathcal{K}_2)\;\longrightarrow\;S\,,$$

an embedding morphism $(X_1 \times_S X_2, \mathcal{K}_1 \times_S \mathcal{K}_2) \to (X_1 \times X_2, \mathcal{K}_1 \times \mathcal{K}_2)$, and projection morphisms

$$(X_1, \mathcal{K}_1) \overset{(\pi_1,\pi_1^\sharp)}{\longleftarrow} (X_1 \times_S X_2, \mathcal{K}_1 \times_S \mathcal{K}_2) \overset{(\pi_2,\pi_2^\sharp)}{\longrightarrow} (X_2, \mathcal{K}_2)\,.$$

Note that $vdim\,(\mathcal{K}_1 \times_S \mathcal{K}_2) = vdim\,\mathcal{K}_1 + vdim\,\mathcal{K}_2 - dim\,S$ when S is a manifold and both $\mathcal{K}_1$ and $\mathcal{K}_2$ are modelled on the category of CW-complexes.

The degeneration-gluing relations of Kuranishi structures.

We are now ready to give the degeneration-gluing relations of Kuranishi structures of the several moduli spaces that occur in the study.

The following bookkeeping graphs are adapted from [Li1: Sec. 4.2]:

Definition 7.1.12 [admissible weighted graph]. Given a relative pair $(Z, L; D)$ with a symplectic/totally-real submanifold, an *admissible weighted graph* Γ for $(Z, L; D)$ is a graph without edges together with the following data:

(1) an ordered collection of *hands*, *fingers*[32], and *legs*; an ordered collection of *weighted roots*; a *relative index function* and two *weight functions* on the vertex set $\mu' : V(\Gamma) \to \mathbb{Z}$, $g : V(\Gamma) \to \mathbb{Z}_{\geq 0}$, and $b : V(\Gamma) \to H_2(Z, L; \mathbb{Z})$; a *weight function* on the ordered set of hands $\gamma : H(\Gamma) \to H_1(L; \mathbb{Z})$ such that $\partial b(v) = \sum_\bullet h_{v,\bullet}$, where $v \in V(\Gamma)$ and the sum is over the ordered subset of hands that are attached to v;

(2) Γ is *relatively connected* in the sense that either $|V(\Gamma)| = 1$ or each vertex in $V(\Gamma)$ has at least one root attached to it.

[32]The order of fingers is lexicographic: first by the order of the hands they are attached to and then by the order within each group that are attached to the same hand.

Definition 7.1.13 [admissible quadruple]. Given a gluing $(Y, L) = (Y_1, L_1) \cup_D (Y_2, L_2)$ of relative pairs from a symplectic cut, let Γ_1 and Γ_2 be a pair of admissible weighted graphs for $(Y_1, L_1; D)$ and $(Y_2, L_2; D)$ respectively. Suppose that Γ_1 and Γ_2 have identical number l of roots, h_1-many and h_2-many hands, n_1-many and n_2-many legs respectively. Let $h = h_1 + h_2$, $n = n_1 + n_2$, $I_{\text{hand}} \subset \{1, \ldots, h\}$ be a set of h_1 elements, and $I_{\text{leg}} \subset \{1, \ldots, n\}$ be a set of n_1 elements. Then $(\Gamma_1, \Gamma_2, I_{\text{hand}}, I_{\text{leg}})$ is called an *admissible quadruple* if the following conditions hold:

(1) the weights on the roots of Γ_1 and Γ_2 coincide: $r_{1,i} = r_{2,i}$, $i = 1, \ldots, l$;

(2) after connecting the i-th root of Γ_1 and the i-th root of Γ_2 for all i, the resulting new graph with h hands, (accompanying fingers), n legs and no roots is connected.

Re-ordering of roots defines an equivalence relation $\sim$ on the set Ω of admissible quadruples. Define $\bar{\Omega} := \Omega/\sim$. Given an admissible quadruple $\eta = (\Gamma_1, \Gamma_2, I_{\text{hand}}, I_{\text{leg}})$, denote by $Per_r(\eta)$ the set of permutations of the roots in Γ_1 that leaves η unchanged.

Note that I_{hand} determines the order of the hands on the graph from gluing paired roots by the unique bijection $\{1, \cdots, h_1\} \amalg \{1, \cdots, h_2\} \to \{1, \cdots, h\}$ such that it preserves the orders of both $\{1, \cdots, h_1\}$ and $\{1, \cdots, h_2\}$ and that the image of $\{1, \cdots, h_1\}$ is I_{hand}. The order of fingers on the glued graph is then determined lexicographically. Similarly, I_{leg} determines the order of legs on the glued graph.

Given an admissible quadruple $\eta = (\Gamma_1, \Gamma_2, I_{\text{hand}}, I_{\text{leg}})$ as above with $(Y, L) = (Y_1, L_1) \cup_D (Y_2, L_2)$ = the degenerate fiber W_0 of W/B, one has

· the *genus function*

$$g(\eta) \;:=\; l + 1 - |V(\Gamma_1 \amalg \Gamma_2)| + \textstyle\sum_{v \in V(\Gamma_1) \cup V(\Gamma_2)} g(v) \;\in\; \mathbb{Z}_{\geq 0}\,,$$

· the *curve-class function*

$$b(\eta) \;:=\; \iota_{1,*}\left(\textstyle\sum_{v \in V(\Gamma_1)} b_{\Gamma_1}(v)\right) + \iota_{2,*}\left(\textstyle\sum_{v \in V(\Gamma_2)} b_{\Gamma_2}(v)\right) \;\in\; H_2(Y, L; \mathbb{Z})\,,$$

where $\iota_i : (Y_i, L_i) \hookrightarrow (Y, L)$, $i = 1, 2$, are the inclusion maps,

· the total *index* $\mu(\eta) = \sum_{v \in V(\Gamma_1)} \mu'(v) \;+\; \sum_{v \in V(\Gamma_2)} \mu'(v)$.

Let $\vec{m}(\eta) := (m_1, \cdots, m_h)$ be the tuple of numbers of fingers attached to hands $\in H(\Gamma_1) \cup H(\Gamma_2)$ and $\vec{\gamma}(\eta)$ be the tuple of values of $\gamma_1 \cup \gamma_2 : H(\Gamma_1) \cup H(\Gamma_2) \to H_1(L; \mathbb{Z})$, both with respect to the order on $H(\Gamma_1) \cup H(\Gamma_2)$ specified by I_{hand}. Define the *type* of η to be

$$|\eta| \;:=\; ((g(\eta),\, h_1 + h_2),\, (n_1 + n_2,\, \vec{m}(\eta)) \,|\, b(\eta),\, \vec{\gamma}(\eta),\, \mu(\eta))\,.$$

For each $\eta = (\Gamma_1, \Gamma_2, I_{\text{hand}}, I_{\text{leg}})$, with l-many roots, such that $|\eta| = ((g, h), (n, \vec{m}) \,|\, \underline{\beta}, \vec{\gamma}, \mu)$, there are five moduli spaces of stable map associated to it:

$$\overline{\mathcal{M}}(Y_1, L_1; D \,|\, \Gamma_1)\,, \qquad \overline{\mathcal{M}}(Y_2, L_2; D \,|\, \Gamma_2)\,, \qquad \overline{\mathcal{M}}(Y_1, L_1; D \,|\, \Gamma_1) \times_{D^l} \overline{\mathcal{M}}(Y_2, L_2; D \,|\, \Gamma_2)\,,$$
$$\text{sub-orbifolds } \overline{\mathcal{M}}((Y_1, L_1; D) \amalg (Y_2, L_2; D) \,|\, \eta) \text{ and } \overline{\mathcal{M}}(Y, L \,|\, \eta) \text{ of } \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y, L \,|\, \underline{\beta}, \vec{\gamma}, \mu)\,.$$

We explain each of these spaces and their standard Kuranishi structures below.

Let Γ be an admissible weighted graph for $(Z, L; D)$., The restriction of the data encoded by Γ to each vertex $v \in |\Gamma|$ specifies a unique type data $((g_v, h_v), (n_v, \vec{m}_v) \,|\, \underline{\beta}_v, \vec{\gamma}_v, \mu'_v; \vec{s}_v)$. Define the moduli space by the direct product:

$$\overline{\mathcal{M}}(Z, L; D \,|\, \Gamma) \;:=\; \prod_{v \in |\Gamma|} \overline{\mathcal{M}}_{(g_v,h_v),(n_v,\vec{m}_v)}(Z, L; D \,|\, \underline{\beta_v}, \vec{\gamma}_v, \mu'_v; \vec{s}_v)\,.$$

A standard Kuranishi structure $\mathcal{K}'_{(Z,L;D\,|\,\Gamma)}$ on $\overline{\mathcal{M}}(Z,L;D\,|\,\Gamma)$ is by definition the direct product of a standard Kuranishi structure on each moduli-space component $\overline{\mathcal{M}}_{(g_v,h_v),(n_v,\vec{m}_v)}(Z,L;D\,|\,\underline{\beta_v},\vec{\gamma}_v,\mu'_v;\vec{s}_v)$. Let l be the number of roots of Γ. Then the saturatedness of the obstruction-space local bundles for each $\overline{\mathcal{M}}_{(g_v,h_v),(n_v,\vec{m}_v)}(Z,L;D\,|\,\underline{\beta_v},\vec{\gamma}_v,\mu'_v;\vec{s}_v)$ implies that there is a virtual bundle map

$$(\mathbf{q}\,,\,\mathbf{q}^\sharp)\ :\ (\overline{\mathcal{M}}(Z,L;D\,|\,\Gamma)\,,\,\mathcal{K}'_{(Z,L;D\,|\,\Gamma)})\ \longrightarrow\ (D^l,\mathcal{K}^{\rm trivial})\,.$$

Apply the above to Γ_1 and Γ_2 from the admissible quadruple with l-many roots, one obtains the fibered-product moduli space $\overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2,L_2;D\,|\,\Gamma_2)$ with a standard Kuranishi structure defined to be the fibered product $\mathcal{K}'_{(Y_1,L_1;D\,|\,\Gamma_1)}\times_{D^l}\mathcal{K}'_{(Y_2,L_2;D_2\,|\,\Gamma_2)}$.

Let

$$\Phi_\eta : \overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2,L_2;D\,|\,\Gamma_2)\ \longrightarrow\ \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$$

be the gluing orbifold map, whose corresponding map at the underlying topological space is given by

$$\begin{aligned}(\,f_1:\Sigma_1\to(Y_{1,[k_1]},L_{1,[k_1]};D_{[k_1]})\,,\ f_2:\Sigma_2\to(Y_{2,[k_2]},L_{2,[k_2]};D_{[k_2]})\,)&\\ \longmapsto\ f=f_1\cup f_2:\Sigma\to(Y_{[k]},L_{[k]})\,,\quad k=k_1+k_2\,,&\end{aligned}$$

where Σ is the gluing $\Sigma_1\cup\Sigma_2$ of Σ_1 and Σ_2 along their paired distinguished marked points; $Y_{[k_1+k_2]}$ is the gluing of $(Y_{1,[k_1]},L_{1,[k_1]};D_{[k_1]})$ and $(Y_{2,[k_2]},L_{2,[k_2]};D_{[k_2]})$ by $D_{[k_1]}\simeq D\simeq D_{[k_2]}$. Denote the image by

$$\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$$

with the induced sub-orbifold structure and the C^∞-topology from $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$; then Φ_η is an orbifold covering map of pure degree $|Per_r(\eta)|$ to $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$. A standard Kuranishi structure on $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$ can be constructed as follows. Since Φ_η is a covering map, a Kuranishi neighborhood $(V_\rho,\Gamma_{V_\rho},E_{V_\rho};s_\rho,\psi_\rho)$ of $\rho\in\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$ can be taken to be a Kuranishi neighborhood of a $\rho'\in\Phi_\eta^{-1}(\rho)$; i.e. via the fibered-product construction. In this way one obtains a system $\mathfrak{N}^{(0)}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}$ of Kuranishi neighborhoods on $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$. Assume that all these neighborhoods are small, then the system $\{\iota_{(k_1,k_2)}\}_{k_1+k_2=k}$ of almost-complex pseudo-embeddings

$$\begin{aligned}&\iota_{(k_1,k_2)}\ :\\ &\big(\,((Y_1[k_1],L_1[k_1];D[k_1])\times A[k_2])\cup_{D[k_1]\times A[k_2]\simeq A[k_1]\times D[k_2]}(A[k_1]\times(Y_2[k_2],L[k_2];D[2]))\,\big)/(A[k_1]\times A[k_2])\\ &\longrightarrow\ W[k]/B[k]\end{aligned}$$

induces a natural embedding of $\mathfrak{N}^{(0)}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}$ into a standard Kuranishi structure $\mathcal{K}$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$. Here, $D[k_1]\times A[k_2]$ and $A[k_1]\times D[k_2]$ are glued via their canonical isomorphisms with $D\times A[k_1]\times A[k_2]=D\times A[k]$, and the pseudo-embedding $A[k]=A[k_1]\times A[k_2]\to B[k]$ is given by $(\vec{\lambda},\vec{\lambda}')\mapsto(\vec{\lambda}\,,\,0\,,\,\vec{\lambda}')$. The transition data from $\mathcal{K}$ then restricts[33] to an transition data on $\mathfrak{N}^{(0)}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}$. By construction, one has an embedding morphism

$$\begin{aligned}\big(\,\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)\ ,\ \mathcal{K}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}\,\big)&\\ \longrightarrow\ (\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,,\,\mathcal{K})\,.&\end{aligned}$$

[33]Note that in general one has to pass to an equivalence to make a system of Kuranishi neighborhoods gluable. However, here the system $\mathfrak{N}^{(0)}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}$ is descended from a covering morphism of a Kuranishi structure. We only need to know whether the transition data also descends in our case. The latter is implied by the existence of a natural embedding of $\mathfrak{N}^{(0)}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}$ into $\mathcal{K}$, (i.e. an embedding at the level of universal maps on universal curves).

With respect to $\mathcal{K}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}$, the covering map Φ_η lifts to a covering morphism

$$(\Phi_\eta,\Phi_\eta^\sharp)\,:\,\big(\,\overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2,L_2;D\,|\,\Gamma_2)\,,\,\mathcal{K}'_{(Y_1,L_1;D\,|\,\Gamma_1)}\times_{D^l}\mathcal{K}'_{(Y_2,L_2;D\,|\,\Gamma_2)}\,\big)$$
$$\longrightarrow\,\big(\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)\,,\,\mathcal{K}_{((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)}\big)\,.$$

One can check that, with these standard Kuranishi structures,

$$\begin{aligned}&vdim\,\big(\,\overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2,L_2;D\,|\,\Gamma_2)\,\big)\\&=\,vdim\,\overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\,+\,vdim\,\overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\,-\,2\,l\,(N-1)\\&=\,vdim\,\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,,\end{aligned}$$

where, recall that, $dim\,Y=2N$. This implies that

$$vdim\,\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)\;=\;vdim\,\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,.$$

Finally, let

$$\overline{\mathcal{M}}(Y,L\,|\,\eta)$$

be the same suborbifold $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$ of $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ but with a Kuranishi structure constructed as follows. Consider the defining embedding morphism

$$(\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,,\,\mathcal{K}_0)\;\longrightarrow\;(\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B,L\,|\,[\beta],\vec{\gamma},\mu)\,,\,\mathcal{K})\,.$$

Let $\mathcal{K}=(\mathfrak{N}^{(0)},\mathfrak{N}^{(1)}$. For $\rho\in\overline{\mathcal{M}}(Y,L\,|\,\eta)$ from gluing $f_1:\Sigma_1\to(Y_{1,[k_1]},L_{1,[k_1]};D_{[k_1]})$ and $f_2:\Sigma_2\to(Y_{2,[k_2]},L_{2,[k_2]};D_{[k_2]})$, one has that $V_\rho\in\mathfrak{N}^{(0)}$ fibers over $B[k]$,where $k=k_1+k_2$. Let $V_{\rho,\eta}\subset V_\rho$ be the preimage of the hyperplane $\{\vec{\lambda}=(\lambda_0,\cdots,\lambda_k)\,:\,\lambda_{k_1}=0\}\subset B[k]$ under this fibration with the multiplicity of the irreducible components of fibers encoded. Then $V_{\rho,\eta}$ is Γ_{V_ρ}-invariant. Define $\Gamma_{V_{\rho,\eta}}=\Gamma_{V_\rho}$, now action on $V_{\rho,\eta}$; $E_{V_{\rho,\eta}}=E_{V_\rho}|_{V_{\rho,\eta}}$; $s_{\rho,\eta}=s_\rho|_{V_{\rho,\eta}}$; and $\psi_{\rho,\eta}=\psi_\rho|_{V_{\rho,\eta}}$. Then the system $\mathfrak{N}^{(0)}_\eta$ of 5-tuples $(V_{\rho,\eta},\Gamma_{V_{\rho,\eta}},E_{V_{\rho,\eta}};s_{\rho,\eta},\psi_{\rho,\eta})$ thus constructed defines a system of Kuranishi neighborhood on $\overline{\mathcal{M}}(Y,L\,|\,\eta)$. The system $\mathfrak{N}^{(1)}$ of transition data in $\mathcal{K}$ restricts to give a system $\mathfrak{N}^{(1)}_\eta$ of transition data for $\mathfrak{N}^{(0)}_\eta$. The pair $\mathcal{K}_\eta:=(\mathfrak{N}^{(0)}_\eta,\mathfrak{N}^{(1)}_\eta)$ thus defines a Kuranishi structure on $\overline{\mathcal{M}}(Y,L\,|\,\eta)$. Kuranishi structures on $\overline{\mathcal{M}}(Y,L\,|\,\eta)$ thus obtained will be called *standard* Kuranishi structures on $\overline{\mathcal{M}}(Y,L\,|\,\eta)$. By construction, one also has:

$$vdim\,\overline{\mathcal{M}}(Y,L\,|\,\eta)\;=\;vdim\,\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)\,.$$

The following theorem that relates these moduli spaces and their standard Kuranishi structures should be compared to [Li2: Corollary 3.13. Lemma 3.14, Theorem 3.15]. It is in effect a re-phrasing of [Li2] in terms of the Fukaya-Ono setting and at the level of Kuranishi structures, rather than of virtual fundamental classes or chains:

Theorem 7.1.14 [degeneration-gluing: Kuranishi structure]. *Regard X as a fiber W_{λ_0} of W/B over $\lambda_0\in B-\{0\}$. Recall the symplectic cut $\xi:(X,L)\to(Y,L)=(Y_1,L_1)\cup_D(Y_2,L_2)$ Given a type $((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)$ of stable maps to (X,L), let $\underline{\beta}=\xi_*([\beta])\in H_2(Y,L;\mathbb{Z})$ and $\bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}$ be the equivalence of admissible quadruples η such that $|\eta|=((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)$. Then, the following statements hold, up to an equivalence of Kuranishi structures:*

(1) *A standard Kuranishi structure $\mathcal{K}_{\lambda_0}$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(X,L\,|\,[\beta],\vec{\gamma},\mu)$ and a standard Kuranishi structure-in-$\mathcal{C}_{\rm spsccw,0}$ $\mathcal{K}_0$ on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ are related as fibers of a standard Kuranishi structure-in-$\mathcal{C}_{\rm spsccw}$ $\mathcal{K}/B$, flat over B.*

(2) *There is a decomposition of moduli space*

$$\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu) \;=\; \cup_{\eta\in\bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}}\overline{\mathcal{M}}(Y,L\,|\,\eta)\,.$$

The two sub-orbifolds $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$ *and* $\overline{\mathcal{M}}(Y,L\,|\,\eta)$ *of* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ *are identical in* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$.

(3) *The restriction* $\mathcal{K}_{0,\eta}$ *of the Kuranishi structure* $\mathcal{K}_0$ *on* $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y,L\,|\,\underline{\beta},\vec{\gamma},\mu)$ *to the component* $\overline{\mathcal{M}}(Y,L\,|\,\eta)$ *is equivalent to the Kuranishi structure* $\mathcal{K}_\eta$ *on* $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$, *except that* $\mathcal{K}_{0,\eta}$ *carries a multiplicity* $m(\eta)$. *Let* $\vec{s}=(s_1,\cdots,s_l)$ *be the weights of the ordered roots in* η; *then* $m(\eta)=m(\vec{s}):=s_1\cdots s_l$. *In notation* $\mathcal{K}_{0,\eta}=m(\eta)\,\mathcal{K}_\eta$.

(4) *Let* $\eta=(\Gamma_1,\Gamma_2,I_{\rm hand},I_{\rm leg})$ *with* l*-many roots. Then, the Kuranishi structure* $\mathcal{K}_\eta$ *on* $\overline{\mathcal{M}}((Y_1,L_1;D)\amalg(Y_2,L_2;D)\,|\,\eta)$ *is locally equivalent to the Kuranishi structure* $\mathcal{K}'_1\times_{D^l}\mathcal{K}'_2$ *on* $\overline{\mathcal{M}}(Y_1,L_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2,L_2;D\,|\,\Gamma_2)$ *under the* $|Per_r(\eta)|$*-fold covering map* Φ_η.

We use the following "formula" to summarize/encapsulate (1), (2), (3), *and* (4)*:*

$$\begin{aligned}[\mathcal{K}_\lambda]\;\leftrightarrow\;[\mathcal{K}_0] \;&=\; \cup_{\eta\in\bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}}[\mathcal{K}_{0,\eta}] \;=\; \cup_{\eta\in\bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}}m(\eta)\,[\mathcal{K}_\eta]\\ &=\; \cup_{\eta\in\bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}}\frac{m(\eta)}{|Per_r(\eta)|}\,\Phi_{\eta\,*}\,[\mathcal{K}'_1\times_{D^l}\mathcal{K}'_2]\,.\end{aligned}$$

Proof. We give only a sketch here and omit the tedious details. Statement (1) is by the definition of $\mathcal{K}_0$. Statement (2) follows by considering the topological types of maps. The multiplicity $m(\eta)$ in Statement (3) arise from the scheme structure of the centra fiber of $\Xi_{\vec{s}}\to\mathbb{C}$. Statement (4) requires a comparison of the fibered product of Kuranishi structures and that from a restriction. Here, as well as whenever we need to justify the equivalence of two standardly constructed Kuranishi structures on a same moduli space in question, is where Siebert's work [Sie1] plays roles again and again. Associated to a Kuranishi structure $\mathcal{K}$ is a fine system $\mathcal{E}_\mathcal{K}$ of saturated obstructed local bundles as sub-fibrations in the related $\check{L}^p$-obstruction space fibration $T^2_{\check{\mathcal{W}}^{1,p}(\cdots)}$ as in Sec. 5.4; and vice versa. To construct the equivalence of two given two Kuranishi structures $\mathcal{K}_1$ and $\mathcal{K}_2$, one constructs an appropriate fine system $\mathcal{E}$ of local bundles that contains both $\mathcal{E}_{\mathcal{K}_i}$ as sub-fibrations.

□

We emphasize that, at the level of Kuranishi structures, the above degeneration-gluing relations under a symplectic cut hold for *both* closed Gromov-Witten theory and open Gromov-Witten theory and by *the same* reason.

Example: Li-Ruan/Li degeneration formula of closed Gromov-Witten invariants.

When L is empty, the domain of maps are closed nodal Riemann surfaces and we resume the moduli space

$$\overline{\mathcal{M}}_{g,n}(W_\lambda,[\beta]) \;:=\; \overline{\mathcal{M}}_{(g,0),(n,0)}(W_\lambda\,|\,[\beta]) \;=\; \amalg_{\beta''\in[\beta]}\overline{\mathcal{M}}_{g,n}(W_\lambda,\beta'')$$

in closed Gromov-Witten theory. The notion of admissible quadruples in Definition 7.1.13 is reduced to the notion of *admissible triples* $\eta=(\Gamma_1,\Gamma_2,I=I_{\rm leg})$ (cf. [Li1: Definition 4.11]) and

its *type* is now a triple of the form $(\hat{g}, \hat{n}; \hat{\underline{\beta}})$. Denote by $\bar{\Omega}_{(g,n,\underline{\beta})}$ the set of equivalence classes of admissible triples η such that $|\eta| = (g, n; \underline{\beta})$. Then, Theorem 7.1.14 reduces to

$$\overline{\mathcal{M}}_{g,n}(Y, \underline{\beta}) \;=\; \cup_{\eta \in \bar{\Omega}_{(g,n;\underline{\beta})}} \overline{\mathcal{M}}(Y \,|\, \eta)\,.$$

and, in the encapsulated form,

$$\begin{aligned} [\mathcal{K}_\lambda] \;\leftrightarrow\; [\mathcal{K}_0] \;&=\; \cup_{\eta \in \bar{\Omega}_{(g,n;\underline{\beta})}} [\mathcal{K}_{0,\eta}] \;=\; \cup_{\eta \in \bar{\Omega}_{(g,n;\underline{\beta})}} m(\eta)\, [\mathcal{K}_\eta] \\ &=\; \cup_{\eta \in \bar{\Omega}_{(g,n;\underline{\beta})}} \tfrac{m(\eta)}{|Per_r(\eta)|}\, \Phi_{\eta *} [\mathcal{K}'_1 \times_{D^l} \mathcal{K}'_2]\,. \end{aligned}$$

A virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(W_\lambda, [\beta])]^{virt}$ of the expected dimension and supported in $s^{-1}_{\rho,\lambda}(0)$ on each Kuranishi neighborhood $V_{\rho;\lambda}$, $\rho \in \overline{\mathcal{M}}_{(g,0),(n,0)}(W_\lambda, [\beta])$ can be constructed[34] via Kuranishi structures $\mathcal{K}_\lambda$. Similarly, for

$$\begin{aligned} &[\overline{\mathcal{M}}_{g,n}(W/B, [\beta])/B]^{virt}\,, \qquad [\overline{\mathcal{M}}_{g,n}(Y, \underline{\beta})]^{virt}\,, \qquad [\overline{\mathcal{M}}(Y \,|\, \eta)]^{virt}\,, \qquad [\overline{\mathcal{M}}(Y_1; D \,|\, \Gamma_1)]^{virt}\,, \\ &[\overline{\mathcal{M}}(Y_2; D \,|\, \Gamma_2)]^{virt}\,, \quad [\overline{\mathcal{M}}(Y_1; D \,|\, \Gamma_1) \times_{D^l} \overline{\mathcal{M}}(Y_2; D \,|\, \Gamma_2)]^{virt}\,, \quad [\overline{\mathcal{M}}((Y_1; D) \amalg (Y_2; D) \,|\, \eta)]^{virt} \end{aligned}$$

that are constructed from Kuranishi structures

$$\mathcal{K}/B\,, \quad \mathcal{K}_0\,, \quad \mathcal{K}_{0,\eta}\,, \quad \mathcal{K}'_1\,, \quad \mathcal{K}'_2\,, \quad \mathcal{K}'_1 \times_{D^l} \mathcal{K}'_2\,, \quad \mathcal{K}_\eta$$

respectively. Since equivalent Kuranishi structures give identical virtual fundamental class, the above degeneration-gluing formula of Kuranishi structures can be reduced[35] to the degeneration/gluing formulas of Li-Ruan [L-R] and Li [Li2][36]:

[34] This step is not trivial. It includes a re-doing of [L-T3] and [Sie2] in the Fukaya-Ono family setting. Readers who are not familiar with ibidem may think of a Kuranishi neighborhood $(V, \Gamma, E_V; s, \psi)$ directly as a "virtual cycle" of the expected dimension in the (usually singular) orbifold local chart $s^{-1}(0) \subset V$ of the moduli space, weighted by $1/|\Gamma|$. Equivalent Kuranishi neighborhoods give equivalent local virtual cycles. Transition data of a Kuranishi structure gives the patching data of these local cycles and defines a *virtual fundamental cycle* on the (usual singular) moduli orbifold space. Equivalent Kuranishi structures define the same virtual fundamental class on the moduli orbifold space.

[35] In the Fukaya-Ono setting, the degeneration formulas of any form in Gromov-Witten theory should be regarded as the consequence of the more fundamental degeneration-gluing relations of Kuranishi structures and an assignment to each moduli space with a Kuranishi structure a virtual fundamental class or chain that is functorial, particularly with respect to restrictions to sub-moduli spaces, fibered product, and covering maps. Recall the layer-structure decompositions of the moduli spaces of stable or relative stable maps and the virtual co-dimension of each stratum. These notions extends to the fiber-products that occur in the problem. The functorial property of a virtual fundamental class $[\mathcal{M}]^{virt}$ implies that $[\mathcal{M}]^{virt}$ is determined by its restriction to the depth-0 (i.e. virtual codimension-0) stratum in the moduli space. As the depth-0 strata that occur in right-hand side of the decomposition $\overline{\mathcal{M}}_{g,n}(Y, \underline{\beta}) \;=\; \cup_{\eta \in \bar{\Omega}_{(g,n;\underline{\beta})}} \overline{\mathcal{M}}(Y \,|\, \eta)$ are disjoint from each other, the union becomes a disjoint union when restricted to depth-0 strata of the moduli spaces in the identity. This disjoint union is then turned into a summation of virtual fundamental classes on these strata when the degeneration-gluing relations of Kuranishi structures are applied. As recovering the whole moduli space by adding in strata of positive depth will extend the virtual fundamental class by only lower-dimensional classes in the strata of positive depth, the summation is not influenced. This gives thus the degeneration/gluing formula at the level of virtual fundamental classes. It is with this aspect that we state, as an example, the result of [L-R] and [Li2] as a corollary.

[36] Note that the degeneration formulas of Li-Ruan and Li are equivalent. Here we use the expression in [Li2]; see [L-R] for the expression in terms of integrals over virtual neighborhoods [Ru] with Thom forms. See also the Appendix of the current work for a discussion on the equivalence of the degeneration formulas of Li-Ruan [L-R], Li [Li2], and the formally different Ionel-Parker [I-P2].

Corollary 7.1.15 [degeneration-gluing: virtual fundamental class]. *$[\overline{\mathcal{M}}_{g,n}(X,[\beta])]^{virt}$ and $[\overline{\mathcal{M}}_{g,n}(Y,\underline{\beta})]^{virt}$ can be realized as the fibers of the flat class $[\overline{\mathcal{M}}_{g,n}(W_\lambda,[\beta])]^{virt}/B$ over B.*

$$\begin{aligned}
[\overline{\mathcal{M}}_{g,n}(Y,\underline{\beta})]^{virt} &= \sum_{\eta\in\bar{\Omega}_{(g,n;\underline{\beta})}} [\overline{\mathcal{M}}(Y\,|\,\eta)]^{virt}\\
&= \sum_{\eta\in\bar{\Omega}_{(g,n;\underline{\beta})}} m(\eta)\,[\overline{\mathcal{M}}((Y_1;D)\amalg(Y_2;D)\,|\,\eta)]^{virt}\\
&= \sum_{\eta=(\Gamma_1,\Gamma_2,I)\in\bar{\Omega}_{(g,n;\underline{\beta})}} \frac{m(\eta)}{|Per_r(\eta)|}\,\Phi_{\eta *}\,[\overline{\mathcal{M}}(Y_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2;D\,|\,\Gamma_2)]^{virt}\\
&= \sum_{\eta=(\Gamma_1,\Gamma_2,I)\in\bar{\Omega}_{(g,n;\underline{\beta})}} \frac{m(\eta)}{|Per_r(\eta)|}\,\Phi_{\eta *}\,\Delta_\eta^!([\overline{\mathcal{M}}(Y_1;D\,|\,\Gamma_1)]^{virt}\times[\overline{\mathcal{M}}(Y_2;D\,|\,\Gamma_2)]^{virt}),
\end{aligned}$$

where, for η with l-many roots, $\Delta_\eta: D^l\hookrightarrow D^l\times D^l$ is the diagonal map and

$$\Delta_\eta^!\ :\ A_*(\overline{\mathcal{M}}(Y_1;D\,|\,\Gamma_1)\times\overline{\mathcal{M}}(Y_2;D\,|\,\Gamma_2))\ \longrightarrow\ A_*(\overline{\mathcal{M}}(Y_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2;D\,|\,\Gamma_2))$$

is the Gysin homomorphism under

$$\begin{array}{ccc}
\overline{\mathcal{M}}(Y_1;D\,|\,\Gamma_1)\times_{D^l}\overline{\mathcal{M}}(Y_2;D\,|\,\Gamma_2) & \longrightarrow & \overline{\mathcal{M}}(Y_1;D\,|\,\Gamma_1)\times\overline{\mathcal{M}}(Y_2;D\,|\,\Gamma_2)\\
\downarrow & & \downarrow\\
D^l & \xrightarrow{\Delta_\eta} & D^l\times D^l\,.
\end{array}$$

The Gromov-Witten invariants of X associated to $(g,n;[\beta])$ are defined by[37]:

$$\begin{array}{rccl}
\Psi^X_{(g,n;[\beta])}\ :\ & H^*(X)^{\times n}\times H^*(\overline{\mathcal{M}}_{g,n}) & \longrightarrow & \mathbb{Q}\\
& (\,\kappa\ ,\ \varsigma\,) & \longmapsto & \Big[ev^*(\kappa)\cup\pi^*_{(g,n)}(\varsigma)\,[\overline{\mathcal{M}}_{g,n}(X,[\beta])]^{virt}\Big]_0\,,
\end{array}$$

where $ev:\overline{\mathcal{M}}_{g,n}(X,[\beta])\to X^n$ is the evaluation map associated to the ordered set of n marked points, $\pi_{(g,n)}:\overline{\mathcal{M}}_{g,n}(X,[\beta])\to\overline{\mathcal{M}}_{g,n}$ is the domain-curve stabilization map[38], and $[\,\cdot\,]_0$ means the degree-0 component of $\cdot$.

Given an admissible weighted graph Γ with n legs and l roots, let $\overline{\mathcal{M}}_\Gamma$ be the moduli space of stables curves with $|V(\Gamma)|$-many connected components in one-one correspondence with $V(\Gamma)$, n ordinary marked points corresponding to legs and l distinguished marked points.corresponding to roots accordingly. The relative Gromov-Witten invariants of the pair (Z,D) associated to an admissible weighted graph Γ with n legs and l roots are defined by

$$\begin{array}{rccl}
\Psi^{(Z,D)}_\Gamma\ :\ & H^*(Z)^{\times n}\times H^*(\overline{\mathcal{M}}_\Gamma) & \longrightarrow & H_*(D^l)\\
& (\,\kappa\ ,\ \varsigma\,) & \longmapsto & \mathbf{q}_*\Big(\,ev^*(\kappa)\cup\pi^*_\Gamma(\varsigma)\,[\overline{\mathcal{M}}(Z;D\,|\,\Gamma)]^{virt}\Big)\quad,
\end{array}$$

where $ev:\overline{\mathcal{M}}_{g,n}(X,[\beta])\to X^n$ is the evaluation map associated to the ordered set of ordinary n marked points, $\pi_\Gamma:\overline{\mathcal{M}}(Z;D\,|\,\Gamma)\to\overline{\mathcal{M}}_\Gamma$ is the domain-curve stabilization map, and $\mathbf{q}:\overline{\mathcal{M}}(Z;D\,|\,\Gamma)\to D^l$ is the evaluation map associated to the ordered set of l distinct marked points.

For an admissible triple $\eta=(\Gamma_1,\Gamma_2,I)$ with (n_1,n_2)-many legs and l-many roots, gluing at the paired distinguished marked points defines an orbifold map $G_\eta:\overline{\mathcal{M}}_{\Gamma_1}\times\overline{\mathcal{M}}_{\Gamma_2}\to\overline{\mathcal{M}}_{g,n}$, where $g=g(\eta)$ and $n=n_1+n_2$. For $\varsigma\in H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$, we assume that the Künneth decomposition

[37] All cohomologies in the definition of Gromov-Witten and relative Gromov-Witten invariants are over $\mathbb{Q}$.

[38] For X smooth, $\pi_{(g,n)}$ is a local complete intersection morphism when extended to a map on Kuranishi neighborhoods.

$G^*_\eta(\varsigma) = \sum_{j\in N_\eta} \varsigma_{\eta,1,j}\boxtimes\varsigma_{\eta,2,j}$ exists. Then the degeneration-gluing formula of Gromov-Witten invariants with respect to X/B is given by: ([Li2])

Corollary 7.1.16 [degeneration-gluing: invariant]. *Let $\kappa\in H^0_c(R^\bullet\pi_*\mathbb{Q}_W)^{\oplus n}$, $\varsigma\in H^*(\overline{\mathcal{M}}_{g,n})$, and $j_i: Y_i\hookrightarrow Y=W_0$, $i=1,2$. Then*

$$\begin{aligned}&\Psi^{W_\lambda}_{(g,n;[\beta])}(\kappa(\lambda),\varsigma)\\ &\quad= \sum_{\eta\in\bar\Omega_{(g,n;\underline{\beta})}} \tfrac{m(\eta)}{|Per_r(\eta)|}\sum_{j\in N_\eta}\left[\Psi^{(Y_1,D)}_{\Gamma_1}(j_1^*\kappa(0),\varsigma_{\eta,1,j})\bullet\Psi^{(Y_2,D)}_{\Gamma_2}(j_2^*\kappa(0),\varsigma_{\eta,2,j})\right]_0,\end{aligned}$$

where $\kappa(\lambda)$ is the restriction of κ to the fiber W_λ of W/B, $\bullet$ is the intersection product on $H_*(D^l)$, $[\,\cdot\,]_0$ is the degree-0 component of $\cdot$.

7.2 A degeneration axiom and a gluing axiom for open Gromov-Witten invariants under a symplectic cut.

When L is non-empty, the (real) codimension-1 boundary on the moduli space $\widetilde{\mathcal{M}}_{(g,h),(n,\vec m)}$ of prestable labelled-bordered Riemann surfaces gives rise to the codimension-1 boundary $\partial\mathcal{K}_{(X,L)}$ on the Kuranishi structure $\mathcal{K}_{(X,L)}$ on the moduli space $\overline{\mathcal{M}}_{(g,h),(n,\vec m)}(X,L\,|\,\beta,\vec\gamma,\mu)$ of stable maps to (X,L). As a Gromov-Witten theory/invariant so far constructed is based on the intersection theory with the functorially constructed virtual fundamental class/chain on the moduli space, the birth-'n-death of chain components along the codimension-1 boundary $\partial\mathcal{K}_{(X,L)}$ of $\mathcal{K}_{(X,L)}$ makes such construction not well-defined unless one has a way to fix the ambiguity. Furthermore, it has been noticed ([K-L]) that to define meaningful open Gromov-Witten invariants and to match with the physicists' computation of open string instantons (e.g. [A-K-V]), a *decoration* α has to be imposed to the Lagrangian submanifold L, to which boundaries of Riemann surfaces/open string world-sheets are mapped. Basic examples of decorations are a *group action* on L, a *framing* on T_*L, and an *involution* on $T_*X|_L$ that leave T_*L fixed, if any of these structures on L exists. Denote a Lagrangian submanifold L with a decoration α by L^α. Thus:

PROBLEM: *To define open Gromov-Witten invariants for (X,L^α).*

Note that in general α on L does not extend to a decoration on X.

With the above problem in mind, the degeneration and gluing of Kuranishi structures studied in this work and the deformation-invariance requirement of open Gromov-Witten invariants propel us to impose the following two *axioms* on open Gromov-Witten invariants.

The *Gromov-Witten invariants* of (X,L^α) associated to $((g,h),(n,\vec m)\,|\,\beta,\vec\gamma,\mu)$ are meant to be the evaluation of a map

$$\Psi^{(X,L^\alpha)}_{((g,h),(n,\vec m)\,|\,\beta,\vec\gamma,\mu)}\ :\ H^*(X)^{\times n}\times H^*(L)^{\times|\vec m|}\times H^*(\overline{\mathcal{M}}_{(g,h),(n,\vec m)})\ \longrightarrow\ \mathbb{Q},$$

where $|\vec m| = m_1+\cdots+m_h$, that satisfies a set of properties[39], e.g. the list in [Ko-M]. The same holds with X replaced by the singular Y. Define also

$$\Psi^{(X,L^\alpha)}_{((g,h),(n,\vec m)\,|\,[\beta],\vec\gamma,\mu)}\ :=\ \sum_{\beta''\in[\beta]}\Psi^{(X,L^\alpha)}_{((g,h),(n,\vec m)\,|\,\beta'',\vec\gamma,\mu)}.$$

[39] Besides the interest in its own right, open Gromov-Witten theory gives a mathematical formulation for the problem of *open string world-sheet instantons* and their enumeration in superstring theory; it is closely related also to *conformal field theory with boundary* and *D-branes*. Some of the properties $\Psi^\bullet_\bullet(\cdots)$ has to satisfy come from these subjects in superstring theory. The following incomplete/intentionally-limited additional stringy literatures only mean to give unfamiliar readers a glimpse of these diverse yet linked topics: [B-C-O-V: Sec's 4, 5.5, 8.2], [H-I-V], [K-K-L-MG], and reviews [Dou], [Ga], [S-F-W], and [T-Z].

Similarly, given an admissible weighted graph Γ with n legs, m fingers, and l roots, the *relative Gromov-Witten invariants* of the relative pair $(Z, L^\alpha; D)$ associated to Γ are meant to be the evaluation of a map

$$\Psi_\Gamma^{(Z,L^\alpha;D)} \ : \ H^*(Z)^{\times n} \times H^*(L)^{\times m} \times H^*(\overline{\mathcal{M}}_\Gamma) \ \longrightarrow \ H_*(D^l)\,,$$

where $\overline{\mathcal{M}}_\Gamma$ is the moduli space of (not necessarily connected) labelled-bordered Riemann surfaces with marked points of combinatorial type specified by Γ. For an admissible quadruple $\eta = (\Gamma_1, \Gamma_2, I_{\rm hand}, I_{\rm leg})$ with (h_1, h_2)-many hands, (n_1, n_2)-many legs, l-many roots, and type $|\eta| = ((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)$ gluing at the paired distinguished marked points defines an orbifold map $G_\eta : \overline{\mathcal{M}}_{\Gamma_1} \times \overline{\mathcal{M}}_{\Gamma_2} \to \overline{\mathcal{M}}_{(g,h),(n,\vec{m})}$. For $\varsigma \in H^*(\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}; \mathbb{Q})$, we assume that the Künneth decomposition $G_\eta^*(\varsigma) = \sum_{j\in N_\eta} \varsigma_{\eta,1,j} \boxtimes \varsigma_{\eta,2,j}$ exists.

Axiom OGW-degeneration. *Let W/B be a degeneration of X associated to a symplectic cut $\xi : X \to Y = Y_1 \cup_D Y_2 = W_0$ and L^α be a decorated Lagrangian submanifold of X disjoint from the cutting locus. The submanifold in W_λ associated to L is denoted also by L. Let $\kappa \in H_c^0(R^\bullet \pi_* \mathbb{Q}_W)^{\oplus n}$, $\upsilon \in H^*(L)^{|\vec{m}|}$, and $\varsigma \in H^*(\overline{\mathcal{M}}_{g,n})$. Then*

$$\Psi^{(W_\lambda,L^\alpha)}_{((g,h),(n,\vec{m})\,|\,[\beta],\vec{\gamma},\mu)}(\kappa(\lambda),\upsilon,\varsigma) \ = \ \Psi^{(Y,L^\alpha)}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}(\kappa(0),\upsilon,\varsigma)\,,$$

where $\underline{\beta} = \xi_*([\beta])$ and $\kappa(\lambda)$ is the restriction of κ to W_λ.

Axiom OGW-gluing. *Gromov-Witten invariants of $(Y, L^\alpha) = (Y_1, L_1^\alpha) \cup_D (Y_2, L_2^\alpha)$ can be expressed in terms of relative Gromov-Witten invariants of $(Y_i, L_i^\alpha; D)$, $i = 1, 2$, by the identity:*

$$\begin{aligned}&\Psi^{(Y,L^\alpha)}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}(\kappa,\upsilon,\varsigma)\\ &= \sum_{\eta\in\bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}} \frac{m(\eta)}{|Per_r(\eta)|} \sum_{j\in N_\eta} \Big[\Psi^{(Y_1,L_1^\alpha;D)}_{\Gamma_1}(j_1^*\kappa, j_1^*\upsilon, \varsigma_{\eta,1,j}) \bullet \Psi^{(Y_2,L_2^\alpha;D)}_{\Gamma_2}(j_2^*\kappa, j_2^*\upsilon, \varsigma_{\eta,2,j})\Big]_0\,,\end{aligned}$$

where $\bullet$ is the intersection product on $H_*(D^l)$, and $[\,\cdot\,]_0$ is the degree-0 component of $\cdot$.

Remark 7.2.1 [*selection of fundamental chains adapted to α – specialization*]. Concerning the ambiguity mentioned in the beginning of this subsection on the choices of virtual fundamental chains, below is how these two axioms are applied to this issue. For simplicity of presentation, we assume that $\xi : (X, L^\alpha) \to (Y, L^\alpha) = (Y_1, L^\alpha) \cup_D (Y_2, \emptyset)$. Suppose that

> [*assumption*] the decoration α is full enough to select in a standard way a class of virtual fundamental chains $[\overline{\mathcal{M}}(Y_1, L^\alpha; D\,|\,\Gamma_1)]^{virt}$ in $\overline{\mathcal{M}}(Y_1, L^\alpha; D\,|\,\Gamma_1)$ associated to a standard Kuranishi structure $\mathcal{K}'_{(Y_1,L^\alpha;D\,|\,\Gamma_1)}$ for all Γ_1 in an $\eta \in \bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{\gamma},\mu)}$,

then it induces a class of virtual fundamental chains on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W_\lambda, L^\alpha\,|\,[\beta],\vec{\gamma},\mu)$ as follows:

- the push-forward of the fibered product of $[\overline{\mathcal{M}}(Y_1, L^\alpha; D\,|\,\Gamma_1)]^{virt}$ with $[\overline{\mathcal{M}}(Y_2; D\,|\,\Gamma_2)]^{virt}$ over D^l weighted by $m(\eta)/|Per_r(\eta)|$ gives rise to a class of virtual fundamental subchains $[\overline{\mathcal{M}}(Y, L^\alpha\,|\,\eta)]^{virt}$ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y, L^\alpha\,|\,\underline{\beta},\vec{\gamma},\mu)$;
- their summation over $\eta \in \bar{\Omega}_{((g,h),(n,\vec{m})\,|\,\underline{\beta},\vec{m},\mu)}$ gives a class of virtual fundamental chains $[\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y, L^\alpha\,|\,\underline{\beta},\vec{\gamma},\mu)]^{virt}$ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y, L^\alpha\,|\,\underline{\beta},\vec{\gamma},\mu)$;
- deform the chains $[\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(Y, L^\alpha\,|\,\underline{\beta},\vec{\gamma},\mu)]^{virt}$ to over $\lambda \neq 0$ by a 2-dimension-higher chain c in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L^\alpha\,|\,[\beta],\vec{\gamma},\mu)$ such that both c and its restriction to $\partial\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L^\alpha\,|\,[\beta],\vec{\gamma},\mu)/B$ are flat over B; this then defines a class of virtual fundamental chains $[\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L^\alpha\,|\,[\beta],\vec{\gamma},\mu)]^{virt}$ in $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W_\lambda, L^\alpha\,|\,[\beta],\vec{\gamma},\mu)$.

In this prescription, $\partial\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L^\alpha \,|\, [\beta], \vec{\gamma}, \mu)/B$ consists of stable maps to the fibers of $(\widehat{W}, \widehat{L})/\widehat{B}$ of the given type such that the domain Σ has either boundary nodes or free marked points landing on $\partial\Sigma$. The requirement of the flatness of the deformation of chains also on the restriction to $\partial\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(W/B, L^\alpha \,|\, [\beta], \vec{\gamma}, \mu)/B$ suppresses the birth-'n-death of chains from the codimension-1 boundary Kuranishi structure of the Kuranishi structure on $\overline{\mathcal{M}}_{(g,h),(n,\vec{m})}(X, L^\alpha \,|\, [\beta], \vec{\gamma}, \mu)/B$. Here, we identify X as some fiber W_{λ_0} of W/B with $\lambda_0 \neq 0$. This process is similar to the *specialization* technique in algebraic geometry.

Definition 7.2.2 [L-isolatable]. We call (X, L^α) *L-isolatable* if there exists a symplectic cut

$$X \;\longrightarrow\; (Y_0\,;\, \amalg_i D_i) \cup_{\cup_i D_i} \amalg_i (Z_i, L_i; D_i)\,,$$

where L_i's are the finitely many connected components of L, such that Z_i is the symplectic manifold determined by L_i with the property that $Z_i - D_i$ is symplecto-isomorphic to a tubular neighborhood of the 0-section of T^*L_i. Here, T^*L is equipped with the canonical symplectic structure.

Under Axiom OGW-degeneration and Axiom OGW-gluing, the problem of the construction of open Gromov-Witten invariants of L-isolatable (X, L^α) is reduced to

Step (2): the construction of relative open Gromov-Witten invariants of $(Z, L^\alpha; D)$ determined by L^α.

Such class of (X, L^α)'s includes those that have occurred in the open/closed string duality.

Appendix. The equivalence of Li-Ruan/Li's degeneration formula and Ionel-Parker's degeneration formula.

The details of [L-R] and [Li1], [Li2], together with Comparison 3.2.4 in Sec. 3.2, imply that the degeneration formula of the (closed) Gromov-Witten invariants derived by A.-M. Li and Y. Ruan in [L-R] and J. Li in [Li1], [Li2] are the same. The Degeneration Axiom and the Gluing Axiom of open Gromov-Witten invariants we propose in Sec. 7.2 are of the Li-Ruan/Li form. This form are formally different[40] from that derived by E.-N. Ionel and T.H. Parker in [I-P1], [I-P2]. Indeed, we can also adopt the discussion of [I-P2: Sec. 12] to give degeneration-gluing axioms of open Gromov-Witten invariants in the Ionel-Parker form, though algebro-geometrically (cf. [Fu: Chap. 10]) it is the Li-Ruan/Li form that we would choose, as it comes from a flat family construction. This leads to the following question:

Q. *Do Li-Ruan/Li and Ionel-Parker give different/independent sets of gluing/degeneration axioms for open Gromov-Witten invariants for a symplectic cut?*

In this appendix, as a not-completely-irrelevant issue to our project, we explain the following conjecture, whose justification will answer the above question *negatively*[41]:

[40] See [L-R: p. 159], [I-P1: p. 48], and [Li1: Sec. 0] for a light comparison by these authors themselves.

[41] Indeed, from the algebro-geometric point of view, *any degeneration/gluing formula for intersection-theoretic type invariants that are constant under flat deformations must be re-derivable from a flat family construction* and *any gluing/degeneration formula of Gromov-Witten invariants different from the one derived by a flat family construction* (*i.e. the Li-Ruan/Li formula in the case of symplectic cut*) *must be convertible to the latter* unless they are indeed dealing with different invariants or different kinds of degenerations.

Conjecture A.1 [Li-Ruan/Li = Ionel-Parker]. *Li-Ruan/Li's degeneration formula and Ionel-Parker's degeneration formula for closed Gromov-Witten invariants are equivalent/convertible to each other. Furthermore, the conversion is induced by*

$$\textit{Li-Ruan / Li formula} \quad \begin{array}{c} \textit{un-rigidifying } Y_{[k]}\textit{'s} \\ \longleftarrow\!\!\!

defined around $s^{-1}(0)$ on each Kuranishi neighborhood from $\widetilde{\mathcal{K}}_0^{(i)}$. (Recall a Kuranishi neighborhood data $(V, \Gamma_V, E_V; s, \psi)$.) We expect also that a resemble to [I-P2: Identity (12.4) in Lemma 12.1]

$$\mathcal{K}_0 \;=\; \tilde{\imath}^{(1)}(\mathcal{K}_0^{(1)}) \;-\; \tilde{\imath}^{(2)}(\mathcal{K}_0^{(2)}) \;+\; \tilde{\imath}^{(3)}(\mathcal{K}_0^{(3)}) \;-\; \cdots$$

holds around $s^{-1}(0)$ on each Kuranishi neighborhood from $\mathcal{K}_0$. This should then reproduce the degeneration/gluing formula in the form of [I-P2].

Since all the difference in [L-R], [Li2] versus [I-P2] that are related to the expression of the degeneration/gluing formula is whether or not and when and where to apply rigidification of targets of maps, we thus make the conjecture.

□

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