# On the Solvability of Magnetic Differential Équations

W. Engelhardt<sup>1</sup>, retired from:

Max-Planck-Institut für Plasmaphysik, IPP-Euratom Association, Garching, Germany

## Abstract

The calculation of both resistive and ideal plasma equilibria amounts to solving a number of magnetic differential equations which are of the type  $\vec{B} \cdot \nabla \Phi = s$ . We apply the necessary and sufficient criterion for the existence of the potential  $\Phi$  and find that a static equilibrium configuration of a magnetically confined plasma does not exist in axi-symmetric toroidal geometry.

#### AMS classification: 76W05, 35F05

#### **Keywords**:

- Magnetohydrodynamics
- Magnetic plasma confinement
- First-order partial differential equations

#### I Introduction

The term 'magnetic differential equation' was coined by Kruskal and Kulsrud [1] in 1958. It arose from an attempt to model a 'resistive plasma equilibrium' with the equations:

$$\vec{j} \times \vec{B} = \nabla p \tag{1}$$

$$rot \vec{B} = \mu_0 \vec{j} \tag{2}$$

$$div\,\vec{B} = 0\tag{3}$$

$$\vec{E} + \vec{v} \times \vec{B} = \eta \, \vec{j} \tag{4}$$

$$rot \,\vec{E} = 0 \tag{5}$$

From (5) follows  $\vec{E} = -\nabla \Phi$  and from (4) after taking the scalar product with  $\vec{B}$ :

$$\vec{B} \cdot \nabla \Phi = -\eta \, \vec{j} \cdot \vec{B} = s \tag{6}$$

This equation was called a magnetic differential equation. Newcomb [2] formulated a criterion for its solvability in 1959:

$$\oint \frac{s \, dl}{\left|\vec{B}\right|} = 0 \tag{7}$$

<sup>&</sup>lt;sup>1</sup>Fasaneriestrasse 8, D-80636 München, Germany, e-mail: wolfgangw.engelhardt@t-online.de

where the integration has to be carried out around closed field lines.

Equations (1 - 5) are the basis for calculating the classical diffusive particle losses in a magnetically confined plasma. Pfirsch and Schlüter [3] used a magnetic model field which did not exactly satisfy equations (1 - 2), but they obtained, nevertheless, an estimate for the diffusion coefficient in a toroidal configuration. Later on Maschke [4] generalized their results for a true equilibrium configuration and confirmed the order of magnitude of the predicted losses. The present state of the art to calculate collisional transport in tokamaks including the effect of trapped particles can be found, e.g., in [5].

In this paper we analyze again the solvability of (6) and find that Newcomb's criterion (7) is necessary, but not sufficient. In general, the integral  $\oint \nabla \phi \cdot d\vec{l}$  taken around any closed loop, which does not surround the central hole of the torus, must vanish, if the potential  $\phi$  is to exist. If we restrict ourselves to axi-symmetric configurations (e.g. tokamak), we see that (7) is only applied to a special class of closed loops, namely to the nested contours of the magnetic surfaces. In Section IV we apply the necessary and sufficient condition (5) on (6) and find that the inhomogeneous part s of the magnetic differential equation (6) must vanish. Since s is given by other requirements, the condition for the existence of the potential cannot be satisfied.

In Section V we reduce the force balance (1) to a magnetic differential equation which is of the same structure as (6). It turns out that the criterion for its solvability can also not be satisfied in general. The reason is that the transformation properties of the cross-product in (1) are at variance with the transformation properties of the pressure gradient. As a consequence we come to the conclusion that the assumed stationary state of a magnetically confined plasma as described by (1 - 5) does not exist.

#### **II** Equilibrium configuration

We introduce a cylindrical coordinate system with unit vectors  $(\vec{e}_R, \vec{e}_{\varphi}, \vec{e}_Z)$  and take the toroidal angle as an ignorable coordinate (axi-symmetry). The magnetic field satisfying (3) may be written as:

$$\vec{B} = \vec{B}_p + \vec{B}_\varphi = \nabla\psi \times \nabla\varphi + \frac{F}{R}\vec{e}_\varphi \tag{8}$$

where  $\psi$  is the poloidal magnetic flux function. From (1) and the poloidal component of (2) follows then:

$$p = p(\psi)$$
,  $F = F(\psi)$  (9)

and the current density may be expressed by

$$\vec{j} = \frac{F'}{\mu_0}\vec{B_p} + \left(Rp' + \frac{FF'}{\mu_0 R}\right)\vec{e_\varphi}$$
(10)

where ' denotes differentiation with respect to  $\psi$ . Inserting this into the toroidal component of (2) yields the Lüst-Schlüter-Grad-Rubin-Shafranov equation [6]:

$$\frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = \Delta^* \psi = -\left(R^2 \mu_0 p' + FF'\right) \tag{11}$$

This equation is usually solved numerically with suitable boundary conditions to find the equilibrium magnetic field in axi-symmetric plasma configurations. There exist also analytical solutions (see Appendix A) which contain free parameters so that the solution can be adapted to realistic boundary conditions. Recently equation (11) has been modified [7] to include approximately the effect of magnetic islands which break axi-symmetry. At this point, however, we assume that a solution of (11) is given for a particular choice of p' and FF'.

**III Standard method for calculating the electric field and the velocity field** The method used in [3] and [4] for calculating the electric field is straightforward. Because of (5) the electric field may be written as:

$$\vec{E} = \vec{E}_p + \vec{E}_\varphi = -\nabla\phi + \frac{U}{2\pi R}\vec{e}_\varphi \tag{12}$$

where U is the loop voltage produced by a transformer. Substituting this into (4) and taking the scalar product with  $\vec{B}$  yields the magnetic differential equation:

$$-\vec{B}_p \cdot \nabla \phi = (\eta j_{\varphi} - E_{\varphi}) \frac{F}{R} + \frac{\eta F'}{\mu_0} B_p^2$$
(13)

with the solution:

$$\phi = \phi_0\left(\psi\right) - \int_{l_0}^l \left[ \left(\eta j_\varphi - E_\varphi\right) \frac{F}{R} + \frac{\eta F'}{\mu_0} B_p^2 \right] \frac{R \, dl}{|\nabla \psi|} \tag{14}$$

where dl denotes a line element on a contour of a magnetic surface  $\psi = const$ . In order to obtain a single-valued potential, the closed integral around a magnetic surface must vanish:

$$\oint \left[ (\eta j_{\varphi} - E_{\varphi}) \frac{F}{R} + \frac{\eta F'}{\mu_0} B_p^2 \right] \frac{R \, dl}{|\nabla \psi|} = 0 \tag{15}$$

This equation reflects Newcomb's condition (7) and puts a constraint on the choice of the functions p' and F' to be used in (11).

The velocity field is obtained by taking the cross-product of (4) with  $\vec{B}$ :

$$\vec{v} = \alpha \left( R, Z \right) \vec{B} + \left( \vec{E} - \eta \, \vec{j} \right) \times \vec{B} \big/ B^2 \tag{16}$$

The arbitrary function  $\alpha$  may be expressed by the divergence of the velocity, thus resulting in a second magnetic differential equation because of (3):

$$\vec{B}_p \cdot \nabla \alpha = div \left( \vec{v} - \left( \vec{E} - \eta \, \vec{j} \right) \times \vec{B} / B^2 \right) \tag{17}$$

The solution is:

$$\alpha = \alpha_0 \left(\psi\right) + \int_{l_0}^{l} div \left(\vec{v} - \left(\vec{E} - \eta \, \vec{j}\right) \times \vec{B} / B^2\right) \frac{R \, dl}{|\nabla \psi|} \tag{18}$$

with the integrability condition:

$$\oint div \, \vec{v} \frac{R \, dl}{|\nabla \psi|} = \oint div \left( \left( \vec{E} - \eta \, \vec{j} \right) \times \vec{B} / B^2 \right) \frac{R \, dl}{|\nabla \psi|} \tag{19}$$

which is usually expressed in the form:

$$\int \oint div \, \vec{v} \, \frac{R \, dl}{|\nabla \psi|} \, d\psi = \oint \left( \vec{E} \times \vec{B} - \eta \, \nabla p \right) \cdot \nabla \psi \, \frac{R \, dl}{B^2 \, |\nabla \psi|} \tag{20}$$

by application of Gauss' theorem.

It appears now that the problem is solved, first by prescribing a flux function satisfying (11) and (15), a divergence of the velocity subject to the constraint (20), and secondly, by evaluating (14) and (18). The solutions may be substituted into (12) and (16) to yield the vector fields  $\vec{E}$  and  $\vec{v}$ . The integration functions  $\phi_0$  and  $\alpha_0$  describe an arbitrary rotation in poloidal and toroidal direction. In the following it is shown that this expectation is not justified, since (15) and (20) turn out to be necessary, but not sufficient conditions.

## IV Application of the necessary and sufficient criterion for the existence of the potential

Before we apply the necessary and sufficient condition (5) for the existence of the potential  $\phi$ , we write (13) in the form:

$$E_R B_R + E_Z B_Z = s$$
 ,  $s = (\eta j_{\varphi} - E_{\varphi}) \frac{F}{R} + \frac{\eta F'}{\mu_0} B_p^2$  (21)

and take the gradient of this equation:

$$B_{R}\frac{\partial E_{R}}{\partial R} + B_{Z}\frac{\partial E_{Z}}{\partial R} + E_{R}\frac{\partial B_{R}}{\partial R} + E_{Z}\frac{\partial B_{Z}}{\partial R} = \frac{\partial s}{\partial R}$$
$$B_{R}\frac{\partial E_{R}}{\partial Z} + B_{Z}\frac{\partial E_{Z}}{\partial Z} + E_{R}\frac{\partial B_{R}}{\partial Z} + E_{Z}\frac{\partial B_{Z}}{\partial Z} = \frac{\partial s}{\partial Z}$$
(22)

Together with (21), (3), and the necessary and sufficient condition (5):

$$\frac{\partial E_R}{\partial Z} = \frac{\partial E_Z}{\partial R} \tag{23}$$

one obtains from (22) two magnetic differential equations for the electric field components:

$$\vec{B}_{p} \cdot \nabla\left(\frac{E_{R}}{R B_{Z}}\right) = \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{s}{B_{Z}}\right) , \quad \vec{B}_{p} \cdot \nabla\left(\frac{E_{Z}}{R B_{R}}\right) = \frac{1}{R} \frac{\partial}{\partial Z} \left(\frac{s}{B_{R}}\right)$$
(24)

In analogy to (14) they have the solutions:

$$E_{R} = R B_{Z} \left[ \int_{l_{0}}^{l} \frac{\partial}{\partial R} \left( \frac{s}{B_{Z}} \right) \frac{dl}{|\nabla \psi|} + f_{1}(\psi) \right]$$
$$E_{Z} = R B_{R} \left[ \int_{l_{0}}^{l} \frac{\partial}{\partial Z} \left( \frac{s}{B_{R}} \right) \frac{dl}{|\nabla \psi|} + f_{2}(\psi) \right]$$
(25)

Inserting this into (21) yields:

$$f_1(\psi) + f_2(\psi) + \int_{l_0}^{l} \left[ \frac{\partial}{\partial R} \left( \frac{s}{B_Z} \right) + \frac{\partial}{\partial Z} \left( \frac{s}{B_R} \right) \right] \frac{dl}{|\nabla \psi|} = \frac{s}{R B_R B_Z}$$
(26)

Since the lower bound  $l_0$  of the integral may be chosen arbitrarily close to the upper bound l, the integral can be made to vanish so that equation (26) leads to the condition:

$$f_2(\psi) = -f_1(\psi) = \phi'_0(\psi) \quad , \qquad s = 0$$
 (27)

as the right-hand-side of (26) is not a function of  $\psi$  alone. Result (27) can also be deduced from the transformation properties of the electric field vector whose components are given in (25). This is demonstrated in Appendix B.

The local condition (27) is more restrictive than Newcombs integral condition (15), and it is not compatible with s as given in (21). Hence, we must conclude that the potential does not exist.

#### V Magnetic differential equations and the force balance

In case of the static force balance (1) scalar multiplication with the magnetic field yields the homogeneous magnetic differential equation:

$$\vec{B} \cdot \nabla p = 0 \tag{28}$$

which satisfies condition (27) so that an ideal equilibrium with  $\eta = 0$  should be possible. The necessary and sufficient condition for the existence of the pressure 'potential' is, however, that the curl of (1) vanishes:  $(\vec{B} \cdot \nabla) \vec{j} = (\vec{j} \cdot \nabla) \vec{B}$ , which again leads to an inhomogeneous magnetic differential equation for the toroidal component of the current density together with (2):

$$\vec{B}_p \cdot \nabla\left(\frac{j_{\varphi}}{R}\right) = -\frac{2B_{\varphi}j_R}{R^2} \tag{29}$$

As shown in the previous Section, it has only the solution  $j_{\varphi} = R f(\psi)$  so that the radial component of the current density must vanish.

This may also be shown by considering the so called 'force-free' situation where the pressure gradient vanishes and (11) becomes:

$$\Delta^* \psi = -FF' = -g\left(\psi\right) \tag{30}$$

Applying Stokes's theorem on this equation by integrating the toroidal current density over the area enclosed by a magnetic surface one has:

$$\oint \frac{|\nabla \psi|}{R} \, dl = \oint g\left(\psi\right) \, \frac{dR \, dZ}{R} \tag{31}$$

With:

$$\psi = \int \frac{F}{g(F)} dF \quad , \qquad \nabla \psi = \frac{F}{g} \nabla F$$
(32)

one obtains from (30):

$$\Delta^* F = -\frac{g^2}{F} - \frac{g}{F} |\nabla F|^2 \frac{d}{dF} \left(\frac{F}{g}\right)$$
(33)

Stoke's theorem applied on this equation gives:

$$\oint \frac{|\nabla F|}{R} \, dl = \oint \left( \frac{g^2}{F} + \frac{g}{F} \, |\nabla F|^2 \, \frac{d}{dF} \left( \frac{F}{g} \right) \right) \, \frac{dR \, dZ}{R} \tag{34}$$

Substitution of (32) into (31) yields on the other hand:

$$\frac{F}{g} \oint \frac{|\nabla F|}{R} \, dl = \oint g \, \frac{dR \, dZ}{R} \tag{35}$$

Elimination of the line integral over the poloidal current density on the left-hand sides of (34) and (35), and using (32) again results in an integral equation:

$$g \oint g \frac{dR \, dZ}{R} = F \oint \left( \frac{g^2}{F} + \frac{1}{F^3} \left( g^2 - F \frac{dg}{d\psi} \right) |\nabla\psi|^2 \right) \frac{dR \, dZ}{R}$$
(36)

which can only be satisfied for g = FF' = 0 in agreement with (27) and (29). We demonstrate this explicitly in Appendix A by adopting a 'Soloviev solution' [8] with g = const.

In view of this result it becomes doubtful whether the condition  $p = p(\psi)$  resulting from (28) in axi-symmetry defines a scalar pressure, or, in other words, whether the cross-product  $\vec{j} \times \vec{B}$ , which is in principle an antisymmetric second-rank tensor, can have the same transformation properties as the polar vector field  $\nabla p$ , at least under certain circumstances.

In order to investigate this question we apply the Laplace operator in Cartesian coordinates on (28):

$$\vec{B} \cdot \nabla (\Delta p) + \nabla p \cdot \Delta \vec{B} + 2\left(\frac{\partial B_x}{\partial x}\frac{\partial^2 p}{\partial x^2} + \frac{\partial B_y}{\partial y}\frac{\partial^2 p}{\partial y^2} + \frac{\partial B_z}{\partial z}\frac{\partial^2 p}{\partial z^2} + k_1\frac{\partial^2 p}{\partial x\partial y} + k_2\frac{\partial^2 p}{\partial x\partial z} + k_3\frac{\partial^2 p}{\partial y\partial z}\right) = 0 \quad (37)$$
$$k_1 = \frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x}, \quad k_2 = \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x}, \quad k_3 = \frac{\partial B_y}{\partial z} + \frac{\partial B_z}{\partial y}$$

The gradient of (28):

$$\frac{\partial}{\partial x} \left( B_x \frac{\partial p}{\partial x} + B_y \frac{\partial p}{\partial y} + B_z \frac{\partial p}{\partial z} \right) = 0$$

$$\frac{\partial}{\partial y} \left( B_x \frac{\partial p}{\partial x} + B_y \frac{\partial p}{\partial y} + B_z \frac{\partial p}{\partial z} \right) = 0$$

$$\frac{\partial}{\partial z} \left( B_x \frac{\partial p}{\partial x} + B_y \frac{\partial p}{\partial y} + B_z \frac{\partial p}{\partial z} \right) = 0$$
(38)

yields three equations which can be used to eliminate the mixed derivatives in (37). The bracket may then be written in the form:

$$\left(2\frac{\partial B_x}{\partial x} + l_x B_x\right)\frac{\partial^2 p}{\partial x^2} + \left(2\frac{\partial B_y}{\partial y} + l_y B_y\right)\frac{\partial^2 p}{\partial y^2} + \left(2\frac{\partial B_z}{\partial z} + l_z B_z\right)\frac{\partial^2 p}{\partial z^2} + \nabla p \cdot \left(\vec{l} \cdot \nabla\right)\vec{B}$$

$$\vec{l} = \left(\frac{k_3 B_x}{B_y B_z} - \frac{k_1}{B_y} - \frac{k_2}{B_z}\right)\vec{e}_x + \left(\frac{k_2 B_y}{B_x B_z} - \frac{k_1}{B_x} - \frac{k_3}{B_z}\right)\vec{e}_y + \left(\frac{k_1 B_z}{B_x B_y} - \frac{k_2}{B_x} - \frac{k_3}{B_y}\right)\vec{e}_z$$

$$(39)$$

It turns out that this expression does not have the transformation properties of a scalar, in contrast to the first two terms in (37). By 'scalar' we refer to a quantity

which does not change its value, when it is expressed in different coordinate systems as a function of space:  $p(\vec{x}) = p(\vec{x}')$ ,  $\vec{x}' = \vec{x}$ ,  $x'_i = a_{ik}x_k$ . The reason for the 'non-scalar' property of (39) is that the directed quantity  $\vec{l}$  cannot be considered as a vector field which maintains its modulus as a scalar, when it is transformed into a rotated coordinate system. The inner product of the last term in (39) will, therefore, depend on the orientation of the coordinate system which would not be the case for the invariant inner product  $\nabla p \cdot (\vec{B} \cdot \nabla) \vec{B}$ , e.g., or for the first two terms in (37). Similar remarks apply to the first three terms in (39) which resemble the Laplacian of the pressure, but the second derivatives have coefficients which are all different so that the sum of these terms is not invariant, as compared to the Laplacian of a scalar field. Consequently, equation (39) leads to an incongruity, when it is transformed into a rotated coordinate system, as it will depend explicitly on the rotational angle.

In order to show this we choose a coordinate system which is rotated around the y-axis by an angle  $\alpha$ :

$$x = x' \cos \alpha - z' \sin \alpha , \quad z = x' \sin \alpha + z' \cos \alpha , \quad y = y'$$
(40)

The transformation rules are:

$$\frac{\partial}{\partial x} = \cos \alpha \, \frac{\partial}{\partial x'} - \sin \alpha \frac{\partial}{\partial z'} \,, \quad \frac{\partial}{\partial z} = \sin \alpha \frac{\partial}{\partial x'} + \cos \alpha \frac{\partial}{\partial z'} \,, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \tag{41}$$

Applying these to (39) one finds that the transformed expression contains not only the components of the magnetic field and of the pressure gradient in the primed system, but in addition the rotational angle  $\alpha$ . This may be most readily verified by transforming the coefficient  $l_y$ . It should be invariant, since the *y*-component of the pressure gradient does not change under the assumed rotation so that the second term in (39) should be the same in the rotated system. Instead one obtains:

$$l'_{y} = \frac{(k'_{2}B_{y'} - k'_{1}B_{z'} - k'_{3}B_{x'})\cos 2\alpha + (k'_{1}B_{x'} - k'_{3}B_{z'} + (\frac{\partial B_{z'}}{\partial z'} - \frac{\partial B_{x'}}{\partial x'})B_{y'})\sin 2\alpha}{(B_{x'}\cos\alpha - B_{z'}\sin\alpha) \ (B_{x'}\sin\alpha + B_{z'}\cos\alpha)}$$
(42)

where the  $k'_n$  are defined as in (37) in terms of primed derivatives of the primed field components. For  $\alpha = 0$  one returns to the expression  $l_y$  as given in (39). Evidently, the expression (39) does not transform like a scalar field which is only the case when the pressure itself, as defined by (28) in the form  $p(\psi)$ , is not a scalar, contrary to our assumption. In Appendix A we show this explicitly for the pressure as given by a Soloviev solution.

From the transformation properties of (39), which are a consequence of the hypothetical equation (1), we infer that a static equilibrium configuration does not exist, since the transformation properties of the vector cross-product are incompatible with those of the gradient of a scalar pressure.

## Conclusion

The stationary equilibrium equations (1 - 5) of a magnetically confined plasma may be formulated in terms of magnetic differential equations. We have shown that these equations have ambiguous solutions in axi-symmetry, unless their inhomogeneous part vanishes. Furthermore, due to the transformation properties of the cross-product  $\vec{j} \times \vec{B}$ , it cannot be set equal to the gradient of a scalar pressure. As a consequence, the set of equations (1 - 5) is not solvable.

The time dependent terms which are omitted in the stationary magneto-hydrodynamic model can apparently not be neglected. Consequently, magnetic confinement of a plasma in toroidal geometry leads inevitably to temporal changes of the pressure and the electromagnetic field. This may not be necessarily a unidirectional temporal evolution, but turbulent fluctuations could lead to an average 'quasi-stationary' state, which would not be strictly axi-symmetric any longer.

#### Acknowledgments

The author is grateful to Professor Pfirsch who carefully read the manuscript and spotted an error in Section IV which subsequently was corrected. Professor Schlüter pointed out in discussions that – according to his own analysis of the Vlasov equation – the static force balance (1) cannot be satisfied when an isotropic pressure is assumed.

## Appendix A

Soloviev has constructed a flux function [8]:

$$\psi = \frac{1}{2} \left( c_0 R^2 + b R_0^2 \right) Z^2 + \frac{a - c_0}{8} \left( R^2 - R_0^2 \right)^2 \tag{A.1}$$

which substituted into (8) yields the poloidal magnetic field components:

$$B_R = -\frac{Z}{R} \left( c_0 R^2 + b R_0^2 \right) , \quad B_Z = c_0 Z^2 + \frac{a - c_0}{2} \left( R^2 - R_0^2 \right)$$
(A.2)

When these are inserted into the toroidal component of (2) one obtains:

$$\mu_0 j_{\varphi} = -a \, R - \frac{b \, R_0^2}{R} \tag{A.3}$$

so that (10) and (11) are satisfied with:

$$p' = -\mu_0 a , \quad FF' = -b R_0^2$$
 (A.4)

Recently, expressions similar to (A.1) were published in [9], which have more adjustable parameters. They allow to model tokamak plasma configurations more realistically than (A.1).

With  $g = -bR_0^2$  equation (36) becomes:

$$g^{2} \oint \frac{dR \, dZ}{R} = g^{2} F \oint \left(\frac{1}{F} + \frac{|\nabla\psi|^{2}}{F^{3}}\right) \frac{dR \, dZ}{R}$$
(A.5)

and from (A.4) follows:

$$F = \sqrt{F_0^2 - 2bR_o^2\psi} \tag{A.6}$$

Converting the left-hand-side of (A.5) into a line integral and performing a partial integration on the first term of the right-hand-side of (A.5) yields with (A.6):

$$g^{2} \int_{R_{1}}^{R_{2}} \frac{Z\left(R,\psi\right)}{R} dR = g^{2} F \left[ \int_{R_{1}}^{R_{2}} \frac{Z\left(R,\psi\right)}{FR} dR + \oint \left( -\frac{bR_{0}^{2}Z}{F^{3}} \frac{\partial\psi}{\partial Z} + \frac{|\nabla\psi|^{2}}{F^{3}} \right) \frac{dR dZ}{R} \right]$$
(A.7)

where  $R_1$  and  $R_2$  are the points where a magnetic surface cuts the mid-plane Z = 0. The Z - coordinate on a magnetic surface is expressed as a function of R and  $\psi$  with (A.1). Collecting terms equation (A.7) becomes with (8):

$$0 = F g^2 \oint \left( b R_0^2 Z B_R + R \left( B_R^2 + B_Z^2 \right) \right) \frac{dR dZ}{F^3}$$
(A.8)

Inserting the magnetic field components as given in (A.2) with a = 0 one finds that the double integral over the cross-section of the plasma inside a magnetic surface does not vanish which requires then g = 0 to satisfy (A.8). This result was already expected from the condition (27) and the magnetic differential equation (29).

Soloviev's solution may help to understand the conclusion following from (42) that  $\vec{B} \cdot \nabla p = 0$  does not define a scalar pressure. When the first expression in (A.4) is integrated, one obtains for the pressure:

$$p = p_0 - \mu_0 a \psi \tag{A.9}$$

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Because of  $\vec{B} = rot \vec{A}$  the flux function  $\psi$  in (8) is related to the toroidal component of the vector potential:  $\psi = RA_{\varphi}$ . This expression may be considered as the Z component of the vector field  $R\vec{e}_R \times A_{\varphi}\vec{e}_{\varphi}$ . An arbitrary function of a single vector component does, however, not transform like a scalar field. Writing equation (A.9) in the form:

$$p = p_0 - \mu_0 a R A_{\varphi} = p_0 - \mu_0 a (x A_y - y A_x)$$
(A.10)

and transforming this expression into a coordinate system which is rotated around the y - axis by an angle  $\alpha$  as in Section V:

$$x = x' \cos \alpha - z' \sin \alpha , \quad z = x' \sin \alpha + z' \cos \alpha , \quad y = y'$$

$$A_x = A_{x'} \cos \alpha - A_{z'} \sin \alpha , \quad A_z = A_{x'} \sin \alpha + A_{z'} \cos \alpha , \quad A_y = A_{y'}$$
(A.11)

one obtains:

$$p = p_0 - \mu_0 a \left[ (x'A_{y'} - y'A_{z'}) \cos \alpha + (y'A_{z'} - z'A_{y'}) \sin \alpha \right]$$
(A.12)

This expression contains not only the coordinates and the components of the vector potential in the primed system, but in addition the rotational angle  $\alpha$ . Hence, the pressure obtained from (A.9) does not transform like a scalar.

#### Appendix B

The derivation of the condition s = 0 in Section IV rested on a particular choice of the lower bound of the integral (26). More generally, result (27) follows also from the vector character of the electric field whose components are given in (25) in cylindrical coordinates. If one formulates equation (21) in spherical coordinates  $(R = r \sin \theta, Z = r \cos \theta, \varphi = \varphi)$ :

$$E_r B_r + E_\theta B_\theta = s \tag{B.1}$$

the method applied in Section IV yields now expressions for the field components in spherical coordinates. Taking the gradient of (B.1) one obtains with (3) and (5) two magnetic differential equations for the electric field components:

$$\vec{B}_{p} \cdot \nabla \left(\frac{E_{r}}{R B_{\theta}}\right) = \frac{1}{R r} \frac{\partial}{\partial r} \left(\frac{s r}{B_{\theta}}\right) , \quad \vec{B}_{p} \cdot \nabla \left(\frac{E_{\theta}}{R B_{r}}\right) = \frac{1}{R r} \frac{\partial}{\partial \theta} \left(\frac{s}{B_{r}}\right)$$
(B.2)

which have the formal solutions:

$$E_{r} = R B_{\theta} \left[ \int_{l_{0}}^{l} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{s r}{B_{\theta}} \right) \frac{dl}{|\nabla \psi|} + f_{3} (\psi) \right]$$
$$E_{\theta} = R B_{r} \left[ \int_{l_{0}}^{l} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{s}{B_{r}} \right) \frac{dl}{|\nabla \psi|} + f_{4} (\psi) \right]$$
(B.3)

Writing  $E_{\theta} = (s - E_r B_r)/B_{\theta}$  and  $E_Z = (s - E_R B_R)/B_Z$  one obtains for the modulus of the electric field from (B.3):

$$E_r^2 + E_{\theta}^2 = R^2 B_p^2 \left( f_3 + I_3 \right)^2 + \frac{s^2}{B_{\theta}^2} - \frac{2 s R B_r}{B_{\theta}} \left( f_3 + I_3 \right), \ I_3 = \int_{l_0}^l \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{s r}{B_{\theta}} \right) \frac{dl}{|\nabla \psi|}$$
(B.4)

and from (25) we have:

 $k_2$ 

$$E_R^2 + E_Z^2 = R^2 B_p^2 \left( f_1 + I_1 \right)^2 + \frac{s^2}{B_Z^2} - \frac{2 s R B_R}{B_Z} \left( f_1 + I_1 \right), \ I_1 = \int_{l_0}^l \frac{\partial}{\partial R} \left( \frac{s}{B_Z} \right) \frac{dl}{|\nabla \psi|}$$
(B.5)

Subtracting both equations one finds:

$$(f_1 + I_1)^2 - (f_3 + I_3)^2 - \frac{2s}{RB_p^2} \left(\frac{B_R}{B_Z} (f_1 + I_1) - \frac{B_r}{B_\theta} (f_3 + I_3)\right) + \frac{s^2}{R^2B_p^2} \left(\frac{1}{B_Z^2} - \frac{1}{B_\theta^2}\right) = 0$$
(B.6)

If one differentiates this equation along the poloidal magnetic field lines, one obtains a linear relationship between the integrals  $I_1$  and  $I_3$ :

$$k_{3} (f_{3} + I_{3}) = k_{1} (f_{1} + I_{1}) + k_{2}$$
(B.7)  

$$k_{1} = \vec{B}_{p} \cdot \nabla I_{1} - \vec{B}_{p} \cdot \nabla \left(\frac{s B_{R}}{R B_{Z} B_{p}^{2}}\right) , \quad \vec{B}_{p} \cdot \nabla I_{1} = \frac{1}{R} \frac{\partial}{\partial R} \left(\frac{s}{B_{Z}}\right)$$

$$k_{3} = \vec{B}_{p} \cdot \nabla I_{3} - \vec{B}_{p} \cdot \nabla \left(\frac{s B_{r}}{R B_{\theta} B_{p}^{2}}\right) , \quad \vec{B}_{p} \cdot \nabla I_{3} = \frac{1}{R r} \frac{\partial}{\partial r} \left(\frac{s r}{B_{\theta}}\right)$$

$$= \vec{B}_{p} \cdot \nabla \left[\frac{s^{2}}{2 R^{2} B_{p}^{2}} \left(\frac{1}{B_{Z}^{2}} - \frac{1}{B_{\theta}^{2}}\right)\right] - \frac{s}{R B_{p}^{2}} \left(\frac{B_{R}}{B_{Z}} \vec{B}_{p} \cdot \nabla I_{1} - \frac{B_{r}}{B_{\theta}} \vec{B}_{p} \cdot \nabla I_{3}\right)$$

Eliminating from (B.6) and (B.7) the integral  $I_3$  one obtains a quadratic equation for  $(f_1 + I_1)$ :

$$(f_1 + I_1)^2 \left(1 - \frac{k_1^2}{k_3^2}\right) - 2 \left(f_1 + I_1\right) \left[\frac{k_1 k_2}{k_3^2} + \frac{s}{R B_p^2} \left(\frac{B_R}{B_Z} - \frac{B_r}{B_\theta} \frac{k_1}{k_3}\right)\right] + \frac{2 s B_r}{R B_\theta B_p^2} \frac{k_2}{k_3} - \frac{k_2^2}{k_3^2} + \frac{s^2}{R^2 B_p^2} \left(\frac{1}{B_Z^2} - \frac{1}{B_\theta^2}\right) = 0$$
(B.8)

Because of (25):  $f_1 + I_1 = E_R/(R B_Z)$  equation (B.8) yields an explicit algebraic expression for  $E_R$  as a function of s and the poloidal magnetic field components. This

result is not compatible with  $E_R$  given as an integral in the first equation of (25), unless condition (27) is satisfied. In this case follows from (B.6):  $f_1^2 = f_3^2 = \phi'_0{}^2$  and the potential becomes a function of  $\psi$  only.

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