

Lieb's simple proof of concavity of $(A, B) \mapsto \text{Tr } A^p K^\dagger B^{1-p} K$ and remarks on related inequalities

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Abstract

A simple, self-contained proof is presented for the concavity of the map $(A, B) \mapsto \text{Tr } A^p K^\dagger B^{1-p} K$. The author makes no claim to originality; this note gives Lieb's original argument in its simplest, rather than its most general, form. A sketch of the chain of implications from this result to concavity of $A \mapsto \text{Tr } e^{K+\log A}$ is then presented. An independent elementary proof is given for the joint convexity of the map $(A, B, X) \mapsto \text{Tr } \int_0^\infty X^\dagger \frac{1}{A+uI} X \frac{1}{B+uI} du$ which plays a key role in entropy inequalities.

1 Introduction

Properties of quantum entropy, particularly the inequality known as strong subadditivity (SSA), play an important role in quantum information theory. The original proof of SSA is based on a concavity result of Lieb, and several approaches to SSA use either this result or one of his related convex trace functions [7]. These results have acquired an undeserved reputation as difficult to prove; indeed, the influential

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book by Nielsen and Chuang [12] states on p. 645 that “no transparent proof of SSA is known.” This note is intended to remedy this situation.

During recent lectures, I presented Lieb’s original proof of the joint concavity of the map $(A, B) \mapsto \text{Tr } A^p K^\dagger B^{1-p} K$ and was reminded just how simple and elegant it really is. Unfortunately, some of this simplicity was lost when published [7] in a form that lent itself to generalizations, such as the concavity of $(A, B) \mapsto \text{Tr } A^p K^\dagger B^q K$ with $p + q \leq 1$. In view of the renewed interest in this result, and frequent reference to the elementary, but long, proof¹ given in an Appendix to Nielsen and Chuang [12], it seemed worth making a simple argument available to a larger audience.

Lieb’s results in [7] include the following three theorems which we consider only for finite dimensional matrices.

Theorem 1 *Let K be a fixed $n \times n$ matrix. For $0 \leq p \leq 1$, the map $(A, B) \mapsto F(A, B) \equiv \text{Tr } A^p K^\dagger B^{1-p} K$ is well-defined on the cone of pairs of positive, semi-definite $n \times n$ matrices A, B . Moreover, for each $p \in (0, 1)$, the function $F(A, B)$ is a concave map from this cone to $[0, \infty)$.*

Theorem 2 *The map $(A, B, K) \mapsto F(A, B, K) \equiv \text{Tr} \int_0^\infty K^\dagger \frac{1}{A+uI} K \frac{1}{B+uI} du$ is well-defined when A, B are positive, semi-definite $n \times n$ matrices, $\ker(A) \subset \ker(K)$ and $\ker(A) \subset \ker(K^\dagger)$. Moreover, $F(A, B, K)$ is jointly convex in A, B, K .*

Theorem 3 *Let K be a fixed, self-adjoint $n \times n$ matrix. The map $A \mapsto F(A) \equiv \text{Tr } e^{K+\log(A)}$ is well-defined on the cone of positive definite matrices. Moreover, $F(A)$ is concave on this cone.*

Lieb’s proof of Theorem 1 uses the fact that the modulus of a function which is analytic and uniformly bounded on a strip is bounded by its supremum on the boundary. In order to make this note self-contained and accessible to readers with varied background, we explain this result and sketch a proof in Appendix A. The three functions in the theorems above satisfy a homogeneity condition of the form $F(\lambda A) = \lambda F(A)$ or $F(\lambda A, \lambda B) = \lambda F(A, B)$, etc. which has useful consequences summarized in Appendix B.

In an earlier review [16] the author emphasized the role of the related concavity of $A \mapsto \text{Tr } e^{K+\log A}$, and presented Epstein’s proof [3] of this result. In fact, both Lieb’s and Epstein’s methods can be used to prove a much larger class of inequalities whose equivalence was established by Lieb in [7]. Epstein gives a more direct route to concavity of $A \mapsto \text{Tr } e^{K+\log A}$; while Lieb’s is simpler for $(A, B) \mapsto \text{Tr } A^p K^\dagger B^{1-p} K$. Epstein’s approach requires some deep results from complex analysis, while Lieb’s

¹Based on an argument of Uhlmann [18] as presented by Simon [17] and Wehrl [20].

argument requires only the maximum modulus principle. However, in the generalization to $\text{Tr } A^p K^\dagger B^q K$, Lieb also uses a result about concave operator functions. The theory of monotone and convex operator functions is closely connected to Epstein's approach and has other applications in quantum information theory. (For some examples and references, see [8].)

In [16], the author showed how to use the concavity of $\text{Tr } e^{K+\log A}$ to obtain simple proofs of the strong subadditivity of quantum entropy and the joint convexity of relative entropy. The advantage to the presentation in [16] is the ease with which equality conditions are obtained. However, the proof of concavity of $(A, B) \mapsto \text{Tr } A^p K^\dagger B^{1-p} K$ given here also yields a short route to the same group of entropy inequalities. By observing that

$$\lim_{p \rightarrow 0} \frac{1}{p} (\text{Tr } A^{1-p} B^p - \text{Tr } A) = -\text{Tr } A(\log A - \log B) \quad (1)$$

one can see that Theorem 1 also yields a simple proof (first noted in [11]) of the joint convexity of relative entropy.

In section 3, Lieb's arguments showing a series of implications from Theorem 1 to Theorems 2 and 3 are sketched. The goal is only to give the reader a feel for the ideas connecting seemingly disparate results. For more details, the presentation in Chapter 3 of [13] is recommended.

In Section 4, a simple new proof of Theorem 2 is presented, together with some remarks about its connection to entropy inequalities.

2 Lieb's proof of Theorem 1

Theorem 1 will first be proved for the special case $A = B$. Concavity then means that for each pair $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$,

$$\lambda_1 \text{Tr } A_1^p K^\dagger A_1^{1-p} K + \lambda_2 \text{Tr } A_2^p K^\dagger A_2^{1-p} K \leq \text{Tr } C^p K^\dagger C^{1-p} K \quad (2)$$

where $C = \lambda_1 A_1 + \lambda_2 A_2$. Observe that $\langle \phi, C\phi \rangle = 0 \Rightarrow \langle \phi, A_1\phi \rangle = \langle \phi, A_2\phi \rangle = 0$ so that (2) holds trivially for $\phi \in \ker(C)$. Hence, it suffices to prove the inequality on $\ker(C)^\perp$ so that we can assume without loss of generality that C is invertible. In finite dimensions this also implies that C^{-1} is bounded. Now define $M = C^{(1-p)/2} K C^{p/2}$ and

$$f_k(p) = \text{Tr } A_k^p C^{-p/2} M^\dagger C^{-(1-p)/2} A_k^{1-p} C^{-(1-p)/2} M C^{-p/2} \quad k = 1, 2. \quad (3)$$

Then (2) is equivalent to

$$f(p) \equiv \lambda_1 f_1(p) + \lambda_2 f_2(p) \leq \text{Tr } M^\dagger M. \quad (4)$$

Observe that the functions above can be analytically continued to the strip $0 \leq \operatorname{Re}(z) \leq 1$.

The next step is to show that each $f_k(z)$ is bounded on this strip. To do this, write $z = x + iy$ with x, y real, and observe that for $A > 0$, $\|A^{iy}\| = 1$ and $\|A^x\| = \sup_{\|\psi\|=1} \langle \psi, A^x \psi \rangle = (\sup_{\|\psi\|=1} \langle \psi, A \psi \rangle)^x = \|A\|^x$ is the (largest eigenvalue of A)^x. Then by repeated application of the inequality $|\operatorname{Tr} XY| \leq \operatorname{Tr} |XY| \leq \|X\|_\infty \operatorname{Tr} |Y|$ (where $\|X\|_\infty \equiv \|X\|$ is the usual operator sup norm as above), one finds that

$$|f_k(z)| \leq \|A_k\| \|C^{-1}\| \operatorname{Tr} M^\dagger M \leq \|A_k\| \|C^{-1}\| \|C\| \operatorname{Tr} K^\dagger K. \quad (5)$$

for $0 \leq x \leq 1$. Thus the functions $f_k(z)$ are uniformly bounded on the strip $0 \leq \operatorname{Re}(z) \leq 1$.

By the maximum modulus principle (Appendix A), $|f_k(z)|$ is bounded by its supremum on the boundary of this strip, i.e., for $z = 0 + iy$ or $z = 1 + iy$. Now,

$$\begin{aligned} f_k(0 + iy) & \quad (6) \\ &= \operatorname{Tr} (A_k^{iy/2} C^{-iy/2} M^\dagger C^{iy/2} C^{-1/2} A_k^{1/2}) (A_k^{-iy} A_k^{1/2} C^{-1/2} C^{iy/2} M C^{-iy/2} A_k^{iy/2}) \end{aligned}$$

has the form $\operatorname{Tr} X^\dagger Y$ which is bounded above by $(\operatorname{Tr} X^\dagger X \operatorname{Tr} Y^\dagger Y)^{1/2}$. Since operators of the form A^{it} are unitary for t real and A positive, one finds

$$|f_k(0 + iy)| \leq \operatorname{Tr} M^\dagger C^{iy/2} C^{-1/2} A_k C^{-1/2} C^{-iy/2} M \quad k = 1, 2. \quad (7)$$

Thus

$$|f(0 + iy)| \leq \lambda_1 |f_1(0 + iy)| + \lambda_2 |f_2(0 + iy)| \quad (8)$$

$$\begin{aligned} & \leq \operatorname{Tr} M^\dagger C^{iy/2} C^{-1/2} (\lambda_1 A_1 + \lambda_2 A_2) C^{-1/2} C^{-iy/2} M \\ & = \operatorname{Tr} M^\dagger C^{iy/2} C^{-iy/2} M = \operatorname{Tr} M^\dagger M \end{aligned} \quad (9)$$

since C was defined as $\lambda_1 A_1 + \lambda_2 A_2$. One can similarly show that $|f(1 + iy)| \leq \operatorname{Tr} M^\dagger M$ which implies (4). This establishes the concavity of $A \mapsto \operatorname{Tr} A^p K^\dagger A^{1-p} K$.

The general case then follows from the observation

$$\operatorname{Tr} A^p K^\dagger B^{1-p} K = \operatorname{Tr} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^p \begin{pmatrix} 0 & K^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^{1-p} \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}. \quad (10)$$

Extension to infinite dimensions: The restriction to finite dimensional matrices was used only to ensure that C^{-1} is bounded on the orthogonal complement of $\ker(C)$ so that (5) gives a uniform bound for $|f_k(z)|$ on the strip $0 \leq \operatorname{Re}(z) \leq 1$. It is worth emphasizing that in this part of the proof it is enough to show that $|f_k(z)|$

satisfies *some* upper bound, which can be rather crude as long as it holds uniformly for all z in the infinite strip. Only for the subsequent estimate on the boundary do we need a precise bound of the form (9), which can be generalized to operators on infinite dimensional spaces.

Therefore, the theorem can be extended to infinite dimensions in several ways. First, observe that $C = \lambda_1 A_1 + \lambda_2 A_2$ implies that for $0 \leq q \leq 1$

$$A_k \leq \lambda_k^{-1} C \Rightarrow A_k^q \leq \lambda_k^{-q} C^q \Rightarrow C^{-q/2} A_k^q C^{-q/2} \leq \lambda_k^{-q} I \quad (11)$$

where the first implication uses the operator monotonicity of the map $A \rightarrow A^q$. Then under the additional hypothesis that K is Hilbert-Schmidt (i.e., $\text{Tr } K^\dagger K < \infty$), one can replace (5) by

$$|f_k(z)| \leq \lambda_k^{-1} \text{Tr } M^\dagger M \leq \lambda_k^{-1} \|C\| \text{Tr } K^\dagger K. \quad (12)$$

Lieb uses the even weaker assumption that $M = C^{q/2} K C^{p/2}$ is Hilbert-Schmidt, and also proves concavity for the map $(A, B) \mapsto \text{Tr } A^p K^\dagger B^q K$ for $0 \leq p+q \leq 1$. For this, he uses the operator concavity of the map $A \mapsto A^t$ for $0 \leq t \leq 1$ to conclude that $C^{-t/2} (\lambda_1 A_1^t + \lambda_2 A_2^t) C^{-t/2} \leq I$.

3 Connecting the concavity theorems

3.1 Non-commutative multiplication and differentiation

The key to connecting the two concavity results mentioned above was Lieb's realization that, for any fixed positive semi-definite matrix A , the following two linear maps on $n \times n$ matrices are inverses of each other, i.e.,

$$X = \Omega_A(K) \equiv \int_0^1 A^p K A^{1-p} dp \quad (13)$$

\Leftrightarrow

$$K = \Omega_A^{-1}(X) \equiv \int_0^\infty \frac{1}{A + uI} X \frac{1}{A + uI} du. \quad (14)$$

This is far from obvious, but can be verified by expanding in a basis of eigenvectors of A . Moreover, Ω_A can be regarded as a non-commutative version of multiplication by A and Ω_A^{-1} as a non-commutative version of multiplication by A^{-1} . Both operators are positive semi-definite with respect to the inner product $\text{Tr } A^\dagger B = \langle A, B \rangle$, i.e., $\text{Tr } K^\dagger \Omega_A(K) \geq 0$ and $\text{Tr } X^\dagger \Omega_A^{-1}(X) \geq 0$.

The operator Ω_A^{-1} arises when one uses the integral representation

$$\log P - \log Q = \int_0^\infty \left[\frac{1}{Q + uI} - \frac{1}{P + uI} \right] du \quad (15)$$

$$= \int_0^\infty \left[\frac{1}{Q + uI} (P - Q) \frac{1}{P + uI} \right] du \quad (16)$$

to compute derivatives which arise in studying entropy. In particular, when $f(x) = \log(A + xK)$ with $K = K^\dagger$ self-adjoint, it follows from (15) that $f'(0) = \Omega_A^{-1}(K)$ and $f''(0) = -2\Upsilon_A(K)$ where

$$\Upsilon_A(K) = \int_0^\infty \frac{1}{A + uI} K^\dagger \frac{1}{A + uI} K \frac{1}{A + uI} du. \quad (17)$$

This leads [13] to the useful (and norm convergent) expansion

$$\log(A + xK) = \log(A) + x\Omega_A^{-1}(K) - x^2\Upsilon_A(K) + \dots \quad (18)$$

3.2 From Theorem 1 to Theorem 2

To show that Theorem 1 implies Theorem 2, first observe that the joint convexity of $X^\dagger \Omega_A^{-1}(X)$ is equivalent to the inequality

$$\begin{aligned} s \operatorname{Tr} X^\dagger \Omega_A^{-1}(X) + (1-s) \operatorname{Tr} Y^\dagger \Omega_B^{-1}(Y) \\ \geq \operatorname{Tr} (sX + (1-s)Y)^\dagger \Omega_{sA+(1-s)B}^{-1}(sX + (1-s)Y) \end{aligned} \quad (19)$$

which can be rewritten as

$$\operatorname{Tr} \begin{pmatrix} X^\dagger & Y^\dagger \end{pmatrix} \begin{pmatrix} s\Omega_A^{-1} & 0 \\ 0 & (1-s)\Omega_B^{-1} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (20a)$$

$$\geq \begin{pmatrix} X^\dagger & Y^\dagger \end{pmatrix} \Omega_{sA+(1-s)B}^{-1} \otimes \begin{pmatrix} s^2 & s(1-s) \\ s(1-s) & (1-s)^2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \quad (20b)$$

We now regard A, B, s as fixed and consider the ratio obtained by dividing (20b) by (20a). Joint convexity holds if this ratio is ≤ 1 for all choices of (X, Y) . This is an optimization problem of the form $\sup_v \frac{\langle v, Cv \rangle}{\langle v, Dv \rangle}$ which is equivalent to the eigenvalue problem $Cx = \lambda Dx$ when C, D are positive semi-definite. For the operators used here, the reduction yields a pair of equations

$$\Omega_{sA+(1-s)B}^{-1}(sX + (1-s)Y) = \lambda \Omega_A^{-1}(X) \quad (21a)$$

$$\Omega_{sA+(1-s)B}^{-1}(sX + (1-s)Y) = \lambda \Omega_B^{-1}(Y) \quad (21b)$$

and we want to show that $\lambda \leq 1$. Define $M = \Omega_{sA+(1-s)B}^{-1}(sX + (1-s)Y)$ and observe that (21) gives three expressions for M since

$$M = \Omega_{sA+(1-s)B}^{-1}(sX + (1-s)Y) = \lambda \Omega_A^{-1}(X) = \lambda \Omega_B^{-1}(Y). \quad (22)$$

Applying the appropriate inverse operator to each of these and combining the last two, one finds

$$sX + (1-s)Y = \Omega_{sA+(1-s)B}(M) \quad (23a)$$

$$\lambda[sX + (1-s)Y] = s\Omega_A(M) + (1-s)\Omega_B(M) \quad (23b)$$

Now Theorem 1 implies

$$s\text{Tr } M^\dagger \Omega_A(M) + (1-s)\text{Tr } M^\dagger \Omega_B(M) \leq \text{Tr } M^\dagger \Omega_{sA+(1-s)B}(M), \quad (24)$$

which then implies

$$(1-\lambda)\text{Tr } M^\dagger [sX + (1-s)Y] = (1-\lambda)\text{Tr } M^\dagger \Omega_{sA+(1-s)B}(M) \leq 0. \quad (25)$$

It then follows from the fact that $\Omega_{sA+(1-s)B}$ is positive semi-definite that $\lambda \leq 1$.

3.3 From Theorem 2 to Theorem 3

Showing that Theorem 2 implies the concavity of the map $A \mapsto \text{Tr } e^{K+\log A}$ is elementary, but tedious, as it is “merely” a matter of computing derivatives with attention to non-commutativity. Let $f(x) = \text{Tr } e^{K+\log(A+xB)}$. Then $f''(0) \leq 0$ implies that $\text{Tr } e^{K+\log A}$ is concave. One can use power series to verify that $\frac{d}{dx} e^{F+xG} = \Omega_{e^F}(G)$. This can then be combined with (18) to yield

$$f''(0) = \text{Tr } \Omega_A^{-1}(B) \Omega_{e^{K+\log A}} [\Omega_A^{-1}(B)] - 2\text{Tr } e^{K+\log A} \Upsilon_A(B). \quad (26)$$

(Chapter 3 of [13] gives a clear exposition of the details.) To show that (26) is negative, one can apply the inequality (44) to $g(x) = G(A + xD, K + xL)$ with $G(A, K) = -\text{Tr } K \Omega_A^{-1}(K)$. One finds

$$g'(0) = -2\text{Tr } D \Upsilon_A(K) + 2\text{Tr } L \Omega_A^{-1}(K) \leq \text{Tr } L \Omega_D^{-1}(L). \quad (27)$$

Now choose $K = B, D = e^{K+\log A}$ and $L = \Omega_{e^{K+\log A}} [\Omega_A^{-1}(B)]$. Then $\text{Tr } L \Omega_D^{-1}(L) = \text{Tr } L \Omega_A^{-1}(B)$. Making the appropriate substitutions in (26) and (27) yields

$$f''(0) \leq 2\left(\text{Tr } L \Omega_D^{-1}(L) - \text{Tr } L \Omega_A^{-1}(B)\right) = 0. \quad (28)$$

Although we have sketched the path from concavity of $\text{Tr } K^\dagger A^p K A^{1-p}$ to concavity of $\text{Tr } e^{K+\log A}$, most of the steps are easily seen to be reversible. With a bit more effort [7], one can show that the reverse implication also holds.

4 Theorem 2: a direct proof and implications

4.1 Proof

Lieb and Ruskai [10] proved the joint operator convexity of the map $(X, A) \mapsto X^\dagger A^{-1} X$. Unfortunately, this does not seem to directly imply the joint convexity of $\text{Tr } X^\dagger \Omega_A^{-1}(X)$. However, the following observation allows one to adapt the argument in [10] to give another proof of the joint convexity of $\text{Tr } X^\dagger \Omega_A^{-1}(X)$. Let L_A and R_A denote left and right multiplication by A so that $L_A(X) = AX$ and $R_A(X) = XA$. By expanding in a basis in which A is diagonal, one can verify that

$$\text{Tr } K^\dagger \Omega_A^{-1}(K) = \text{Tr} \int_0^\infty K^\dagger (L_A + tR_A)^{-1}(K) \frac{1}{1+t} dt, \quad (29)$$

and

$$\text{Tr } A \Upsilon_A(K) = \text{Tr} \int_0^\infty K^\dagger (L_A + tR_A)^{-1}(K) \frac{1}{(1+t)^2} dt. \quad (30)$$

The operator $(L_A + tR_A)^{-1}$ can be regarded as another non-commutative version of multiplication by A^{-1} . It is positive semi-definite with respect to the Hilbert-Schmidt inner product since both L_A and R_A are self-adjoint and positive semi-definite. For example, $\text{Tr} [L_A(W)]^\dagger X = \text{Tr} (AW)^\dagger X = \text{Tr } W^\dagger AX = \text{Tr } W^\dagger L_A(X)$ and $\text{Tr } X^\dagger L_A(X) \geq 0$.

Lemma 4 *For each fixed $t \geq 0$ the map $(A, K) \mapsto \text{Tr } K^\dagger (L_A + tR_A)^{-1}(K)$ is jointly convex.*

Proof: Let $M_j = (L_{A_j} + tR_{A_j})^{-1/2}(K_j) - (L_{A_j} + tR_{A_j})^{1/2}(\Lambda)$. Then

$$\begin{aligned} 0 &\leq \sum_j \text{Tr } M_j^\dagger M_j \\ &= \sum_j \text{Tr } K_j^\dagger (L_{A_j} + tR_{A_j})^{-1}(K_j) - \text{Tr} (\sum_j K_j^\dagger) \Lambda \\ &\quad - \text{Tr } \Lambda^\dagger (\sum_j K_j) + \text{Tr } \Lambda^\dagger \sum_j (L_{A_j} + tR_{A_j}) \Lambda. \end{aligned} \quad (31)$$

Next, note that $\sum_j (L_{A_j} + tR_{A_j})(W) = \sum_j (A_j W + tW A_j) = (\sum_j A_j) W + tW (\sum_j A_j) = L_{\sum_j A_j}(W) + tR_{\sum_j A_j}(W)$ for any matrix W . Therefore, inserting the choice $\Lambda = (L_{\sum_j A_j} + tR_{\sum_j A_j})^{-1}(\sum_j K_j)$ in (31) yields

$$\text{Tr} (\sum_j K_j)^\dagger (L_{\sum_j A_j} + tR_{\sum_j A_j})^{-1}(\sum_j K_j) \leq \sum_j \text{Tr } K_j^\dagger (L_{A_j} + tR_{A_j})^{-1}(K_j). \quad (32)$$

for any $t \geq 0$. The joint convexity then follows from the replacement $A \rightarrow \lambda_j A$, $K \rightarrow \lambda_j K$ with $\lambda_j > 0$ and $\sum_j \lambda_j = 1$. Alternatively, one can simply observe that $\text{Tr} \lambda K^\dagger (L_{\lambda A} + t R_{\lambda A})^{-1} (\lambda K) = \lambda \text{Tr} K^\dagger (L_A + t R_A)^{-1} (K)$ and use the first observation in Appendix B to reduce the joint convexity to (32). **QED**

The joint convexity of both $\text{Tr} K^\dagger \Omega_A^{-1}(K)$ and $\text{Tr} A \Upsilon_A^{-1}(K)$ follow immediately from Lemma 4 and the integral representations above. In particular, inserting (32) in (29) yields

$$\text{Tr} (\sum_j K_j)^\dagger \Omega_{\sum_j A_j}^{-1} (\sum_j K_j) \leq \sum_j \text{Tr} K_j^\dagger \Omega_{A_j}^{-1} (K_j) \quad (33)$$

which proves Theorem 2 when $A = B$. The general case then follows as in (10). The choice $t = 0$ in Lemma 4 yields the operator convexity of $X^\dagger A^{-1} X$ proved in [10].

4.2 From Theorem 2 to entropy inequalities

The von Neumann entropy is defined as $S(P) = -\text{Tr} P \log P$ and the relative entropy as $H(P, Q) \equiv \text{Tr} P (\log P - \log Q)$ with P, Q positive semi-definite and $\ker(Q) \subset \ker(P)$. (One usually assumes that $\text{Tr} P = \text{Tr} Q = 1$, but this is not strictly necessary.) Several special cases of the joint convexity of $H(P, Q)$ follow immediately from Theorem 2.

Using (18), one can show that $\frac{d}{dx} S(A + xK) \Big|_{x=0} = -\text{Tr} K \log(A)$, when $\text{Tr} K = 0$ and that the second derivative satisfies

$$-\Delta(A, K) \equiv \frac{d^2}{dx^2} S(A + xK) \Big|_{x=0} \quad (34)$$

$$= -\text{Tr} K \Omega_A^{-1}(K) + \text{Tr} A \Upsilon_A(K) \quad (35)$$

$$= -\text{Tr} \int_0^\infty K^\dagger (L_A + t R_A)^{-1} (K) \frac{t}{(1+t)^2} dt \quad (36)$$

where we used the integral representations (29) and (30) together with the simple fact $\frac{1}{1+t} - \frac{1}{(1+t)^2} = \frac{t}{(1+t)^2}$. Lemma 4 implies that $\Delta(A, K)$ is jointly convex. One also has

$$\frac{\partial^2}{\partial a \partial b} H(P + aA, P + bB) \Big|_{a=b=0} = \text{Tr} A^\dagger \Omega_P^{-1}(B). \quad (37)$$

Now consider density matrices defined on a tensor product of two or three spaces. First, suppose that $\gamma_{12} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is block diagonal with A, B in \mathcal{H}_2 so that $S(\gamma_{12}) = S(A) + S(B)$ and $\gamma_2 = \text{Tr}_1 \gamma_{12} = A + B$. Then

$$S(\gamma_{12}) - S(\gamma_2) = S(A) + S(B) - S(A + B) \quad (38)$$

Now consider the perturbed density matrix $\gamma_{12} = \begin{pmatrix} A + xK & 0 \\ 0 & B + xL \end{pmatrix}$ and let $g(x) = S(A + xK) + S(B + xL) - S[A + B + x(K + L)]$. Then

$$g''(0) = -\Delta(A, K) - \Delta(B, L) + \Delta(A + B, K + L) \leq 0 \quad (39)$$

because $\Delta(A, K)$ is jointly convex in (A, K) . This proves the concavity of the map $\gamma_{12} \mapsto S(\gamma_{12}) - S(\gamma_2)$ in the special case of block diagonal matrices.

The situation above can be compared to $\rho_{123} = \begin{pmatrix} t\rho'_{12} & 0 \\ 0 & (1-t)\rho''_{12} \end{pmatrix}$. In this case, the strong subadditivity inequality in the form $S(\rho_{123}) - S(\rho_{23}) \leq S(\rho_{12}) - S(\rho_2)$ is precisely the concavity of $\rho_{12} \mapsto S(\rho_{12}) - S(\rho_2)$ with $\rho_{12} = t\rho'_{12} + (1-t)\rho''_{12}$ arbitrary. In the previous paragraph, ρ'_{12} and ρ''_{12} were also required to be block diagonal.

The joint convexity of the less familiar symmetrized relative entropy follows immediately from the joint convexity of $\text{Tr } X^\dagger \Omega_A^{-1}(X)$. Using (29) one easily finds

$$H(P, Q) + H(Q, P) = \text{Tr} \int_0^\infty \left[(P - Q) \frac{1}{Q + uI} (P - Q) \frac{1}{P + uI} \right] du. \quad (40)$$

It then follows immediately from Theorem 2 with $K = P - Q$ that $H(P, Q) + H(Q, P)$ is jointly convex in P, Q . This implies that at least one of $H(P, Q)$ and $H(Q, P)$ is jointly convex in P, Q .

Although Theorem 1 gives a short route to special cases of the joint convexity of relative entropy, proving the general case seems to require one of the paths through Theorem 1 or Theorem 3. Bauman [2] also recognized that there is a connection between the concavity of $S(\rho_{12}) - S(\rho_2)$ for block diagonal ρ_{12} and joint convexity of $\text{Tr } X^\dagger \Omega_A^{-1}(X)$ for self-adjoint X ; and proved the latter for 2×2 matrices.

4.3 Further remarks

The argument used to prove Lemma 4 is remarkably simple² (and can be applied to $L_{A_j} + tR_{B_j}$ to give a direct proof of the general case of Theorem 2.) However, the insight needed to rewrite $\text{Tr } K^\dagger \Omega_A^{-1}(K)$ in the form (29) comes from the very powerful relative modular operator formalism originally developed by Araki [1] to extend relative entropy to infinite dimensional operator algebras which might not even have a trace. The utility of this formalism in finite dimensional situations was not realized until much later. For an exposition see Ohya and Petz [13]. Lemma 4 can also be used to prove the joint convexity of any operator which has an integral

²In quant-ph/0604206 the argument presented in Section 4.1 is used to give a direct proof of the joint convexity of relative entropy.

representation as in (29) or (30) with a positive weight, and the proof presented here is similar to one presented in [8] to prove monotonicity of a generalized form of relative entropy and related inequalities within the theory of monotone Riemannian metrics developed by Petz [14].

In view of connections to the Bures metric and quantum channels, we make a few further remarks. Monotone means decreasing under completely positive trace preserving maps, and (29) is the Riemannian metric associated with the usual relative entropy $H(P, Q)$. A large class of such metrics, generalizations of relative entropy, and of the operator Ω_A^{-1} can be obtained by replacing the weight $\frac{1}{1+t}$ in (29) by one coming from a subclass of monotone operator functions. The bound,

$$R_A^{-1} + L_A^{-1} \geq \Omega_A^{-1} \geq (R_A + L_A)^{-1}, \quad (41)$$

which holds as an operator inequality is also satisfied by these generalizations of Ω_A^{-1} , and the upper and lower bounds are special cases of these generalizations. This is of interest because any of these Riemannian metrics can be used to define a geodesic distance $D(P, Q)$ between two density matrices. The Bures metric given by $[D_{\text{Bures}}(P, Q)]^2 = 2[1 - \text{Tr}(\sqrt{P}Q\sqrt{P})^{1/2}]$ is precisely the geodesic distance associated with $\text{Tr} X^\dagger (R_A + L_A)^{-1} X$. The importance of the Bures metric, which is closely related to the fidelity, in quantum information has been emphasized by Uhlmann [19]. Unfortunately, no closed form expression for the geodesic distance associated with (29) is known. See [5, 6] for bounds and a discussion of the geodesic distances for the Wigner-Yanase-Dyson entropies, which are closely related to the function considered in Theorem 1.

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A Maximum modulus principle

The maximum modulus principle given in most elementary texts, states that the modulus of an analytic function can not have a local maximum on a bounded open set unless it is a constant. It then follows that a function $f(z)$ which is analytic on a bounded set and continuous on its closure achieves the supremum of $|f(z)|$ on the boundary.

Now consider a function $f(z)$ which is analytic on the strip $\{z : 0 < \text{Re } z < 1\}$ and continuous and uniformly bounded on its closure $\{z : 0 \leq \text{Re } z \leq 1\}$. Let $M = \sup\{|f(z)| : 0 \leq \text{Re } z \leq 1\}$. The maximum modulus principle for a strip

[5] says that M is equal to the supremum of $|f(z)|$ on the boundary, i.e., $M = \sup\{|f(z)| : z = 0 + iy \text{ or } z = 1 + iy\}$.

If the supremum is actually attained at some point $\tilde{z} = a + iy$, i.e., $|f(\tilde{z})| = M$, the result follows easily from the theorem for bounded regions. The possibility of $a \neq 0, 1$ can be excluded by considering $f(z)$ on the rectangular region $\mathcal{Q}_b = \{z : 0 \leq \operatorname{Re} z \leq 1 \text{ and } -b \leq \operatorname{Im} z \leq b\}$ with $b > y$. Since f cannot have a relative maximum on \mathcal{Q}_b , one has a contradiction unless f constant. If $a = 0$ or $a = 1$, then \tilde{z} lies on the boundary of the strip and the assertion holds. To prove the result if the supremum is not attained, consider the functions $f_\varepsilon(z) \equiv z^{-\varepsilon} f(z)$. Since $\lim_{y \rightarrow \pm\infty} |f_\varepsilon(x + iy)| = 0$, the maximum modulus $M_\varepsilon = \sup\{|f_\varepsilon(z)| : 0 \leq \operatorname{Re} z \leq 1\}$ is attained for some $z_\varepsilon = iy$ or $z_\varepsilon = 1 + iy$. But $|f_\varepsilon(z)| \leq |f(z)|$, and $M = \lim_{\varepsilon \rightarrow 0} M_\varepsilon = \sup\{|f(z)| : z = 0 + iy \text{ or } z = 1 + iy\}$.

B Homogenous concave functions

Most of the functions considered here are homogenous of degree one, i.e., $g(\mu A) = \mu g(A)$. In this case concavity is equivalent to superadditivity. To see this observe that homogeneity and concavity at $\frac{1}{2}$ imply

$$\frac{1}{2}g(A + B) = g\left(\frac{1}{2}[A + B]\right) \geq \frac{1}{2}g(A) + \frac{1}{2}g(B). \quad (42)$$

Conversely, if g is homogenous and superadditive

$$g[xA + (1 - x)B] \geq g(xA) + g[(1 - x)B] = xg(A) + (1 - x)g(B) \quad (43)$$

One extremely useful property of homogenous concave functions is the following derivative inequality [15].

$$\lim_{x \rightarrow 0} \frac{g(A + xB) - g(A)}{x} \leq g(B) \quad (44)$$

which follows easily from $g(A + xB) \leq g(A) + xg(B)$.

C Erratum to quant-ph/0404126 by M.B. Ruskai

Since this note appeared, people who have presented the proof in class have reported some minor errors and omissions in the argument presented in the note. The problems and necessary modifications are described in detail below.

First, the cyclicity of the trace and procedure described above (5) yield

$$|f_k(z)| \leq \|A_k\|^{1-x} \|C^{-1}\|^{1-x} \text{Tr} |MC^{-(x+iy)/2} A_k^{x+iy} C^{-(x+iy)/2} M^\dagger| . \quad (45)$$

For $y \neq 0$ one can *not* remove the $||$ because the quantity inside is not positive semi-definite. From a pedagogical point of view, the simplest argument may be to let $G(z) = C^{-z/2} A_k^z C^{-z/2}$ and write $f_k(z) = \text{Tr} M^\dagger G_k(1-z) M G_k(z)$. Then use (49) to obtain

$$|f_k(z)| \leq (\text{Tr} M^\dagger G_k(1-z) G_k(1-\bar{z}) M)^{1/2} (\text{Tr} M^\dagger G_k(\bar{z}) G_k(z) M)^{1/2}. \quad (46)$$

Then, the cyclicity of the trace and procedure described above (5) yields

$$\begin{aligned} \text{Tr} M^\dagger G_k(\bar{z}) G_k(z) M &= \text{Tr} G_k(\bar{z}) G_k(z) M M^\dagger \\ &\leq \|C^{-1}\|^{2x} \|A_k\|^{2x} \text{Tr} M^\dagger M. \end{aligned} \quad (47)$$

Combining this with a similar estimate for $\text{Tr} M^\dagger G_k(1-z) G_k(1-\bar{z}) M$ and taking the square root gives the desired result.

Another alternative is to realize that one only needs a bound independent of z , not one with the precise form (5). The polar decomposition theorem implies that one can write any operator as $X = V|X|$ with V unitary and $|X| = \sqrt{X^\dagger X}$ so that $|X| = V^\dagger X$. Using this in (45) yields

$$|f_k(z)| \leq \|A_k\| \|C^{-1}\| \text{Tr} |M^\dagger V^\dagger M| \leq \|A_k\| \|C^{-1}\| \|M\| \text{Tr} |M|. \quad (48)$$

Although the unitary V may depend on z , the last bound on the right is independent of z . One could also use (49) to show $\text{Tr} |M^\dagger V^\dagger M| \leq \text{Tr} M^\dagger M \equiv \|M\|_2^2$; however, the weaker bound above will suffice.

Second, to bound (6) one needs the inequality

$$|\text{Tr} X^\dagger Y| \leq (\text{Tr} X^\dagger X)^{1/2} (\text{Tr} Y^\dagger Y)^{1/2} \leq \frac{1}{2} (\text{Tr} X^\dagger X + \text{Tr} Y^\dagger Y) \quad (49)$$

and (7) should be replaced by

$$\begin{aligned} |f_k(0+iy)| &\leq \frac{1}{2} (\text{Tr} M^\dagger C^{iy/2} C^{-1/2} A_k C^{-1/2} C^{-iy/2} M \\ &\quad + \text{Tr} M^\dagger C^{-iy/2} C^{-1/2} A_k C^{-1/2} C^{iy/2} M) \quad k = 1, 2. \end{aligned} \quad (50)$$

Then (8) becomes

$$\begin{aligned} |f(0+iy)| &\leq \lambda_1 |f_1(0+iy)| + \lambda_2 |f_2(0+iy)| \\ &\leq \frac{1}{2} (\text{Tr} M^\dagger C^{iy/2} C^{-1/2} (\lambda_1 A_1 + \lambda_2 A_2) C^{-1/2} C^{-iy/2} M \\ &\quad + \text{Tr} M^\dagger C^{-iy/2} C^{-1/2} (\lambda_1 A_1 + \lambda_2 A_2) C^{-1/2} C^{iy/2} M) \\ &= \frac{1}{2} (\text{Tr} M^\dagger C^{iy/2} C^{-iy/2} M + \text{Tr} M^\dagger C^{-iy/2} C^{iy/2} M) = \text{Tr} M^\dagger M. \end{aligned}$$

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